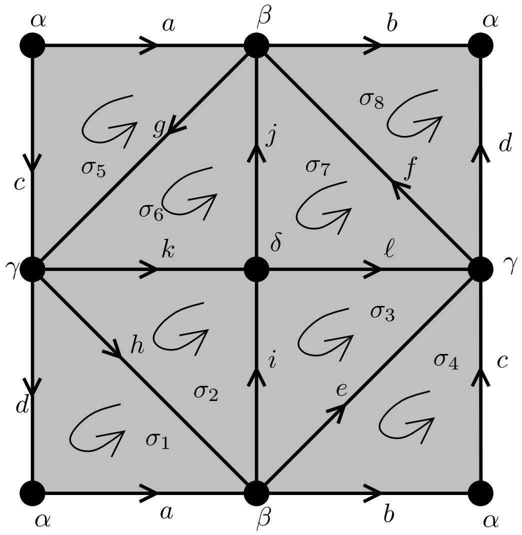
PROBLEM SET 9
Due: 13.07.2023

Instructions

Problems marked with $(*)$ will be graded. Solutions may be written up in German or English and should be handed in before the Übung on the due date. For problems without $(*)$, you do not need to write up your solutions, but it is highly recommended that you think through them before the next Thursday lecture.

Problems

1. The following picture shows a simplicial complex $K = (V, S)$ whose associated polyhedron $|K|$ is homeomorphic to the Klein bottle.



There are four vertices $V = \{\alpha, \beta, \gamma, \delta\}$, twelve 1-simplices labeled by letters $a, \dots, \ell$, and eight 2-simplices labeled σ_i for $i = 1, \dots, 8$. The picture also shows a choice of orientation for each of the 2-simplices[1] (circular arrows represent a cyclic ordering of the vertices) and 1-simplices (arrows point from the first vertex to the last). Assume every 0-simplex is chosen to have the positive orientation.

(a) Write down $\partial \sigma_i$ explicitly for each $i = 1, \dots, 8$.

(b) $(*)$ Prove that $H_2^\Delta(K; \mathbb{Z}_2) \cong \mathbb{Z}_2$, and write down a specific 2-cycle that generates it.
Hint: If this were an oriented triangulation, there would be an obvious way to find a 2-cycle with integer coefficients (we did it for $\mathbb{T}^2$ in lecture). The use of $\mathbb{Z}_2$ coefficients is meant to make up for the fact that the triangulation is not oriented.

(c) $(*)$ Prove that $H_2^\Delta(K; \mathbb{Z}) = 0$.
Hint: Consider how the coefficients of individual 1-simplices in $\partial \sum_{i=1}^8 c_i \sigma_i \in C_1(K; \mathbb{Z})$ are determined. Show that if $\sum_{i=1}^8 c_i \sigma_i$ is a cycle, then $c_1 = c_2$, $c_2 = c_3$ and so forth, but also $c_1 + c_8 = 0$.

(d) Show that the 1-cycle $c + d$ represents a nontrivial homology class $[c + d]$ in both $H_1^\Delta(K; \mathbb{Z})$ and $H_1^\Delta(K; \mathbb{Z}_2)$, but satisfies $2[c + d] = 0 \in H_1^\Delta(K; \mathbb{Z})$ and $[c + d] = 0 \in H_1^\Delta(K; \mathbb{Q})$.[2]

[1] Notice however that this does not define an oriented triangulation, as the chosen orientations of neighboring 2-simplices are not always compatible with each other. The Klein bottle does not admit an oriented triangulation.

[2] In any abelian group G, there is an obvious definition of mg for any $m \in \mathbb{Z}$ and $g \in G$, so e.g. $2[c + d] := [c + d] + [c + d]$.

2. $(*)$ Show that for the 1-point space $\{\text{pt}\}$ and any coefficient group G, singular homology satisfies

$$H_n(\{\text{pt}\}; G) \cong \begin{cases} G & \text{for } n = 0, \\ 0 & \text{for } n \neq 0. \end{cases}$$

Hint: For each integer $n \geqslant 0$, there is exactly one singular n-simplex $\Delta^n \to \{\text{pt}\}$, so the chain groups $C_n(\{\text{pt}\}; G)$ are all naturally isomorphic to G. What is $\partial : C_n(\{\text{pt}\}; G) \to C_{n-1}(\{\text{pt}\}; G)$?

3. In this problem, we prove that $H_1(X; \mathbb{Z})$ for a path-connected space X is isomorphic to the abelianization of its fundamental group. Fix a base point $x_0 \in X$ and abbreviate $\pi_1(X) := \pi_1(X, x_0)$, so elements of $\pi_1(X)$ are represented by paths $\gamma : I \to X$ with $\gamma(0) = \gamma(1) = x_0$. Identifying the standard 1-simplex

$$\Delta^1 := \left\{ (t_0, t_1) \in \mathbb{R}^2 \;\middle|\; t_0 + t_1 = 1,\ t_0, t_1 \geqslant 0 \right\}$$

with $I := [0, 1]$ via the homeomorphism $\Delta^1 \to I : (t_0, t_1) \mapsto t_0$, every path $\gamma : I \to X$ corresponds to a singular 1-simplex $\Delta^1 \to X$, which we shall denote by $\tilde{h}(\gamma)$ and regard as an element of the singular 1-chain group $C_1(X; \mathbb{Z}) = \bigoplus_{\sigma \in \mathcal{K}_1(X)} \mathbb{Z}$. Show that $\tilde{h}$ has each of the following properties:

   (a) If $\gamma : I \to X$ satisfies $\gamma(0) = \gamma(1)$, then $\partial \tilde{h}(\gamma) = 0$.

   (b) For any constant path $e : I \to X$, $\tilde{h}(e) = \partial \sigma$ for some singular 2-simplex $\sigma : \Delta^2 \to X$.

   (c) $(*)$ For any paths $\alpha, \beta : I \to X$ with $\alpha(1) = \beta(0)$, the concatenated path $\alpha \cdot \beta : I \to X$ satisfies $\tilde{h}(\alpha) + \tilde{h}(\beta) - \tilde{h}(\alpha \cdot \beta) = \partial \sigma$ for some singular 2-simplex $\sigma : \Delta^2 \to X$.
   Hint: Imagine a triangle whose three edges are mapped to X via the paths α, β and $\alpha \cdot \beta$. Can you extend this map continuously over the rest of the triangle?

   (d) If $\alpha, \beta : I \to X$ are two paths that are homotopic with fixed end points, then $\tilde{h}(\alpha) - \tilde{h}(\beta) = \partial f$ for some singular 2-chain $f \in C_2(X; \mathbb{Z})$.
   Hint: If you draw a square representing a homotopy between α and β, you can decompose this square into two triangles.

   (e) Applying $\tilde{h}$ to paths that begin and end at the base point x_0, deduce that $\tilde{h}$ determines a group homomorphism $h : \pi_1(X) \to H_1(X; \mathbb{Z}) : [\gamma] \mapsto [\tilde{h}(\gamma)]$.

We call $h : \pi_1(X) \to H_1(X; \mathbb{Z})$ the **Hurewicz homomorphism**. Notice that since $H_1(X; \mathbb{Z})$ is abelian, $\ker h$ automatically contains the commutator subgroup $[\pi_1(X), \pi_1(X)] \subset \pi(X)$ (see Problem Set 6 #2), thus h descends to a homomorphism on the abelianization of $\pi_1(X)$,

$$\Phi : \pi_1(X) \big/ [\pi_1(X), \pi_1(X)] \to H_1(X; \mathbb{Z}).$$

We will now show that this is an isomorphism by writing down its inverse. For each point $p \in X$, choose arbitrarily a path $\omega_p : I \to X$ from x_0 to p, and choose ω_{x_0} in particular to be the constant path. Regarding singular 1-simplices $\sigma : \Delta^1 \to X$ as paths $\sigma : I \to X$ under the usual identification of I with Δ^1, we can then associate to every singular 1-simplex $\sigma \in C_1(X; \mathbb{Z})$ a concatenated path

$$\widetilde{\Psi}(\sigma) := \omega_{\sigma(0)} \cdot \sigma \cdot \omega_{\sigma(1)}^{-1} : I \to X$$

which begins and ends at the base point x_0, hence $\widetilde{\Psi}(\sigma)$ represents an element of $\pi_1(X)$. Let $\Psi(\sigma)$ denote the equivalence class represented by $\widetilde{\Psi}(\sigma)$ in the abelianization $\pi_1(X)/[\pi_1(X), \pi_1(X)]$. This uniquely determines a homomorphism[3]

$$\Psi : C_1(X; \mathbb{Z}) \to \pi_1(X)/[\pi_1(X), \pi_1(X)] : \sum_i m_i \sigma_i \mapsto \sum_i m_i \Psi(\sigma_i).$$

   (f) $(*)$ Show that $\Psi(\partial \sigma) = 0$ for every singular 2-simplex $\sigma : \Delta^2 \to X$, and deduce that Ψ descends to a homomorphism $\Psi : H_1(X; \mathbb{Z}) \to \pi_1(X)/[\pi_1(X), \pi_1(X)]$.

   (g) Show that $\Psi \circ \Phi$ and $\Phi \circ \Psi$ are both the identity map.

   (h) For a closed surface Σ_g of genus $g \geqslant 2$, find an example of a nontrivial element in the kernel of the Hurewicz homomorphism $\pi_1(\Sigma_g) \to H_1(\Sigma_g)$. *Hint: See Problem Set 7 #2.*

[3]Since $\pi_1(X)/[\pi_1(X), \pi_1(X)]$ is abelian, we are adopting the convention of writing its group operation as addition, so the multiplication of an integer $m \in \mathbb{Z}$ by an element $\Psi(\sigma) \in \pi_1(X)/[\pi_1(X), \pi_1(X)]$ is defined accordingly.