TAKE-HOME MIDTERM
Due: 6.07.2023

Instructions

The purpose of this assignment is two-fold:

- It gives the instructors a chance to gauge your understanding more directly than usual, and give feedback.
- It provides an opportunity to improve your final grade in the course.

In the absence of regular problem sets for the next two weeks, it also contains a little bit of current material from the lectures.

To receive feedback and/or credit, you must submit your written solutions by **Thursday, July 6 at 11:15**. Submissions can be on paper or electronic via the moodle. The solutions will be discussed in the Übung on the due date.

You are free to use any resources at your disposal and to discuss the problems with your comrades, but **you must write up your solutions alone**. Solutions may be written up in German or English, this is up to you.

A score of 75 points or better will boost your final exam grade according to the formula that was indicated in the course syllabus. The number of points assigned to each part of each problem is meant to be approximately proportional to its importance and/or difficulty.

If a problem asks you to prove something, then unless it says otherwise, a **complete argument** is typically expected, not just a sketch of the idea. Partial credit may sometimes be given for incomplete arguments if you can demonstrate that you have the right idea, but for this it is important to write as clearly as possible. Less complete arguments can sometimes be sufficient, e.g. a clear and convincing picture is often a better way to prove that two spaces are homotopy equivalent than by writing down explicit maps and homotopies (use your best judgement). Unless stated otherwise, you are free to make use of all results that have appeared in the lecture notes or in problem sets, without reproving them. When using a result from a problem set or the lecture notes, say explicitly which one.

If you get stuck on one part of a problem, it may often still be possible to move on and do the next part. You are free to ask for clarification or hints via e-mail/moodle or in office hours or Übungen; of course we reserve the right not to answer such questions.

Problems

1. [45 pts total] For two topological spaces X and Y, let $C(X,Y)$ denote the space of continuous maps $f : X \to Y$, endowed with the compact-open topology; recall (from Problem Set 4) that the latter is generated by a subbase consisting of all sets $\mathcal{U}_{K,V} \subset C(X,Y)$ of the form

$$\mathcal{U}_{K,V} := \{ f \in C(X,Y) \mid f(K) \subset V \} \tag{1}$$

where $K \subset X$ is compact and $V \subset Y$ is open. In the special case where Y is a metric space, the compact-open topology can also be characterized usefully by the condition that sequences $f_n \in C(X,Y)$ converge if and only if they converge uniformly over all compact subsets of X.

   (a) [20 pts] Show that if X and Z are any topological spaces and Y is a locally compact Hausdorff space, then the map

$$\Phi : C(X,Y) \times C(Y,Z) \to C(X,Z), \qquad \Phi(f,g) := g \circ f$$

is continuous.

Hint: The result of Problem Set 4 #2 is helpful for this.

Recall next that a **topological group** is a group G with a topology for which the group operations

$$G \times G \to G : (g,h) \mapsto gh, \qquad G \to G : g \mapsto g^{-1}$$

are continuous maps, and a continuous **left action** of G on a topological space X is a continuous map

$$G \times X \to X : (g,p) \mapsto gp$$

that satisfies the relations $g(hp) = (gh)p$ for all $g,h \in G$ and $p \in X$ and $ep = p$ for every $p \in X$ and the identity element $e \in G$. Given any space X, one natural candidate for a topological group with an action on X is

$$\text{Homeo}(X) = \left\{ f \in C(X,X) \;\middle|\; f \text{ is bijective and } f^{-1} \in C(X,X) \right\},$$

where the group operation is defined via composition of maps, and the action on X is given by

$$\text{Homeo}(X) \times X \to X : (f,p) \mapsto f(p). \tag{2}$$

It turns out however that one has to be a bit careful with the definition of the topology on Homeo(X) in order for this to work.[1] Since Homeo(X) is a subset of $C(X,X)$, a natural thing to try is putting the compact-open topology on $C(X,X)$ and then endowing Homeo(X) with the resulting subspace topology—if X is a metric space, then this will again just mean that a sequence $f_n \in \text{Homeo}(X)$ converges if and only if it converges uniformly on compact subsets. We will see below that this works if X is compact and Hausdorff, but it does not always work if X is Hausdorff and only *locally* compact.

(b) [15 pts] Let $\mathcal{T}_H$ denote the topology on $C(X,X)$ with subbase consisting of all sets of the form $\mathcal{U}_{K,V}$ and $\mathcal{U}_{X \setminus V, X \setminus K}$, as in Equation (1), where again $K \subset X$ can be any compact subset and $V \subset X$ any open subset. Notice that if X is compact and Hausdorff, then for any V open and K compact, $X \setminus V$ is compact and $X \setminus K$ is open, thus $\mathcal{T}_H$ is again simply the compact-open topology. But if X is not compact or Hausdorff, $\mathcal{T}_H$ may be stronger than the compact-open topology. Show that if X is locally compact and Hausdorff and $\text{Homeo}(X) \subset C(X,X)$ is endowed with the subspace topology induced by the topology $\mathcal{T}_H$, then Homeo(X) is indeed a topological group, and its action on X as defined in Equation (2) is continuous.
Hint: Notice that $f(K) \subset V$ if and only if $f^{-1}(X \setminus V) \subset X \setminus K$. Use this to show directly that $f \mapsto f^{-1}$ is continuous, and then to reduce the rest to what was proved already in part (a).

(c) [10 pts] Here is an example showing that if X is not compact, then the compact-open topology is not always strong enough to make Homeo(X) a topological group. The space

$$X := \{0\} \cup \left\{ e^n \;\middle|\; n \in \mathbb{Z} \right\} \subset \mathbb{R}$$

inherits a natural topology from $\mathbb{R}$ that matches the discrete topology at most points—the exception is the point $0 \in X$, whose neighborhoods always contain infinitely many other points of X. Clearly X is Hausdorff (because $\mathbb{R}$ is), and one can easily check that it is locally compact, but it is not compact. The following defines a sequence of homeomorphisms $f_k \in \text{Homeo}(X)$: for each $k \in \mathbb{N}$, let

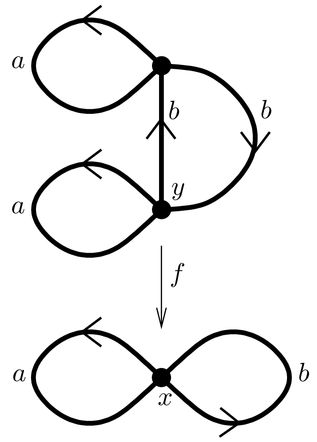
$$f_k(0) := 0 \qquad \text{and} \qquad f_k(e^n) := \begin{cases} e^{n-1} & \text{for } n > k, \\ e^{-k} & \text{for } n = k, \\ e^n & \text{for } -k < n < k, \\ e^{n-1} & \text{for } n \le -k. \end{cases}$$

Show that in the compact-open topology on $C(X,X)$, the sequence $f_k \in C(X,X)$ converges to the identity map on X, but the sequence $f_k^{-1} \in C(X,X)$ does not. Deduce from this that Homeo(X) with the compact-open topology is not a topological group.

[1] It would of course work if we put the discrete topology on Homeo(X), but that is usually not the most natural choice. For example, if X is something simple like the unit interval $[0,1]$, then there are natural ways to define convergence for a sequence of continuous maps $f_n : [0,1] \to [0,1]$, but the discrete topology will not detect them—in the discrete topology, nonconstant sequences never converge.

Remark: The real problem with part (c) turns out to be that X is not locally connected—one can show that for any locally compact Hausdorff space X that is also locally connected, $\text{Homeo}(X)$ *with the compact-open topology is a topological group after all.*[2]

2. [35 pts total] The picture below describes a base-point preserving covering map $f : (Y, y) \to (X, x)$, where $X \cong S^1 \vee S^1$, the two generators of $\pi_1(X, x) \cong \mathbb{Z} * \mathbb{Z}$ are labeled by loops a and b based at x, and the preimages of these loops in Y are given the same labels. Let us write elements of $\pi_1(X, x)$ accordingly as words in the letters a and b.



   (a) [15 pts] What is the subgroup $f_* (\pi_1(Y, y)) \subset \pi_1(X, x)$? Describe it as $\langle S \rangle$ or $\langle S \rangle_N$, meaning the smallest subgroup or normal subgroup respectively containing some specific subset $S \subset \pi_1(X, x)$. Find also a specific element $w \in \pi_1(X, x)$ such that $w \notin f_* (\pi_1(Y, y))$.

   (b) [20 pts] Draw a similar picture to describe a covering map $g : (Z, z) \to (X, x)$ such that $g_* (\pi_1(Z, z)) = \langle a^2, b^2, ab, ba \rangle_N$.
   Hint: First determine what the quotient group $\pi_1(X, x)/\langle a^2, b^2, ab, ba \rangle_N$ is, and use this to deduce the degree of the cover g. Then consider loops in X based at x, and determine whether their lifts to Z starting at the base point should close up or not.

3. [20 pts] Describe (in whatever way you can) a compact and path-connected topological space X whose fundamental group has the finite presentation

$$\pi_1(X) \cong \{a, b, c \mid abc = cba\}.$$

   Is the space you've described a manifold?

[2]If you're interested, R. Arens, Topologies for homeomorphism groups, *Amer. J. Math.* **68** (1946) 593–610 contains a quite clever proof of this fact.