Catastrophe Theory: Talk 2

Luna Cieliebak

1. Germs: Right equivalence, Jacobian ideal and codimension

Def. 1.1: Right equivalence

Two function germs $f, g : (\mathbb{R}^n, 0) \to \mathbb{R}$ are right equivalent (or R-equivalent) if there is a diffeomorphism germ ϕ of $(\mathbb{R}^n, 0)$ such that

$$f = g \circ \phi.$$

They are R^+-equivalent if there is some constant $a \in \mathbb{R}$ such that

$$f = g \circ \phi + a$$

We then write $f \sim_R g$ or $f \sim_{R^+} g$.

R-equivalence allows us to consider function germs independent of the particular coordinates used.

Example 1.2

(1) The germs $f, g : (\mathbb{R}, 0) \to \mathbb{R}$, $f(x) = x^2 - x^4$, $g(x) = x^2$ are R-equivalent:

We require a change of coordinates $y = \phi(x)$ such that

$f(x) = (g \circ \phi)(x) = g(y)$.

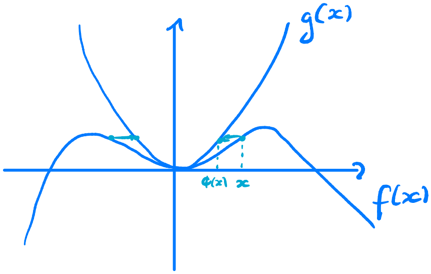
Solving $x^2 - x^4 = y^2$ gives us $y = x\sqrt{1 - x^2} =: \phi(x)$.

The inverse function theorem tells us that ϕ is a diffeomorphism in a neighbourhood of 0.

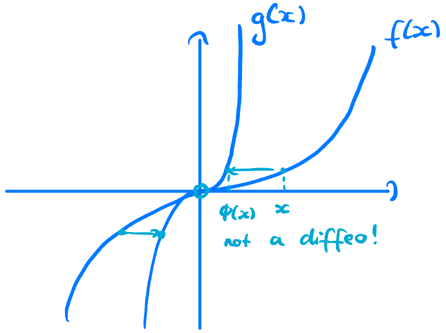
[check the preconditions for the inverse function theorem:

ϕ is smooth and

$d\phi(x) = x \frac{1}{2}(1 - x^2)^{-\frac{1}{2}} + \sqrt{1 - x^2}$, $d\phi(0) = 1 \neq 0$]



(2) The germs $f, g : (\mathbb{R}, 0) \to \mathbb{R}$, $f(x) = x^3$, $g(x) = x^5$ are not $\mathcal{R}$-equivalent:
Suppose there is a diffeomorphism germ ϕ such that $f = g \circ \phi$.
Then $x^3 = \phi(x)^5$, that is $\phi(x) = x^{\frac{3}{5}}$. This map is not differentiable at 0, so not a diffeomorphism.



Remark 1.3

(1) In the example 1.2 (2), $\phi(x) = x^{\frac{3}{5}}$ is not a diffeomorphism.
But it is a homeomorphism as ϕ and ϕ^{-1} are continuous.
This marks the difference between the study of topology (using homeomorphisms) and differential topology (using diffeomorphisms).

(2) $\mathcal{R}$-equivalence preserves critical points in the following sense:
Suppose $f : U \to \mathbb{R}$, $g : V \to \mathbb{R}$ are smooth functions with $U, V \subset_{\text{open}} \mathbb{R}^n$ and let $\phi : U \to V$ a diffeomorphism such that $f = g \circ \phi$.
Then f has a critical point at $x \in U$ if and only if g has a critical point at $\phi(x)$.

$$\left[\begin{array}{l}
f \text{ has a critical point at } x \in U \\
\iff df(x) = 0 \\
\iff d(g \circ \phi)(x) = 0 \\
\iff dg(\phi(0)) \cdot \underbrace{d\phi(x)}_{\det(d\phi(x)) \neq 0} = 0 \quad \Rightarrow d\phi(x) \text{ has full rank} \\
\iff dg(\phi(x)) = 0 \\
\iff \phi(x) \text{ is a critical point of } g
\end{array}\right]$$

For $\mathcal{R}$-equivalent function germs $f, g : (\mathbb{R}^n, 0) \to \mathbb{R}$ with a diffeomorphism germ ϕ of $(\mathbb{R}^n, 0)$ such that $f = g \circ \phi$, this translates to:
f has a critical point at 0 if and only if g has a critical point at $\phi(0) = 0$.

Def. 1.4: The Jacobian ideal

Let $f \in \mathcal{E}_n$ (i.e. $f: (\mathbb{R}^n, 0) \to \mathbb{R}$ a smooth function germ). The Jacobian ideal Jf is the ideal in $\mathcal{E}_n$ generated by the partial derivatives of f:

$$Jf := \left\langle \frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \dots, \frac{\partial f}{\partial x_n} \right\rangle \lhd \mathcal{E}_n$$

The Jacobian ideal will help us classify critical points as they are only determined by the partial derivatives.

Remark 1.5:

$f \in \mathcal{E}_n$ has a critical point at the origin if and only if each of the generators $\frac{\partial f}{\partial x_i}$ of Jf belongs to m_n, and hence if and only if $Jf \subset m_n$.

Remark 1.6:

The definition of the Jacobian ideal is independent of the coordinates used: Let $\Phi: (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$ be a change of coordinates and $\hat{x} = \Phi(x)$.

Then $\frac{\partial (f \circ \Phi)}{\partial \hat{x}_i} = \frac{\partial f}{\partial x_i}$.

Example 1.7:

Let $f(x) = x_1^2 + \dots + x_n^2 \in \mathcal{E}_n$. Then $Jf = \langle 2x_1, \dots, 2x_n \rangle = m_n$

Def. 1.8: Codimension

A germ $f \in m_n^2$ is of finite codimension if the Jacobian ideal Jf is of finite codimension in m_n. In this case, we define

$$\operatorname{codim}(f) := \dim(m_n / Jf).$$

Remark 1.9:

$\mathcal{R}^+$- equivalent germs have the same codimension.

(see Montaldi, Problem 4.9)

Example 1.10:

Let $f(x) = x_1^2 + x_2^2 + \dots + x_n^2 \in m_n^2$.

Then $Jf = m_n$ and $\operatorname{codim}(f) = \dim(m_n / m_n) = 0$.

2. Nondegenerate critical points

Recall that a function germ $f \in m_n^2$ has a nondegenerate critical point (at 0) if $d^2f(0)$ is nondegenerate.

Proposition 2.1: (4.9 in Montaldi)
A germ $f \in m_n$ has a nondegenerate critical point (at 0) if and only if $Jf = m_n$

Remark 2.2
By definition, $Jf = m_n \Leftrightarrow \operatorname{codim} f = 0$. So the proposition above actually tells us that nondegenerate critical points are precisely those of codimension 0.

Lemma 2.3
(1) For a quadratic form $f: \mathbb{R}^n \to \mathbb{R}$, $f(x) = x^T A x$, where $A \in \mathbb{R}^{n \times n}$, the first and second derivatives are given by $df(x) = x^T (A + A^T)$ and $d^2 f(x) = A + A^T$.
(2) For $f(x) = x^T A(x) x$, where $A(x)$ is smooth, we get $df(0) = 0$ and $d^2 f(0) = A(0) + A(0)^T$.

Proof of 2.3
The proof of Lemma 2.3 is simple calculus:
Let $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$, $f(x) = x^T A(x) x$

$$f(x) = x^T A(x) x = \sum_{i=1}^n \Big(\sum_{j=1}^n a_{ij}(x)\, x_i x_j \Big)$$

$$df(x) = \begin{pmatrix} \sum_{i=1}^n \Big(\sum_{j=1}^n \frac{\partial a_{ij}}{\partial x_1} x_i x_j \Big) + \sum_{j=1}^n a_{1j} x_j + \sum_{i=1}^n a_{i1} x_i \\ \vdots \\ \sum_{i=1}^n \Big(\sum_{j=1}^n \frac{\partial a_{ij}}{\partial x_n} x_i x_j \Big) + \sum_{j=1}^n a_{nj} x_j + \sum_{i=1}^n a_{in} x_i \end{pmatrix}$$

for $k, l \in \{1, \ldots, n\}$ we have

$$\big(d^2 f(x)\big)_{kl} = \sum_{i=1}^n \Big(\sum_{j=1}^n \frac{\partial a_{ij}}{\partial x_k \partial x_l} x_i x_j \Big) + \sum_{j=1}^n \frac{\partial a_{lj}}{\partial x_k} x_j + \sum_{i=1}^n \frac{\partial a_{il}}{\partial x_k} x_i + \sum_{j=1}^n \frac{\partial a_{kj}}{\partial x_l} x_j + a_{kl}(x) + \sum_{i=1}^n \frac{\partial a_{ik}}{\partial x_l} x_i + a_{lk}(x).$$

If $A(x) = A$ is constant, then $f(x) = x^T A x$ is a quadratic form and the expressions above can be simplified to

$$df(x) = \begin{pmatrix} \sum_{j=1}^{n} a_{1j} x_j + \sum_{i=1}^{n} a_{i1} x_i \\ \vdots \\ \sum_{j=1}^{n} a_{nj} x_j + \sum_{i=1}^{n} a_{in} x_i \end{pmatrix} = \begin{pmatrix} \sum_{i=1}^{n} (a_{1i} + a_{i1}) x_i \\ \vdots \\ \sum_{i=1}^{n} (a_{ni} + a_{in}) x_i \end{pmatrix} = x^T (A + A^T)$$

$(d^2 f(x))_{kl} = a_{kl} + a_{lk}$, $d^2 f(x) = A + A^T$.

For a matrix $A(x)$, we can simplify the expressions at the origin:

$df(0) = 0$

$(d^2 f(0))_{kl} = a_{kl}(0) + a_{lk}(0)$, $d^2 f(0) = A(0) + A(0)^T$.

Proof of 2.1

"$\Longrightarrow$": We prove the first implication with the help of Nakayama's lemma.

Assume that f has a nondegenerate critical point at 0.
Then we already know that $J_f \subset m_n$ (see earlier remark)
So it suffices to show that $m_n \subset J_f$.

Since the critical point (at 0) is nondegenerate,
$Q := d^2 f(0)$ is a nondegenerate symmetrical matrix.

By Taylor's theorem with remainder, we get

$$f(0 + x) = \underbrace{f(0)}_{=0} + \underbrace{df(0)(x)}_{=0} + \frac{1}{2} x^T Q x + h(x) \quad \text{with } h \in m_n^3$$

[The remainder is given by $h(x) = \sum_{|\alpha| = 3} \frac{\partial^\alpha f(cx)}{\alpha!} x^\alpha$ for some $c \in (0,1)$.
Thus, all partial derivatives of h of order less than 3 vanish at 0
and $h \in m_n^3$]

We can use Lemma 2.3 to see that

$$df(x) = \frac{1}{2} x^T (Q + Q^T) + dh(x) = \frac{1}{2} x^T \cdot 2Q + dh(x) = x^T Q + dh(x).$$

$$= \begin{pmatrix} \sum_{j=1}^{n} q_{1j} x_1 \\ \vdots \\ \sum_{j=1}^{n} q_{nj} x_n \end{pmatrix} + \begin{pmatrix} \frac{\partial h}{\partial x_1} \\ \vdots \\ \frac{\partial h}{\partial x_n} \end{pmatrix}$$

So for each $i \in \{1, \dots, n\}$, we get

$\frac{\partial f}{\partial x_i} = \sum_{j=1}^{n} q_{ij} x_j + \frac{\partial h}{\partial x_i}$ with q_{ij} the entries of Q.

Now, we can use the equation above to investigate the Jacobian ideal. Since $\frac{\partial h}{\partial x_i} \in m_n^2$, we get

$$Jf + m_n^2 = \langle \sum_{j=1}^{n} q_{ij} x_j \mid i \in \{1, \dots, n\} \rangle + m_n^2 .$$

claim: $I := \langle \sum_{j=1}^{n} q_{ij} x_j \mid i \in \{1, \dots, n\} \rangle = m_n$

"$\subset$": $I \subset m_n$ is clear since every polynomial
$p_i(x) = \sum_{j=1}^{n} q_{ij} x_j$ is smooth and vanishes at 0.

"$\supset$": Since Q is nondegenerate, we can write $A := Q^{-1}$.
Then for $k \in \{1, \dots, n\}$:

$$x_k = \sum_{j=1}^{n} \underbrace{\Big(\sum_{i=1}^{n} a_{ki} q_{ij} \Big)}_{= \begin{cases} 1, & i=j=k \\ 0, & \text{else} \end{cases}} x_j = \sum_{i=1}^{n} a_{ki} \Big(\sum_{j=1}^{n} q_{ij} x_j \Big) \in I$$

because $AQ = \mathbb{1}_n$

So all $x_1, \dots, x_n \in I$ and therefore $m_n = \langle x_1, \dots, x_n \rangle \subset I$.

Using $I = m_n$ (and $m_n^2 \subset m_n$), we now have $m_n = Jf + m_n^2$.
It follows from Nakayama's lemma that $m_n \subset Jf$ and thus $m_n = Jf$.

[check the preconditions for Nakayama's lemma:
$\mathcal{E}_n$ is a ring,
m_n is an ideal s.t. $a \in m_n \Rightarrow a+1$ is a unit in $\mathcal{E}_n$:
$a \in m_n \Rightarrow a = \sum_{j=1}^{n} a_j x_j$ for some coefficients a_j

$\Rightarrow \frac{1}{1+a} = \frac{1}{1 + \sum_{j=1}^{n} a_j x_j}$, $0 \in \mathrm{dom}\left(\frac{1}{1+a}\right)$

$\Rightarrow \frac{1}{1+a} \in \mathcal{E}_n$, $1+a$ is a unit in $\mathcal{E}_n$

$m_n, Jf \lhd \mathcal{E}_n$,
m_n is finitely generated]

"$\Leftarrow$": We prove this implication by contraposition.

If f has no critical point at all at the origin, then $Jf \not\subset m_n$ (see earlier remark).

Now suppose that f has a critical point at 0 and this point is degenerate.

Using Taylor's theorem and $Q := d^2 f(0)$ we can write, as in the first part of the proof,

$f(x) = \frac{1}{2} x^T Q x + h(x)$

with $h \in m_n^3$ and Q now degenerate.

Let $m < n$ be the rank of Q. Then we can choose a basis in $\mathbb{R}^n$ s.t.

$Q = \begin{pmatrix} \hat{Q} & 0 \\ 0 & 0 \end{pmatrix}$

with $\hat{Q}$ an invertible and symmetric $m \times m$-matrix.

We now write the corresponding coordinates as $(\hat{x}_1, \dots, \hat{x}_m, y_1, \dots, y_{n-m})$.

Then

$$f(\hat{x}, y) = \frac{1}{2} (\hat{x}^T, y^T) \begin{pmatrix} \hat{Q} & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \hat{x} \\ y \end{pmatrix} + h(\hat{x}, y)$$
$$= \frac{1}{2} (\hat{x})^T \hat{Q} \hat{x} + h(\hat{x}, y)$$

In these coordinates, we can calculate the partial derivatives of f like we did in the first part of the proof:

$$\frac{\partial f}{\partial \hat{x}_i} = \sum_{j=1}^{m} \hat{q}_{ij} \hat{x}_j + \frac{\partial h}{\partial \hat{x}_i} \qquad \text{for } i \in \{1, \dots, m\}$$

$$\frac{\partial f}{\partial y_j} = \frac{\partial h}{\partial y_j} \qquad \text{for } j \in \{1, \dots, n-m\}$$

Again, $\frac{\partial h}{\partial \hat{x}_i}, \frac{\partial h}{\partial y_j} \in m_n^2$.

Thus $Jf + m_n^2 = \langle \sum_{j=1}^{m} \hat{q}_{ij} \hat{x}_j \mid i \in \{1, \dots, m\} \rangle + m_n^2$, so in particular

$y_j \notin Jf$ for $j \in \{1, \dots, n-m\}$ as y_j is neither in $\langle \sum_{j=1}^{m} \hat{q}_{ij} \hat{x}_j \mid i \in \{1, \dots, m\} \rangle$

nor in m_n^2 (as $\frac{\partial y_j}{\partial y_j}(0) = 1 \neq 0$). But $y_j \in m_n$ and thus $Jf \subsetneq m_n$. $\square$

Proposition 2.4 (4.10 in Montaldi)

Suppose $f:(\mathbb{R}^n,0)\to(\mathbb{R},0)$ has a nondegenerate critical point (at 0).
Then there is a change of coordinates ϕ such that

$$(f\circ\phi)(x) = \tfrac{1}{2}x^T d^2f(0)x$$

The proposition can also be understood as 'f is right equivalent to its quadratic part (Taylor series to degree 2)'.

Proof:

$f(0)=0$ and f has a critical point at 0. Thus, $f\in m_n^2$.
We know from the Corollary to Hadamard's lemma that m_n^2 is generated by the monomials of degree 2 in $x_1,\dots,x_n$.

So we can write

$$f(x)=\sum_{i,j=1}^{n}\varphi_{ij}(x)x_ix_j = x^T\,\Psi(x)\,x$$

with smooth functions $\varphi_{ij}\in\mathcal{E}_n$ and $\Psi(x)=(\varphi_{ij}(x))$ can be chosen to be symmetric.

[if $\varphi_{lk}(x)\neq\varphi_{kl}(x)$ for some $l,k\in\{1,\dots,n\}$, we can simply exchange the coefficient for those entries with $\hat\varphi_{lk}(x),\hat\varphi_{kl}(x):=\tfrac{1}{2}(\varphi_{lk}(x)+\varphi_{kl}(x))$ and the coefficient for x_lx_k remains $\hat\varphi_{lk}(x)+\hat\varphi_{kl}(x)=2\cdot\hat\varphi_{kl}(x)=\varphi_{lk}(x)+\varphi_{kl}(x)$]

Using Lemma 2.3, we get $d^2f(0)=\Psi(0)+\Psi(0)^T=2\Psi(0)$.
Since f has a nondegenerate critical point at 0, $\Psi(0)=\tfrac{1}{2}d^2f(0)$ is a nondegenerate matrix.

Nearby nondegenerate quadratic forms are similar, so for each x near 0 there is an invertible matrix P_x with $P_0=\mathbb{1}$ such that $\Psi(x)=P_x^T\,\Psi(0)\,P_x$.

It follows from the inverse function theorem that the map $x \mapsto P_x x$ is a diffeomorphism germ at the origin.

[Check the preconditions for the inverse function theorem:
$\varphi(x) := P_x x$ is smooth and
$d\varphi(0) = P_0 = \mathbb{1}$, therefore $\det(d\varphi_0) = 1 \neq 0$]

So we can define new coordinates $y = P_x x$ and write
$f(x) = x^T \Psi(x) x = x^T P_x^T \Psi(0) P_x x = y^T \Psi(0) y$.
By defining ϕ via $x = \phi(y)$, the inverse of $y = P_x x$, we get the required statement
$(f \circ \phi)(y) = f(x) = y^T \Psi(0) y = \frac{1}{2} y^T d^2 f(0) y$.

Corollary 2.5 (Morse Lemma)

If $f: (\mathbb{R}^n, 0) \to (\mathbb{R}, 0)$ has a nondegenerate critical point at 0, then there is a change of coordinates ϕ such that

$$(f \circ \phi)(x) = \pm x_1^2 \pm x_2^2 \pm \dots \pm x_n^2$$

Remark 2.6

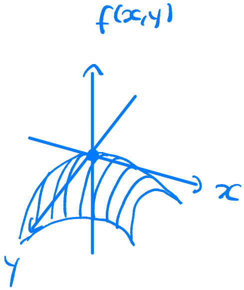
For a nondegenerate critical point of f, the number of negative squares in the Morse lemma is the index of the critical point.

Proof of 2.5

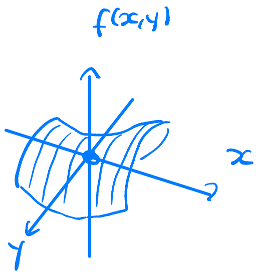
Any quadratic form can be diagonalised by a change of basis, and if it is nondegenerate, the diagonal terms can be made equal to ± 1. If we apply this to $x \mapsto x^T \left(\frac{1}{2} d^2 f(0)\right) x$, we get the required statement.

Example 2.7:

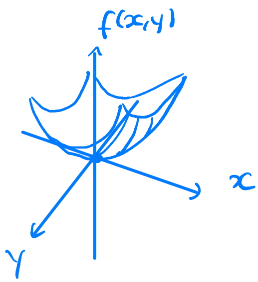
Suppose $f \in m_2^2$ has a nondegenerate critical point at the origin. Then the Morse Lemma tells us that, after a change of coordinates, $f(x,y)$ can only be of one of the following three forms:



$$f(x,y) = -x^2 - y^2$$



$$f(x,y) = x^2 - y^2$$



$$f(x,y) = x^2 + y^2$$

3. Splitting Lemma

Example 3.1

$f(x,u) = x^2 + 2xu^2$

Here, 0 is a critical point of f, but $d^2 f = \begin{pmatrix} 2 & 4u \\ 4u & 4x \end{pmatrix}$ is degenerate at 0.

If we write

$f(x,u) = (x+u^2)^2 - u^4$ and put $X(x,u) = x + u^2$, we get

$$f(x,u) = f(X,u) = \underbrace{X^2}_{\text{nondegenerate part}} \underbrace{-u^4}_{\text{degenerate part}}$$

Now, $d_X^2 f(0) = 2 \neq 0$ and $d_u^2 f(0) = 0$

We have already seen some nice properties of nondegenerate critical points. For classifying degenerate critical points, we want to 'split' the nondegenerate part from the degenerate part as seen in the example above. This is the idea of the Splitting Lemma.

Theorem 3.2 (Splitting lemma)

Let $f \in m_{n+k}$, which we write as $f(x,u)$ for $x \in \mathbb{R}^n, u \in \mathbb{R}^k$.

Suppose the restriction $f|_{\mathbb{R}^n \times \{0\}}$ has a nondegenerate critical point (at $x=0$). Then there is a change of coordinates in a neighbourhood of the origin $(x,u) = (x(X,u),u)$ such that

$$f(x(X,u),u) = Q(X) + h(u),$$

where $Q = (\frac{1}{2} d^2 f)|_{\mathbb{R}^n \times \{0\}}$ and h is a smooth function.

Furthermore, the 'remainder function' h can be found implicitly as follows:

For each u near 0 there is a unique point $x = \chi(u)$ such that $d_x f(\chi(u),u) = 0$. ($d_x f(x,u)$ is the differential of f with respect to only x)

Then $h(u) = f(\chi(u),u)$.

Remark 3.3

(1) This result is sometimes called the parametrized Morse lemma, where the variables u take the place of parameters.

(2) One can further simplify Q by diagonalising it so that $Q(x)$ takes the form $\sum \pm x_i^2$

(3) $x(u)$ can be found in principle by using the implicit function theorem. This means that the Taylor series of h can be found to any given order.

Before we prove 3.2, we take a look at a typical application of the Splitting lemma.

Corollary 3.4

If $f \in m_n$ has a critical point at the origin with Hessian matrix of corank k (i.e. the dimension of its kernel) then there are coordinates $x \in \mathbb{R}^{n-k}$ and $u \in \mathbb{R}^k$ such that

$$f = Q(x) + h(u),$$

where Q is a nondegenerate quadratic form and $h \in m_k^3$

Proof of 3.4

Since the Hessian matrix of f has corank k, we can choose a basis so that it takes the form

$\begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix}$ with A an invertible symmetric $(n-k) \times (n-k)$ matrix.

We write the corresponding coordinates as $(x_1, \dots, x_{n-k}, u_1, \dots, u_k)$.

In these new coordinates, we can write

$d_x^2 f(0) = A$, $d_u^2 f(0) = 0$, $d_u d_x f(0) = 0$.

Now, the preconditions of the Splitting lemma are satisfied.

[$f|_{\mathbb{R}^k \times \{0\}}$ has a nondegenerate critical point at $x=0$ as $d_x^2 f(0) = A$ is nondegenerate]

So after a further change of coordinates $(x,u) \mapsto (X(x,u),u)$, we can write f in the form

$f(x,u) = Q(X) + h(u)$.

There remains to show that $h \in m_k^3$. We already know that $h \in m_k^2$ as f has a critical point at 0.

Furthermore, we can represent the quadratic form Q by a symmetric matrix $\hat{Q}$, such that $Q(X) = X^T \hat{Q} X$.

From Lemma 2.3, we know that $d^2 Q = \hat{Q} + \hat{Q}^T = 2\hat{Q}$.

Then we have

$$d^2 f(0) = \begin{pmatrix} d^2 Q(0) & 0 \\ 0 & d^2 h(0) \end{pmatrix} = \begin{pmatrix} 2\hat{Q} & 0 \\ 0 & d^2 h(0) \end{pmatrix}.$$

Since $\hat{Q}$ is invertible of rank m, we can conclude that

$m = \operatorname{rank}(d^2 f(0)) = m + \operatorname{rank}(d^2 h(0))$

So $d^2 h(0) = 0$ and thus $h \in m_k^3$.

<u>Proof of 3.2</u>:

We begin by finding the map $u \mapsto x(u)$ referred to at the end of the statement.

The map $\varphi: (x,u) \mapsto d_x f(x,u)$ is of rank m at the origin, because $d\varphi = [d_x^2 f, d_u d_x f]$ and the first $m \times m$ block of $d\varphi$ is $d_x^2 f$ which is invertible at the origin as f has a nondegenerate critical point there.

It follows from the implicit function theorem that $\varphi(x,u) = 0$ can be solved uniquely for x as a (continuously differentiable) function of u, defining $u \mapsto x(u)$.

[check the preconditions for the implicit function theorem:
$\varphi: \mathbb{R}^{m+k} \to \mathbb{R}^m$ is continuously differentiable,
$d_x \varphi = d_x^2 f$ is invertible]

That is, in a neighbourhood of the origin,

$$d_x f(x,u) = 0 \iff x = \chi(u).$$

Now, we change the coordinates by defining $y = x - \chi(u)$. The map

$$(x,u) \mapsto (x - \chi(u), u)$$

is a diffeomorphism the map itself and its inverse $(y,u) \mapsto (y + \chi(u), u)$ are continuously differentiable.

Let $g(y,u) = f(x,u) = f(y + \chi(u), u)$.

Then $d_y g = 0$ if and only if $y = 0$.

$[d_y g(y,u) = 0 \iff d_x f(y + \chi(u), u) = 0 \iff y + \chi(u) = \chi(u) \iff y = 0]$

Now for each fixed value of u (near 0) there is a function $g_u(y) := g(y,u)$ that has a nondegenerate critical point at the origin:

$dg_u(y) = d_y g(u,y) = 0$ for $y = 0$ as seen above.

$d^2 g_u(0)$ is invertible and so too is $d^2 g_u(0)$ for sufficiently small values of u by continuity.

We now mimic the proof of the Morse lemma with the variables u as parameters.

Let $\mathcal{M}_x$ be the ideal of functions $f \in \mathcal{E}_{m+k}$ such that $f(0,u) \equiv 0$.

Then $\mathcal{M}_x = \langle x_1, \dots, x_m \rangle \subset \mathcal{E}_{m+k}$ by Hadamard's lemma.

Define $F(y,u) := g(y,u) - g(0,u)$. Then $F \in \mathcal{M}_x$ $(F(0,u) = g(0,u) - g(0,u) \equiv 0)$.

So we can write

$$F(y,u) = \sum_{i,j=1}^{m} \varphi_{ij}(y,u)\, y_i y_j = y^T \Phi(y,u) y.$$

Nearby symmetric and nondegenerate matrices are similar, so for each (y,u) near 0 there is an invertible matrix $P_{(y,u)}$ such that

$$\Phi(y,u) = P_{(y,u)}^T\, \Phi(0,0)\, P_{(y,u)} \quad (\text{and thus } P_{(0,0)} = \mathbb{1})$$

It follows from the inverse function theorem that $\tau : (y,u) \mapsto (P_{(y,u)}y, u)$ is a diffeomorphism germ.

[Check the preconditions for the inverse function theorem:
τ is smooth,

$$\tau(y,u) = \left(\begin{pmatrix} \sum_{j=1}^m p_{1j}(y,u) y_1 \\ \vdots \\ \sum_{j=1}^m p_{mj}(y,u) y_m \end{pmatrix}, \begin{pmatrix} u_1 \\ \vdots \\ u_k \end{pmatrix} \right) \text{ and } d\tau(y,u) = \left(\begin{pmatrix} \sum_{j=1}^m \frac{\partial p_{1j}}{\partial y_1} y_1 + \sum_{j=1}^m p_{1j}(y,u) \\ \vdots \\ \sum_{j=1}^m \frac{\partial p_{mj}}{\partial y_1} y_m + p_{m1}(y,u) y_1 \end{pmatrix} \cdots \begin{pmatrix} \sum_{j=1}^m \frac{\partial p_{1j}}{\partial u_k} y_1 \\ \vdots \\ \sum_{j=1}^m \frac{\partial p_{mj}}{\partial u_k} y_m \end{pmatrix} \\ \vdots \qquad \ddots \\ \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix} \cdots \begin{pmatrix} 0 \\ \vdots \\ 1 \end{pmatrix} \right)$$

$$\text{so } d\tau(0,0) = \begin{pmatrix} \begin{pmatrix} \overbrace{\sum_{j=1}^m p_{1j}(0,0)}^{=1 \text{ because } P_{(0,0)} = \mathbb{1}} \\ 0 \\ \vdots \\ 0 \end{pmatrix} & \cdots & \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix} \\ \vdots & \ddots & \vdots \\ \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix} & & \begin{pmatrix} 0 \\ \vdots \\ 1 \end{pmatrix} \end{pmatrix} \text{ and } \det(d\tau(0)) = 1 \neq 0$$
]

Write $(X,u) = (P_{(y,u)}y, u)$. Then
$F(y,u) = y^T \Psi(y,u) y = y^T P^T_{(y,u)} \Psi(0,0) P_{(y,u)} y = X^T \Psi(0,0) X$.

Write $Q(x) = X^T \Psi(0,0) X$, so that
$g(y,u) = g(0,u) + F(y,u) = g(0,u) + Q(X)$.

But $y=0$ corresponds to $x = \chi(u)$ so that $g(0,u) = f(\chi(u),u) = h(u)$.

So finally we have
$f(x,u) = g(y,u) = F(y,u) + g(0,u) = Q(X) + h(u)$ $\square$

Proposition 3.5
Let $f \in \mathcal{E}_{m+k}$ with $f(x,u) = \sum_{i=1}^m \pm x_i^2 + h(u)$ where $x \in \mathbb{R}^m$, $u \in \mathbb{R}^k$.
Then $\operatorname{codim}(f) = \operatorname{codim}(h)$.

For the proof, we use the following algebraic lemma.

Lemma 3.6
Let R, S be rings and $J \trianglelefteq S$. Suppose $\phi : R \to S$ is a surjective homomorphism and let $I = \phi^{-1}(J)$. Then $I \trianglelefteq R$ and ϕ induces an isomorphism $\bar{\phi} : R/_I \to S/_J$, defined by $\bar{\phi}(r+I) = \phi(r) + J$

Proof of 3.6:

If we compose ϕ with the natural homomorphism $S \to S/J$, $s \mapsto s+J$, we get a ring homomorphism $\varphi: R \to S/J$, $r \mapsto \phi(r)+J$.

Then the first isomorphism theorem for rings tells us that $\ker\varphi$ is an ideal and provides an isomorphism $\operatorname{im}\varphi \cong R/\ker\varphi$.
In our case, $\ker\varphi = \varphi^{-1}(J) = \phi^{-1}(J) = I$ and $\operatorname{im}\varphi = S/J$ because ϕ is surjective.
So $S/J \cong R/I$ with an isomorphism $\Phi: R/I \to S/J$, $r+I \mapsto \phi(r)+J$.

Proof of 3.5

The Jacobian ideal of $f(x,u) = \sum_{i=1}^{m} \pm x_i^2 + h(u)$ is given by
$Jf = \langle x_1, \dots, x_m, \frac{\partial h}{\partial u_1}, \dots, \frac{\partial h}{\partial u_k} \rangle$. The map $\phi: \mathcal{E}_{m+k} \to \mathcal{E}_k$, $\phi(g)(u) = g(0,u)$ defines a homomorphism.
This homomorphism is surjective as for any given $h \in \mathcal{E}_k$, the germ $H(x,u) := h(u)$ satisfies $\phi(H) = h$.

Let $g \in \mathcal{E}_{m+k}$. Then $g \in Jf$ if and only if $\phi(g) \in Jh$

"$\Rightarrow$": Suppose that $g \in Jf$. Then we can write
$g(x,u) = \sum_{i=1}^{m} \varphi_i(x,u)\, x_i + \sum_{j=1}^{k} \psi_j(x,u) \frac{\partial h}{\partial u_j}(u)$ for some $\varphi_i, \psi_j \in \mathcal{E}_{m+k}$.
So $\phi(g)(u) = g(0,u) = \sum_{j=1}^{k} \psi_j(0,u) \frac{\partial h}{\partial u_j}(u)$ and thus $\phi(g) \in Jh$

"$\Leftarrow$": Suppose $g_2 := \phi(g) \in Jh$.
Then we can write
$g(x,u) = g_1(x,u) + g_2(u)$ with $g_1 \in \mathcal{E}_{m+k}$ such that $g_1(0,u) = 0$
Thus, $g_1 \in \{ g(x,u) \in \mathcal{E}_{m+k} \mid g(0,u) = 0 \}$.
From Hadamard's lemma we know that
$\{ g(x,u) \in \mathcal{E}_{m+k} \mid g(0,u) = 0 \} = \langle x_1, \dots, x_m \rangle$.
So $g_1 \in \langle x_1, \dots, x_m \rangle$ and $g = g_1 + g_2 \in Jf$.

Now, we can use Lemma 3.6 for $\phi: \mathcal{E}_{m+k} \to \mathcal{E}_k$, $Jh \trianglelefteq \mathcal{E}_k$, $Jf = \phi^{-1}(Jh)$.
So there is an isomorphism $\Phi: \mathcal{E}_{m+k}/Jf \to \mathcal{E}_k/Jh$ and in particular,
$\operatorname{codim}(f) = \dim(\mathcal{E}_{m+k}/Jf) = \dim(\mathcal{E}_k/Jh) = \operatorname{codim}(h)$.