Part I:

Motivation & Statement of the

Malgrange preparation Theorem.

I - (1) Motivate & State Malgrange's Preparation Thm.

- "A map germ at point q is an equivalence class of germ equivalent maps" (locally same on nbhd of q).
- $\mathcal{E}_n$: the set of all germs at the origin of smooth functions $f: \mathbb{R}^n \to \mathbb{R}$.

* Motivation: Algebra / Germ Language

- $\mathcal{E}_n$ to $\mathcal{E}_p$ module structure

  Given a smooth map germ $\phi: (\mathbb{R}^n, 0) \to (\mathbb{R}^p, 0)$, ϕ induces the homomorphism $\phi^*: \mathcal{E}_p \to \mathcal{E}_n$ via $\phi^* h = \underbrace{h \circ \phi}_{\mathcal{E}_n}$, $h \in (\mathbb{R}^p, 0) \to \mathbb{R}$. $h \in \mathcal{E}_p$

  $$(\mathbb{R}^n, 0) \xrightarrow{\phi} (\mathbb{R}^p, 0) \xrightarrow{h} \mathbb{R}.$$

  An $\mathcal{E}_n$ module A becomes an $\mathcal{E}_p$ module via ϕ^*:

  let $x \in \mathcal{E}_p$, $a \in A$ s.t. $xa = (\phi^* x) a \in A$.

- F.g. as a $\mathcal{E}_p$ module problem (simple counter-example)

  counter. $\phi: (x, y) \mapsto (x)$, $\phi^*: \mathcal{E}_1 \to \mathcal{E}_2$

  $A = \mathcal{E}_2$ is a f.g $\mathcal{E}_2$ by $\langle 1 \rangle \in \mathcal{E}_2$.

  A is not f.g. as a $\mathcal{E}_1$-module, $\nexists$ $\langle a_1, \dots, a_k \rangle \in A$ s.t. $\mathcal{E}_1 \langle a_1, \dots, a_k \rangle = A$.

* The Preparation Thm: Statement

Theorem 16.1 (Malgrange–Mather preparation theorem). *Let $\phi: (\mathbb{R}^n, 0) \to (\mathbb{R}^p, 0)$ be the germ of a smooth map, and let A be a finitely generated $\mathcal{E}_n$-module for which $A/I_\phi A$ is finite–dimensional. Then A is finitely generated as an $\mathcal{E}_p$-module. More precisely, let $\{u_1 \dots, u_r\} \subset A$ be a cobasis for $I_\phi A$ in A. Then A is generated by $\{u_1 \dots, u_r\}$ as an $\mathcal{E}_p$-module.*

Explicitly, to say A is generated by $\{u_1 \dots, u_r\}$ as an $\mathcal{E}_p$-module means that for each $a \in A$ there are $h_1, \dots, h_r \in \mathcal{E}_p$ for which

$$a = (h_1 \circ \phi) u_1 + \cdots + (h_r \circ \phi) u_r.$$

In general, the h_j are not uniquely determined.

Part II:

The Versality Theorem:

proof using the Preparation theorem.

II-(1): Reminders & State Versality Thm.
Intuition for Versality Thm. Pf.

II-(2): A Technical Tool: Lemma 1

II-(3): Prove Versality Thm.

- Versal unfoldings.
- Initial Speeds.

Definition 7.7.

Let $f_0 \in \mathcal{E}_n$ be a function-germ, and $F\colon (\mathbb{R}^n \times \mathbb{R}^a, (0,0)) \to \mathbb{R}$ be an unfolding of f_0. The unfolding F is ***versal*** if given any other unfolding $G\colon (\mathbb{R}^n \times \mathbb{R}^b, (0,0)) \to \mathbb{R}$ of f_0 there is a map germ $\phi\colon (\mathbb{R}^b, 0) \to (\mathbb{R}^a, 0)$ such that G is equivalent to $\phi^* F$. ⋆

The equivalence here is of course $\mathcal{R}^+_{\mathbf{un}}$-equivalence.

Not only did Thom introduce the notion of versal unfolding, he provided an easily computable way to recognize whether a given unfolding is versal, and indeed to construct a versal unfolding of any germ of finite codimension.

Given an unfolding $F(x, u)$ (with $u \in \mathbb{R}^b$), the ***initial speeds*** of the unfolding are defined to be,

$$\dot{F}_j(x) = \frac{\partial F}{\partial u_j}(x, 0), \quad (j = 1, \ldots, b).$$

These are elements of $\mathcal{E}_n$. Let $\dot{F} \subset \mathcal{E}_n$ be the vector subspace spanned by the initial speeds:

$$\dot{F} = \mathbb{R}\left\{\dot{F}_1, \ldots, \dot{F}_b\right\}.$$

II-(1): Reminders & State Versality Thm. Intuition for Versality Thm. Pf.

lomins

Necessary Reminders Before Thm 7.8 Pf

- R^+_{un} Equivalence

Definition 7.6.

Two families $F, G\colon \mathbb{R}^n \times \mathbb{R}^a \to \mathbb{R}$ are ***equivalent*** if there is a diffeomorphism $\Phi\colon \mathbb{R}^n \times \mathbb{R}^a \to \mathbb{R}^n \times \mathbb{R}^a$ of the form

$$\Phi(x, u) = (\phi(x, u), \psi(u))$$

and a function-germ $C\colon \mathbb{R}^a \to \mathbb{R}$ such that

$$F(x, u) = G(\phi(x, u),\ \psi(u)) + C(u).$$

This equivalence is called ***$\mathcal{R}^+_{\mathrm{un}}$-equivalence***. ⭑

The subscript 'un' in $\mathcal{R}^+_{\mathrm{un}}$ is of course for 'unfolding'. In essence therefore, for each value of u, the function $G_{\psi(u)}$ is $\mathcal{R}^+$-equivalent to F_u, via the change of coordinates ϕ_u (here $\phi_u(x) = \phi(x, u)$), and the constant $C(u)$.

It is important to emphasize that the change in parameters does not involve the state variable x.

- Versal unfoldings.

- Initial Speeds.

Definition 7.7.

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Not only did Thom introduce the notion of versal unfolding, he provided an easily computable way to recognize whether a given unfolding is versal, and indeed to construct a versal unfolding of any germ of finite codimension.

Given an unfolding $F(x, u)$ (with $u \in \mathbb{R}^b$), the ***initial speeds*** of the unfolding are defined to be,

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These are elements of $\mathcal{E}_n$. Let $\dot{F} \subset \mathcal{E}_n$ be the vector subspace spanned by the initial speeds:

$$\dot{F} = \mathbb{R}\left\{\dot{F}_1, \ldots, \dot{F}_b\right\}.$$

– Infinitesimally versal unfolding

Let $f \in \mathcal{E}_x$ be a germ (of finite $\mathbb{R}^+$-codimension) and let an unfolding of f,

$$F(x;u) = f(x) + \sum_{j=1}^{\ell} u_j \underbrace{\gamma_j(x)}_{\in \mathcal{E}_n}$$

be an $\mathbb{R}^+$ infinitesimally versal unfolding of f, i.e.,

$$Jf + \mathbb{R}\{\gamma_0, \gamma_1, \dots, \gamma_\ell\} = \mathcal{E}_n,$$

w/ $\gamma_0 = 1$, $\gamma_i = \frac{\partial F}{\partial x_i}\Big|_{u_i=0}$, $\forall i \in [1, \ell]$.

Statement of the Versality Thm

Thm. Versality (Thm 7.8)

Every infinitesimally versal unfolding is versal.

(it's just easily confusing): unfolding of f.

$F: \mathbb{R}^n \times \mathbb{R}^a \to \mathbb{R}$ is versal if for any other unfolding $G: \mathbb{R}^n \times \mathbb{R}^b \to \mathbb{R}$, $\exists\ \phi: \mathbb{R}^b \to \mathbb{R}^a$ s.t.

$$G = \phi_* F$$

$\underset{(x,u)}{F: \mathbb{R}^n \times \mathbb{R}^a} \to \mathbb{R}$ is infinitesimally versal if

$$\mathcal{E}_n = Jf + \mathbb{R}\left\{1, \frac{\partial F}{\partial u_1}(x,0), \ldots, \frac{\partial F}{\partial u_a}(x,0)\right\}.$$

if for any other unfolding $G: \mathbb{R}^n \times \mathbb{R}^b \to \mathbb{R}$, $\exists\ \phi: \mathbb{R}^b \to \mathbb{R}^a$, $C: \mathbb{R}^b \to \mathbb{R}$ s.t.

$$G = \phi_* F + C(u)$$

some parameters

i.e. we are taking partials over all possible unfoldings G that are equivalent to F in the $\mathbb{R}^+$ inf. versal unfolding sense.

*** Motivation.**

– What is Versality Thm's point?

provide an explicit for versal unfolding

*** Question:**

Is it obvious that versal unfolding $\Rightarrow$ infinitesimally versal unfolding?

II-(2) Warm-Up: Lemma for Versality Theorem (10mins)

Lemma = Statement, observation, big picture.

(ii). *More generally, suppose* $F(x;u,v)$ *is an unfolding of* f *for which* F_1 *is infinitesimally* $\mathcal{R}^+$*-versal, where* $F_1(x;u) := F(x;u,0)$*, then*

$$J_xF + \mathcal{E}_{u,v}\left\{1, \frac{\partial F}{\partial u_1}, \ldots, \frac{\partial F}{\partial u_\ell}\right\} = \mathcal{E}_{x,u,v}.$$

More info than def. of inf $\mathcal{R}^+$versal!

Outline:

- $A = \mathcal{E}_{x,u,v} / J_xF$ - $\mathcal{E}_{x,u,v}$-module, $\phi: \mathbb{R}^{n+\ell+k} \to \mathbb{R}^{\ell+k}$, $(x,u,v) \mapsto (u,v)$
- $A / I_\phi A \cong \mathcal{E}_x / Jf$
- Inf. unfolding $\Rightarrow$ cobasis of Jf = cobasis of $I_\phi A$
- Preparation Thm. $\Rightarrow$ A f.g. over $\mathcal{E}_{u,v}$ by $\langle\ \rangle$.

* Lemma1: Proof

$$\begin{cases} A = \mathcal{E}_{x,u,v}/J_x F \\ A/\underbrace{I_\phi A}_{\text{Preparation Thm.}} \cong \mathcal{E}_x/Jf \end{cases}$$

(ii). *More generally, suppose* $F(x;u,v)$ *is an unfolding of* f *for which* F_1 *is infinitesimally* $\mathcal{R}^+$*-versal, where* $F_1(x;u) := F(x;u,0)$*, then*

$$J_x F + \mathcal{E}_{u,v}\left\{1, \frac{\partial F}{\partial u_1}, \ldots, \frac{\partial F}{\partial u_\ell}\right\} = \mathcal{E}_{x,u,v}.$$

Comments: Note that we're proving basically the existence of a cobasis of $J_x F$ over $\mathcal{E}_{u,v}$ for $\mathcal{E}_{x,u,v}$. This is finite generatedness and interplay between modules! We smell the preparation theorem here, which is indeed the case.

Pf.

Consider a $\mathcal{E}_{x,u,v}$ module, $A = \dfrac{\mathcal{E}_{x,u,v}}{J_x F}$ along w/ the projection,

$$\phi : \mathbb{R}^{n+\ell+k} \to \mathbb{R}^{\ell+k}$$
$$(x,u,v) \mapsto (u,v),$$

then under ϕ and by the homomorphism thm. from evaluation map.

$$\varphi : \frac{\mathcal{E}_{x,u,v}}{J_x F} \longrightarrow \frac{\mathcal{E}_x}{Jf}$$

$$\left(\check{f}(x,u,v) + \left\langle\frac{\partial F(x,u,v)}{\partial x_j}\right\rangle_{j=1}\right) \mapsto \check{f}(x, u=0, v=0) + \left\langle \overbrace{\frac{\partial F(x,u=0,v=0)}{\partial x_j}}^{\partial f}\right\rangle_{j=1},$$

we deduce

$$A/\underbrace{I_\phi A}_{\ker \varphi.} \cong \frac{\mathcal{E}_x}{Jf}. \qquad (1)$$

On the other hand, recall F_1 is infinitesimally versal, i.e.,

$$Jf + \mathbb{R}\{\gamma_0, \ldots, \gamma_\ell\} = \mathcal{E}_x, \qquad (2)$$

s.t. $\gamma_0 = 1$, $\gamma_i = \dfrac{\partial F}{\partial u_i}\Big|_{u=0,v=0} = \dfrac{\partial F_1}{\partial u_i}\Big|_{u=0}$ $\forall i > 0$. Now (1) and (2) imply that $\{1, \frac{\partial F}{\partial u_1}, \ldots, \frac{\partial F}{\partial u_\ell}\}$ is a cobasis of $I_\phi A$ in A.

Then, by the preparation thm. (maybe plus the remark if you're precise), the set $\{1, \frac{\partial F}{\partial u_1}, \dots, \frac{\partial F}{\partial u_\ell}\}$ generates A over $\mathcal{E}_{u,v}$, whence we conclude,

$$\mathcal{E}_{x,u,v} = J_x \mathcal{F} + \mathcal{E}_{u,v}\{1, \frac{\partial F}{\partial u_1}, \dots, \frac{\partial F}{\partial u_\ell}\}.$$

$\blacksquare$

End comment:

I think this pf is quite neat and short. Solely from the fact that $F_1(x,u)$ is infinitesimally $\mathcal{R}^+$ versal, and by finding structural equivalence between modules, we can learn something about the structure of $\mathcal{E}_{x,u,v}$!

II - (3): Prove Versality Thm. (15 mins)

Outline of the Pf. of Thm 7.8.

Brief story: Let $f \in \mathcal{E}_x$, $F(x,u)$ infinitesimally versal unfolding of f. WTS G any other unfolding, then G equivalent to an unfolding induced from F.

Step 1. Construction of H, $\check{F}$ unfolding, then Claim & prove

"Lemma": $H, \check{F}$ equivalent $\Rightarrow$ G equiv. to an unfolding induced from F.

Step 2. show $H, \check{F}$ equivalent

- Construct "chains" of unfoldings H_j based on H.
- Build equivalence between H_j, suffices to prove for the end of such chain then go by way of "iteration"; i.e.:

$$\begin{array}{c} H_0 \cdots \leftarrow \cdots \leftarrow H_{k-2} \leftarrow H_{k-1} \xleftarrow{\text{equivalence}} H_k = H \\ \| \\ \check{F} \end{array}$$

(we prove this)

- H and $\check{F}$ are equivalent.

- Warning: this step requires a technical tool (Prop 1), which we prove earlier as warm-up.

Serious Business: pf of the Versality Thm.

Thm. Infinitesimal Versality $\Rightarrow$ Versality.

$\Leftrightarrow$ Let F be an infinitesimal versal unfolding of f, then given any other unfolding of f, say G, then G is equivalent to an unfolding induced from F.

(keep track: $F(x,u): \mathbb{R}^n \times \mathbb{R}^\ell$, $G(x,\lambda): \mathbb{R}^n \times \mathbb{R}^k$).

Outline (proof):

- Reduce to proving $H(x,u,\lambda)$ equivalent to $\tilde{F}(x,u,\lambda) = F(x,u)$
- Reduce to proving $H_k(x,u,\lambda)$ equivalent to $H_{k-1}(x,u,\lambda_1,\dots,\lambda_{k-1},0)$
  - Apply Prop1 to $\{\underbrace{H_k}_{\text{unf.}}, \underbrace{H_0(x,u)}_{\text{inf. versal}} = H_k(x,u,0)\}$
  - $$\frac{\partial H_k}{\partial \lambda_k} = \sum_{i=1}^n a_i(x,u,\lambda)\cdot\frac{\partial H_k}{\partial x_i} + b_0(u,\lambda) + \sum_{j=1}^{\ell} b_j(u,\lambda)\cdot\frac{\partial H_k}{\partial u_j}$$
  - $\Phi_t = (\phi(x,u,\lambda), \psi(u,\lambda), \lambda', \lambda_k - t)$
  - $\int \frac{d}{dt} H_k \circ \Phi_t = \int b_0(\Phi_t)$
  - $$\begin{cases} H_k \circ \Phi_t - H_k = C_t(u,\lambda) \\ H_k \circ \Phi_t = H_k + C_t(u,\lambda) \end{cases}$$

$$\lambda_k = t \Rightarrow H_{k-1} \circ \Phi_{\lambda_k} = H_k + C_{\lambda_k}(u,\lambda)$$

Ex: $\tilde{\Phi} = (\phi(x,u,\lambda), \psi(u,\lambda), \lambda', \lambda_k)$

$$\Rightarrow H_{k-1} \circ \tilde{\Phi} = H_{k-1} \circ \Phi_{\lambda_k}$$
$$= H_k + C_{\lambda_k}(u,\lambda). \quad \square$$

Serious Business: pf of the Versality Thm.

Thm. Infinitesimal Versality $\Rightarrow$ Versality.

$\Leftrightarrow$ Let F be an infinitesimal versal unfolding of f, then given any other unfolding of f, say G, then G is equivalent to an unfolding induced from F.

(keep track: $F(x,u): \mathbb{R}^n \times \mathbb{R}^l$, $G(x,\lambda): \mathbb{R}^n \times \mathbb{R}^k$).

Pf.

Let $G(x, \underbrace{\lambda}_{\in \mathbb{R}^k})$ be any unfolding of f, and it suffices to show the equivalence between G and any unfolding induced from F.

To get a bit room for flexibility, form another unfolding of f:

$$H := F \oplus G : \mathbb{R}^n \times \mathbb{R}^l \times \mathbb{R}^k \to \mathbb{R}$$

$$H(x,u,\lambda) = F(x,u) + G(x,\lambda) - f(x).$$

Now we have more "freedom" by this H construction. Indeed, we have

Lemma:

Let G be an unfolding of f; then for $\hat{F}(x,u,\lambda) = F(x,u)$,

$H = F \oplus G$ is R^+_{un} equivalent to $\hat{F}$ $\Rightarrow$ G equivalent to an unfolding induced from F.

* proof is IOU (later).

So now it suffices to show H is R^+_{un} equivalent to $\hat{F}$. Construct the unfoldings,

$$H_j(x,u,\lambda_1,\dots,\lambda_k) = H(x,u,\lambda_1,\dots,\lambda_j,0,\dots,0),$$

$j \in \{0,\dots,k\}$, where $H_0 = \hat{F}$, $H_k = H$. In particular we prove equivalence via a chain:

$$H_k \sim H_{k-1} \sim \cdots \sim H_1 \sim H_0,$$

by simply proving H_j is R^+_{un}-equivalent to H_{j-1}, $j > 0$.

Exercise: A side rmk is F inf. versal $\Rightarrow$ H_j are inf. versal by def.

As w/ the Montaldi book notation, to simplify notation we prove the

case:

<u>WTS</u>. H_k is R^+ equivalent to H_{k-1}.

and follow notation $\lambda = (\lambda', \lambda_k)$.

<u>Pf</u>.

Note that $\overbrace{H_k(x,u,\lambda)}^{\text{"}F\text{"}}$ is an unfolding of f for which $\overbrace{H_0}^{\text{"}F_0\text{"}}(x,u) = H_k(x,u,0)$ is infinitesimally R^+-versal. So we apply Lemma 1 from above, i.e.,

$$\mathcal{E}_{x,u,\lambda} = J_x H_k + \mathcal{E}_{u,\lambda}\left\{1, \frac{\partial H_k}{\partial u_1}, \dots, \frac{\partial H_k}{\partial u_\ell}\right\}$$

$\Rightarrow \exists\ a_i(x,u,\lambda)$, $i \in \{1,\dots,n\}$

$\exists\ b_i(u,\lambda)$, $j \in \{0,\dots,\ell\}$ s.t.

$$(*) \quad \underbrace{\frac{\partial H_k}{\partial \lambda_k}}_{\in\, \mathcal{E}_{x,u,\lambda}} = \sum_{i=1}^{n} a_i(x,u,\lambda)\frac{\partial H_k}{\partial x_i} + \underbrace{b_0(u,\lambda)}_{\text{from } \phi_0 = 1} + \sum_{j=1}^{k} b_j(u,\lambda)\frac{\partial H_k}{\partial u_j}$$

Focus on the vector field associated to $(*)$,

$$(**) \quad v = -\frac{\partial}{\partial \lambda_k} + \sum_i a_i(x,u,\lambda)\frac{\partial}{\partial x_i} + \sum_j b_j(u,\lambda)\frac{\partial}{\partial u_j}$$

Note H_k is smooth so (I) view this $(**)$ as a vector field on a flow w/ the form

$$\begin{cases} \Phi_t & \text{flow on the vector field } v \\ v = \displaystyle\sum_{i=1}^{n+\ell+k+1} f_i(x)\frac{\partial}{\partial x_i}, & f_i \text{ the } i\text{th of } f(x) = \left.\frac{d}{dt}\right|_{t=0} \Phi_t(x) \end{cases}$$

and integrating $(**)$ vector field produces a $\underbrace{\text{diffeomorphism}}_{\substack{\text{smoothness of } H_k \\ \text{allows to not worry about continuity}}}$ of form,

$$\Phi_t(x,u,\lambda) = (\phi_t(x,u,\lambda), \gamma_t(u,\lambda), \underbrace{\lambda'}, \lambda_k - t)$$

and denote,

$$\Psi_t(u,\lambda) = (\gamma_t(u,\lambda), \lambda', \lambda_k - t),$$

(comment: Good to think about vector field & integration result's coordinate correspondance. If you're not convinced, see at the end of notes a detailed computation.)

where Φ_t, Ψ_t are diffeomorphism germs.

Now rearranging terms in $(*)$ gives,

$$(1) \quad \frac{-\partial H_k}{\partial \lambda_k} + \sum_i a_i(x,u,\lambda)\frac{\partial H_k}{\partial x_i} + \sum_{j>1} b_j(u,\lambda)\frac{\partial H_k}{\partial u_j} = -b_0(u,\lambda).$$

By the associated vector field integration step,

$$m \quad (2) \quad \frac{d}{dt}(H_k \circ \Phi_t(x,u,\lambda)) = -b_0(\Psi_t(u,\lambda))$$

(Comment: Recall f_i's in $v = \sum_{i=1}^{n+\ell+k+1} f_i(\hat{x})\frac{\partial}{\partial \hat{x}_i}$ vector field are the i-th component of $\frac{d}{dt}\big|_{t=0}\Phi_t(t)$. Now "evaluate" both sides of (1) along the "integrated result" to get (2). "evaluation of vector field" on flow)

Now integrating from $t=0$ (2) gives

$$m \quad H_k \circ \Phi_t(x,u,\lambda) - \underbrace{H_k(x,u,\lambda)}_{?} = C_t(u,\lambda),$$

] derived based on b_0, a smooth function.

and evaluating at $t=\lambda_k$ gives,

$$(3) \quad H_k \circ \Phi_{\lambda_k}(x,u,\lambda) - H_k(x,u,\lambda) = C_{\lambda_k}(u,\lambda)$$

and note the last entry of Φ_{λ_k} is 0, so, (depency on k removed)

$$m \quad H_k \circ \Phi_{\lambda_k}(x,u,\lambda) = H_{k-1}\circ\Phi_{\lambda_k}(x,u,\lambda)$$

(not a fullrank, no ? inverse ?).

Note that Φ_{λ_k} is not a diffeomorphism due to the "degeneracy" coming from the last 0 entry, so a slight twist fixes the problem,

$$m \quad \check{\Phi}(x,u,\lambda) = (\phi_{\lambda_k}(x,u,\lambda), \psi_{\lambda_k}(u,\lambda), \lambda', \lambda_k),$$

and you can use the inverse function Thm to prove that $\check{\Phi}$ is the germ of a diffeomorphism.

By construction,

$$m \quad H_{k-1}\circ\check{\Phi} = H_{k-1}\circ\Phi_{\lambda_k}.$$

Hence w/ (3),

$$m \quad H_{k-1}\circ\check{\Phi} = H_k \circ \Phi_{\lambda_k}(x,u,\lambda) = H_k + C_{\lambda_k}(u,\lambda),$$

where C is smooth. $\breve{\Phi}$ gives the equivalence relation diffeomorphism between unfoldings H_k and H_{k-1}.

Repeat the above process for $j = k, k-1, \dots, 1$, we conclude that $H_k = \breve{F}$ is $\mathcal{R}^{+}_{un}$ equivalent to $H_0 = H$. Hence by Lemma, we have shown infinitesimal versality $\Rightarrow \mathcal{R}^{+}_{un}$ equivalence. $\boxtimes$

IOU proof for Lemma (if time permits)

Lemma :

Let G be an unfoling ; then for $\tilde{F}(x,u,\lambda)=F(x,u)$,

$H=F\oplus G$ is R^{+}_{un} equivalent to $\tilde{F}$ $\Rightarrow$ F equivalent to G

Pf (outline)

- $H \overset{R^{+}}{\sim} \tilde{F} \sim F$
- evaluate at $u=0$

Pf. H and $\tilde{F}$ are R^{+}_{un} equivalent implies that

$$\begin{cases} \exists \text{ diffeomorphism } \Phi(x,u,\lambda)=(\phi(x,u,\lambda),\psi(u,\lambda),\chi(u,\lambda)) \\ \exists \text{ a smooth function } C(u,\lambda) \end{cases}$$

s.t.

$$\begin{aligned} H(x,u,\lambda) &= \tilde{F}(\phi(x,u,\lambda),\psi(u,\lambda),\chi(u,\lambda))+C(u,\lambda) \\ &= F(\phi(x,u,\lambda),\psi(u,\lambda))+C(u,\lambda) \end{aligned}$$

Evaluating at $u=0$ we have

$$G(x,\lambda)=H(x,0,\lambda)=F(\phi(x,0,\lambda),\psi(0,\lambda))+C(\lambda).$$

$\blacksquare$

Part III:

Proof of Malgrange Preparation Theorem

(1) Malgrange's Preparation Thm : Statements & Comments 10 mins

(2) Outline of Pf. 5 mins

(3) Preparation Thm Pf: Skeleton version 20 mins

(4) If time: Preparation Thm pf – Proposition 1 10 mins max.

III-(1) Malgrange's Preparation Thm.: Statements & Comments

- State Thm w/ full
- ϕ^* (maps) Rmk
- Comment on cobasis $\Leftrightarrow$ generating set (Rmk)

III - (1) Malgrange's Preparation Thm.: Statements & Comments (10 mins)

* Statement

Theorem 16.1 (Malgrange–Mather preparation theorem). *Let $\phi : (\mathbb{R}^n, 0) \to (\mathbb{R}^p, 0)$ be the germ of a smooth map, and let A be a finitely generated $\mathcal{E}_n$-module for which $A/I_\phi A$ is finite–dimensional. Then A is finitely generated as an $\mathcal{E}_p$-module. More*

* Comment on notation "I_ϕ"

- "A map germ at point q is an equivalence class of germ equivalent maps" (locally same on nbhd of q).
- $\mathcal{E}_n$: the set of all germs at the origin of smooth functions $f : \mathbb{R}^n \to \mathbb{R}$.

Remark: (i) $\phi : (\mathbb{R}^n, 0) \to (\mathbb{R}^p, 0)$

$$\phi^* : \mathcal{E}_p \to \mathcal{E}_n, \quad \phi^* h = h \circ \phi$$

- Montaldi: $I_\phi = \langle \phi(x_1, \dots, x_n) \rangle \triangleleft \mathcal{E}_n$ generated by components of ϕ
- Bröcker: $\phi^*(m(p)) = \phi^*(\langle x_1, \dots, x_p \rangle)$ by Hadamard's Lemma

($\phi^* m(p)$ can be identified w/ $m(p)$ via ϕ^*).

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*Claim: $\phi^*(m(p)) \subseteq I_\phi \Rightarrow$ Then proving the Malgrange preparation Thm. w/ $A/\phi^*(m(p))A$ implies the case that of $A/I_\phi A$.

Pf.

$$\phi^*(m(p)) = \phi^*(\langle x_1, \dots, x_p \rangle)$$
$$= \{ f \circ \phi \mid f \in \langle x_1, \dots, x_p \rangle \triangleleft \mathcal{E}_p \} \subset \mathcal{E}_n$$
$$\subseteq \langle \phi(x_1, \dots, x_n) \rangle \triangleleft \mathcal{E}_n. \quad \square$$

"exercise"

We will use Bröcker's $\phi^* m(p)$, as it is the more intuitive notation for a crucial step in the proof.

precisely, let $\{u_1 \ldots, u_r\} \subset A$ be a cobasis for $\underbrace{I_\phi A}_{\phi^* m(p) A}$ in A. Then A is generated by $\{u_1 \ldots, u_r\}$ as an $\mathcal{E}_p$-module.

Explicitly, to say A is generated by $\{u_1 \ldots, u_r\}$ as an $\mathcal{E}_p$-module means that for each $a \in A$ there are $h_1, \ldots, h_r \in \mathcal{E}_p$ for which

$$a = (h_1 \circ \phi)u_1 + \cdots + (h_r \circ \phi)u_r.$$

In general, the h_j are not uniquely determined.

Claim. Given A a f.g. $\mathcal{E}_p$-module, TFAE,

(i) $\{u_1, \ldots, u_r\}$ generate A,

(ii) $\underbrace{\{\bar{u}_1, \ldots, \bar{u}_r\}}_{\text{"cobasis of } \mathfrak{a}A \text{ in } A\text{"}}$ proj. of $\{u_1, \ldots, u_r\}$ generate $A/\mathfrak{a}A$

where $\mathfrak{a} \subseteq$ Jacobian ideal of A.

$\Rightarrow$ Obvious

$\Leftarrow$ Nakayama's. let $N = \langle u_1, \ldots, u_r \rangle$, then by (ii).

$$N + \mathfrak{a}A = A$$

$$\Rightarrow N = A. \quad \blacksquare$$

* E.g.

Let $\phi: \mathbb{R} \to \mathbb{R}$, $\phi(x) = x^2$. Let $A = \mathcal{E}_1$. Then $I_\phi = (x^2)$

and obviously $A / I_\phi A = \mathbb{R}\{1, x\}$ is the cobasis of $I_\phi A$ in A.

Malgrange $\Rightarrow$ $\{1, x\}$ generates A as $\mathcal{E}_1$

$\Rightarrow$ $\forall f \in A$, $f = h_1(x^2) + h_2(x^2)\, x$, for some $h_1, h_2 \in \mathcal{E}_1$.

III - (2) Outline of pf. of Preparation Thm. (5 mins)

- So we have A, a f.g. $\mathcal{E}_n$ module s.t. $A/\phi^*m(p)\cdot A$ is finite dimensional, we want to show A a f.g. $\mathcal{E}_p$ module.

- How do we approach? Recall identification of the $\mathcal{E}_n$-module A as an $\mathcal{E}_p$-module is via:

$$\phi^* : \mathcal{E}_p \to \mathcal{E}_n, \text{ induced by } \phi : (\mathbb{R}^n, 0) \to (\mathbb{R}^p, 0).$$

- So we see constructing smooth germs between $(\mathbb{R}^n, 0)$, $(\mathbb{R}^p, 0)$, should be a way to prove our claim.

The big picture behind the proof is:

- Take germ $\phi : (\mathbb{R}^n, 0) \to (\mathbb{R}^p, 0)$, to find a dissection point of the problem we decompose ϕ:

$$\boxed{\phi : (\mathbb{R}^n, 0) \xrightarrow[\hat{f} = (id, f)]{} (\mathbb{R}^n \times \mathbb{R}^p, 0) \xrightarrow[\check{g} = (g_1, \dots, g_n)]{} (\mathbb{R}^p, 0)}$$

- For short hand, we denote

" $A/\phi^*(m(p))\cdot A$ finite $\Rightarrow$ A f.g. $\mathcal{E}(p)$ module "

as simply $M(\phi)$, where $\phi : (\mathbb{R}^n, 0) \to (\mathbb{R}^p, 0)$ is identification for A.

Hard work

- We show that $M(\hat{f})$ and $M(\check{g})$. We then show that $M(\hat{f})$ and $M(\check{g}) \Rightarrow M(\check{g} \circ \hat{f}) = M(\hat{\phi})$.

- We are happy.

III - (3) (Psedo) Proof of the Thm.

Prepare: two propositions (i.e. Prove the Malgrange-preparation Thm in each case).

Proposition 1. Let $h : (\mathbb{R} \times \mathbb{R}^p, 0) \to (\mathbb{R}^p, 0)$ be a germ, then $M(h)$.

$$(t, x) \mapsto x$$

Proposition 2. Let $f : (\mathbb{R}^n, 0) \to (\mathbb{R}^p, 0)$ be a germ w/ rank n,
then $M(f)$.

* The proof for Prop 1 & Prop 2 are at the end of the notes.
They might not be covered in the talk.

Decomposition

Decompose $\check{\phi}: (\mathbb{R}^n, 0) \to (\mathbb{R}^p, 0)$ as,

$$(\mathbb{R}^n, 0) \xrightarrow[\check{f} = (id, \check{\phi})]{} (\mathbb{R}^n \times \mathbb{R}^p, 0) \xrightarrow[\check{g} = \check{g}_n]{} (\mathbb{R}^p, 0)$$

Claim 1: $M(\check{f})$

- $\check{f}$ is the obvious projection w/ rank n ($\ker \check{f} = 0$). By Proposition 2, $M(\check{f})$.

Claim 2: $M(\check{g})$

- $\check{g}$ is not that obvious. Recall that if we have germ (Proposition 1)

$$h : (\mathbb{R} \times \mathbb{R}^p, 0) \to (\mathbb{R}^p, 0)$$
$$(t, x) \mapsto x$$

then $M(h)$. To see knowing $M(h)$ is necessary & sufficient for $M(\check{g})$, note $\check{g}$ is a composition of h's defined w.r.t the n number of $\mathbb{R}$'s in the domain of $\check{g}: (\underbrace{\mathbb{R} \times \cdots \times \mathbb{R}}_{n \text{ copies}} \times \mathbb{R}^p, 0) \to (\mathbb{R}^p, 0)$ st,

where $\check{g}_1 : (\mathbb{R} \times \mathbb{R}^p, 0) \to (\mathbb{R}^p, 0)$

$\check{g}_2 : (\mathbb{R} \times \mathbb{R} \times \mathbb{R}^p, 0) \to (\mathbb{R} \times \mathbb{R}^p, 0)$

$\vdots$

$\check{g}_n : (\mathbb{R}^n \times \mathbb{R}^p, 0) \to (\mathbb{R}^{n-1} \times \mathbb{R}^p, 0)$

so that,

$$\check{g} = \check{g}_1 \circ \cdots \circ \check{g}_n$$

$$\check{g} : \mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R}^{n-1} \times \mathbb{R}^p \to \cdots \to \mathbb{R} \times \mathbb{R}^p \to \mathbb{R}^p.$$

where $\forall i \in [1, n]$, $\check{g}_i \Rightarrow M(\check{g}_i)$ by Proposition 1. Then we have a "chain" of modules w/ f.g. relations :

$$\mathcal{E}_{n+p} \xrightarrow[\substack{A \text{ f.g. as-module} \\ \text{by } M(\check{g}_n)}]{} \mathcal{E}_{(n-1)+p} \to \cdots \mathcal{E}_{p+1} \xrightarrow[M(\check{g}_1)]{} \mathcal{E}_p$$

$\Rightarrow M(\check{g})$ by chained f.g. module relation above. $\blacksquare$

Claim 3: Under the hypothesis of the preparation Thm (A f.g. $\mathcal{E}^n$ mod. $A/\phi^*_{m(p)}$ f.d.)$^\Delta$

$M(\check{f})$ and $M(\check{g}) \Rightarrow$ "$M(\check{g}\circ\check{f}) = M(\phi)$"

Goal: $\begin{cases} ① \ A \text{ a f.g. } \mathcal{E}_n \text{ module} \Rightarrow A \text{ a f.g. } \mathcal{E}_{n+p} \text{ module} \\ ② \ A \text{ a f.g. } \mathcal{E}_{p+n} \text{ module} \Rightarrow A \text{ a f.g. } \mathcal{E}_p \text{ module.} \end{cases}$

$\begin{cases} \check{f}: (\mathbb{R}^n, 0) \to (\mathbb{R}^n \times \mathbb{R}^p, 0) \\ \check{g}: (\mathbb{R}^n \times \mathbb{R}^p, 0) \to (\mathbb{R}^p, 0) \end{cases}$

Assumptions: Assume A is finitely generated over $\mathcal{E}(n)$. Moreover,

$$A/(\check{g}\circ\check{f})^* m(p)\cdot A = A/\check{f}^*(\check{g}^* m(p))\cdot A$$

is finite dimensional ($\mathbb{R}$).

①

$$\check{g}^* m(p) < m(p+n) \Rightarrow \check{f}^*\check{g}^* m(p) < \check{f}^* m(p+n)$$

$$\Rightarrow A/\check{f}^* m(p+n)\cdot A \text{ is finite dimensional}$$

$$\overset{M(\check{f})}{\Longrightarrow} \underline{A \text{ is f.g. over } \mathcal{E}(p+n)} \ ☆$$

② On the other hand, note $A/\check{g}^* m(p)\cdot A = A/\check{f}^*\check{g}^* m(p)\cdot A$,

$\mathcal{E}^p \to \mathcal{E}^n$

"equal" up to representation

$\check{g}^* \langle x_1, \dots, x_p \rangle = I_{\check{g}}$

$\check{f}^*\check{g}^* \langle x_1, \dots, x_p \rangle = I_{\check{g}\circ\check{f}}$.

which is finite dimensional over $\mathbb{R}$. Hence by $M(\check{g})$, A (a $\mathcal{E}(p+n)$ module) is a finitely generated $\mathcal{E}(p)$-module. Note we have:

$\mathcal{E}(n)$-module A f.g. over $\mathcal{E}(p+n)$

$\mathcal{E}(p+n)$-module A f.g. over $\mathcal{E}(p)$

$\Rightarrow$ Under $(\check{g}\circ\check{f})^*$, $\mathcal{E}(n)$-module A is f.g. over $\mathcal{E}(p)$.

■

III - (4) $\boxed{\text{Proof of the Thm: Prop1 + Prop2}}$

A Tool

- One Step of the proof requires the concept of regular germ & Division Lemma: (for Proposition 1 proof).

*** Def (Regular)**

A smooth germ $f: (\mathbb{R}\times\mathbb{R}^n, 0) \to \mathbb{R}$, $(t,x) \mapsto f(t,x)$ is p-regular (w.r.t t), if $f|_{\mathbb{R}\times\{0\}} \in \mathfrak{m}(1)^p$ and $\notin \mathfrak{m}(1)^{p+1}$; i.e.,

$$0 = f(0,0) = \cdots = \frac{\partial^{p-1}}{\partial t^{p-1}} f(0,0), \quad \frac{\partial^p}{\partial t^p} f(0,0) \neq 0.$$

*** Division Lemma**

Let $f, g \in \mathcal{E}(n+1)$ be germs s.t. f is p-regular. Then $\exists$ a $Q \in \mathcal{E}(n+1)$ and germs $h_j \in \mathcal{E}(n)$, $j = 1, \cdots, p$ s.t.

$$g = Qf + \sum_{j=1}^{p} h_j(x)\, t^{p-j}.$$

Prepare: two propositions (i.e. Prove the Malgrange-preparation Thm in each case).

Proposition 1. Let $h:(\mathbb{R}\times\mathbb{R}^p,0)\to(\mathbb{R}^p,0)$, $(t,x)\mapsto x$ be a germ, then $M(h)$.

Pf. By assumption, we may choose finitely many $\alpha_1,\dots,\alpha_\ell\in A$ that generate A over $\mathcal{E}(p+1)$ and $\underbrace{A/h^*m(p)\cdot A}_{\text{as a real vector space.}}$. Then $\forall\alpha\in A$,

$$\alpha=\sum_{j=1}^{\ell}c_j\alpha_j+b \qquad (c_j\in\mathbb{R},\ b\in(h^*m(p)\cdot A))$$

(Consider $\alpha\in A$ then by generators defined before, $\alpha=\sum_{j=1}^{\ell}c_j\alpha_j$ over $\mathbb{R}$, $c_j\in\mathbb{R}$. Note $\alpha-\sum_{j=1}^{\ell}c_j\alpha_j=0\Rightarrow\ \in h^*m(p)\cdot A$, over projection $A\to\dfrac{A}{h^*m(p)\cdot A}$

$$\Rightarrow\ \alpha=\sum_{j=1}^{\ell}c_j\alpha_j+b,\quad b\in h^*m(p)\cdot A.)$$

$$=\sum_{j=1}^{\ell}c_ja_j+\sum y_kb_k \qquad (y_k\in h^*m(p),\ b_k\in A)$$

$$=\sum c_ja_j+\underbrace{\sum y_k\sum r_{kj}}_{z_j\in h^*m(p)\cdot\mathcal{E}(p+1).}a_j \qquad (r_{kj}\in\mathcal{E}(p+1))$$

$$\boxed{\alpha=\sum_{j=1}^{\ell}\underset{\in\mathbb{R}}{c_j}a_j+\sum_{j=1}^{\ell}z_ja_j}$$

In particular, consider the case $\alpha=t\alpha_i$, s.t.,

$$t\alpha_i=\sum_{j=1}^{\ell}(c_{ij}+z_{ij})\,\alpha_j \qquad \in\mathbb{R}$$

$$\Leftrightarrow\ \underset{(\ell\times\ell)}{(t\delta_{ij}-c_{ij}-z_{ij})}\cdot\underset{(\ell\times1)}{a}=0,\qquad ((\delta_{ij})\text{ the identity matrix}) \quad \underline{(1)}$$

$$a=\begin{pmatrix}\alpha_1\\ \vdots\\ \alpha_\ell\end{pmatrix}=\ell\times1$$

and denote $b_{ij}=t\delta_{ij}-c_{ij}-z_{ij}$. (now a matrix $\in\mathbb{R}^{\ell\times\ell}$)

Now let $(B_{ij}) \in Mat_{\ell \times \ell}(\mathbb{R})$ be the adjugate of (b_{ij}), i.e.,

(B_{ij}) is the transpose of (b_{ij})'s cofactor matrix. Then:

$$(B_{ij}) = adj(b_{ij}) \Rightarrow (B_{ij}) \cdot (b_{ij}) = \det(b_{ij}) \cdot (\delta_{ij}) \quad (2)$$

By linear algebra, (1)+(2) implies $\det(b_{ij}) \cdot \alpha = 0$.

quick com. algebra.

Note $\Delta \cdot \alpha = 0 \Rightarrow \Delta \cdot A = 0 \Rightarrow A$ is $\dfrac{\mathcal{E}(p+1)}{\Delta \cdot \mathcal{E}(p+1)}$-module.

Also note $\det(b_{ij}) = \Delta$

is a function in $(t, x) \in \mathbb{R} \times \mathbb{R}^p$; if let $x = 0$, then Δ

is in t alone. This allows to conclude Δ is q-regular w.r.t.

t at $(t, 0)$, for some $q \leq \ell$ (note $\Delta(0,0) = 0$, hence ...↑).

Then by the Division Lemma (see page), since we have,

- Δ is q-regular;
- Let $\gamma \in \mathcal{E}(p+1)$; then by Division Lemma, $\exists\, Q \in \mathcal{E}(p+1)$, germs $h_j \in \mathcal{E}(p)$, $j = 1, \dots, q$ s.t.,

$$\underbrace{\gamma}_{\in \mathcal{E}(p+1)} = \underbrace{Q \cdot \Delta}_{\in \Delta \cdot \mathcal{E}(p+1)} + \underbrace{\sum_{j=1}^{q} h_j(x) t^{q-j}}_{\mathcal{E}(p+1)/\Delta \cdot \mathcal{E}(p+1)}$$

$\Rightarrow \{t, t^2, \dots, t^{q-1}\}$ is a cobasis of $\Delta \cdot \mathcal{E}(p+1)$ in $\mathcal{E}(p)$

$\Rightarrow \mathcal{E}(p+1)/\Delta \cdot \mathcal{E}(p+1)$ f.g. over $\mathcal{E}(p)$.

Then

$$A \xrightarrow{\text{f.g. over}} \frac{\mathcal{E}(p+1)}{\Delta \cdot \mathcal{E}(p+1)} \xrightarrow{\text{f.g. over}} \mathcal{E}(p)$$

$\Rightarrow A$ is f.g. over $\mathcal{E}(p) \Leftrightarrow M(h)$. QED.

Proposition 2. Let $\check{f}: (\mathbb{R}^n, 0) \to (\mathbb{R}^p, 0)$ be a germ w/ rank n, then $M(\check{f})$.

Pf.

By rank-nullity theorem, $\check{f}$ can be expressed in terms of coordinates,

$$(x_1, \dots, x_n) \mapsto (x_1, \dots, x_n, 0, \dots, 0)$$

from which we conclude, for a canonical embedding of $\mathbb{R}^n \subset \mathbb{R}^p$, any smooth germ $(\mathbb{R}^n, 0) \to \mathbb{R}$ can be extended to $(\mathbb{R}^p, 0)$. Hence $\check{f}^*: \mathcal{E}(p) \to \mathcal{E}(n)$ is surjective.

$\Rightarrow$ $\exists$ finitely many generators of A over $\mathcal{E}(n)$ that are (representation of) generators of A over $\mathcal{E}(p)$.

$\blacksquare$

Theorem 2.1. (*Mather Division Theorem*). *Let F be a smooth real-valued function defined on a nbhd of* 0 *in* $\mathbf{R} \times \mathbf{R}^n$ *such that* $F(t, 0) = g(t)t^k$ *where* $g(0) \neq 0$ *and* g *is smooth on some nbhd of* 0 *in* $\mathbf{R}$. *Then given any smooth real-valued function* G *defined on a nbhd of* 0 *in* $\mathbf{R} \times \mathbf{R}^n$, *there exist smooth functions* q *and* r *such that*

(i) $G = qF + r$ *on a nbhd of* 0 *in* $\mathbf{R} \times \mathbf{R}^n$, *and*

(ii) $r(t, x) = \sum_{i=0}^{k-1} r_i(x)t^i$ *for* $(t, x) \in \mathbf{R} \times \mathbf{R}^n$ *near* 0.

Notes. (1) The Malgrange Preparation Theorem which states that there exists a smooth q with $q(0) \neq 0$ such that $(qF)(t, x) = t^k + \sum_{i=0}^{k-1} \lambda_i(x)t$ follows from 2.1 in precisely the same way that Theorem 1.1 follows from Theorem 1.2.

Theorem 1.1. (*Weierstrass Preparation Theorem*). *Let F be a complex-valued holomorphic function defined on a nbhd of* 0 *in* $\mathbf{C} \times \mathbf{C}^n$ *satisfying:*

(a) $F(w, 0) = w^k g(w)$ *where* $(w, 0) \in \mathbf{C} \times \mathbf{C}^n$ *and* g *is a holomorphic function of one variable in some nbhd of* 0 *in* $\mathbf{C}$, *and*

(b) $g(0) \neq 0$.

Then there exists a complex-valued holomorphic function q *defined on a nbhd of* 0 *in* $\mathbf{C} \times \mathbf{C}^n$ *and complex-valued holomorphic functions* $\lambda_0, \ldots, \lambda_{k-1}$ *defined on a nbhd of* 0 *in* $\mathbf{C}^n$ *such that*

(i) $(qF)(w, z) = w^k + \sum_{i=0}^{k-1} \lambda_i(z)w^i$ *for all* (w, z) *in some nbhd of* 0 *in* $\mathbf{C} \times \mathbf{C}^n$, *and*

(ii) $q(0) \neq 0$.

Remark. The reader may well ask what such a theorem is good for. Before we proceed we point out one trivial consequence. Given a nonzero holomorphic function F of $n + 1$ complex variables, we may assume (by a linear change of coordinates) that $F = F(w, z)$ is in the form above. Then the Weierstrass Preparation Theorem states that the zero set of F equals the zero set of the function

$$w^k + \sum_{i=0}^{k-1} \lambda_i(z)w^i$$