<u>Talk</u>: Jet bundles & PDRs, general formulations of the h-principle.

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A "natural" framework to formulate the h-principle is the so-called Jet bundle.

I begin with jet bundle.

In what follows (unless stated), V, W and X are manifolds of dimension n, q and $n+q$ respectively.

§1 Jet bundle

1.1 Jets over $\mathbb{R}^n$

Given a function $f: \mathbb{R}^n \longrightarrow \mathbb{R}^q$ & $r \geq 0$ an integer, the r-jet of f at a point $x \in \mathbb{R}^n$ is the function given by

$$J_f^r(x) = \left(f(x), f'(x), \ldots, f^{(r)}(x)\right)$$

where $f^{(s)}$ consists of all partial derivatives $D^\alpha f$, $\alpha = (\alpha_1, \ldots, \alpha_n)$ such that $|\alpha| = \sum_{i=1}^n \alpha_i = s$.

example: given a function

$$f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$$

$$(x,y) \longmapsto (f_1(x,y), f_2(x,y)),$$

the 1-jet of f at (x,y) is given by

$$J^1_f((x,y)) = (f_1, f_2, \partial_x f_1, \partial_x f_2, \partial_y f_1, \partial_y f_1)$$

at (x,y).

It is then natural to think of $J^1_f(x,y)$ as a point of $\mathbb{R}^2 \times \mathbb{R}^4$.

Thus, we can consider J^r_f as a function from $\mathbb{R}^n$ to some higher dimensional euclidean space. More precisely, let $d_r = d(n,r)$ be the number of all partial derivatives D^α of order r

of a function $\mathbb{R}^n \longrightarrow \mathbb{R}$ &

$N_r = N(n,r) = 1 + d_1 + \cdots + d_r$, then

the r-jet $J_f^r(x)$ of a function

$f: \mathbb{R}^n \longrightarrow \mathbb{R}^q$ is a point in the space

$$\mathbb{R}^q \times \mathbb{R}^{qd_1} \times \cdots \times \mathbb{R}^{qd_r} \cong \mathbb{R}^{qN_r}.$$

<u>Remark</u>: $d_r = \dfrac{(n+r-1)!}{(n-1)\,r!}$ and $N_r = \dfrac{(n+r)!}{n!\,r!}$.

This leads to the following definition.

<u>Definition</u>: The r-jet bundle $J^r(\mathbb{R}^n, \mathbb{R}^q)$

is defined as $\mathbb{R}^n \times \mathbb{R}^{qN_r}$ i.e.

$$J^r(\mathbb{R}^n, \mathbb{R}^q) = \mathbb{R}^n \times \mathbb{R}^{qN_r}$$

as the trivial fiber bundle

$\mathbb{R}^n \times \mathbb{R}^{qN_r} \longrightarrow \mathbb{R}^n$ over $\mathbb{R}^n$ whose fiber is $\mathbb{R}^{qN_r}$.

With this defination, the r-jet of a function $f : \mathbb{R}^n \longrightarrow \mathbb{R}^q$ is a section

$$\begin{aligned} \mathbb{R}^n &\longrightarrow J^r(\mathbb{R}^n, \mathbb{R}^q) \\ x &\longmapsto (x, J^r_f(x)) \end{aligned}$$

of $J^r(\mathbb{R}^n, \mathbb{R}^q)$ over $\mathbb{R}^n$.

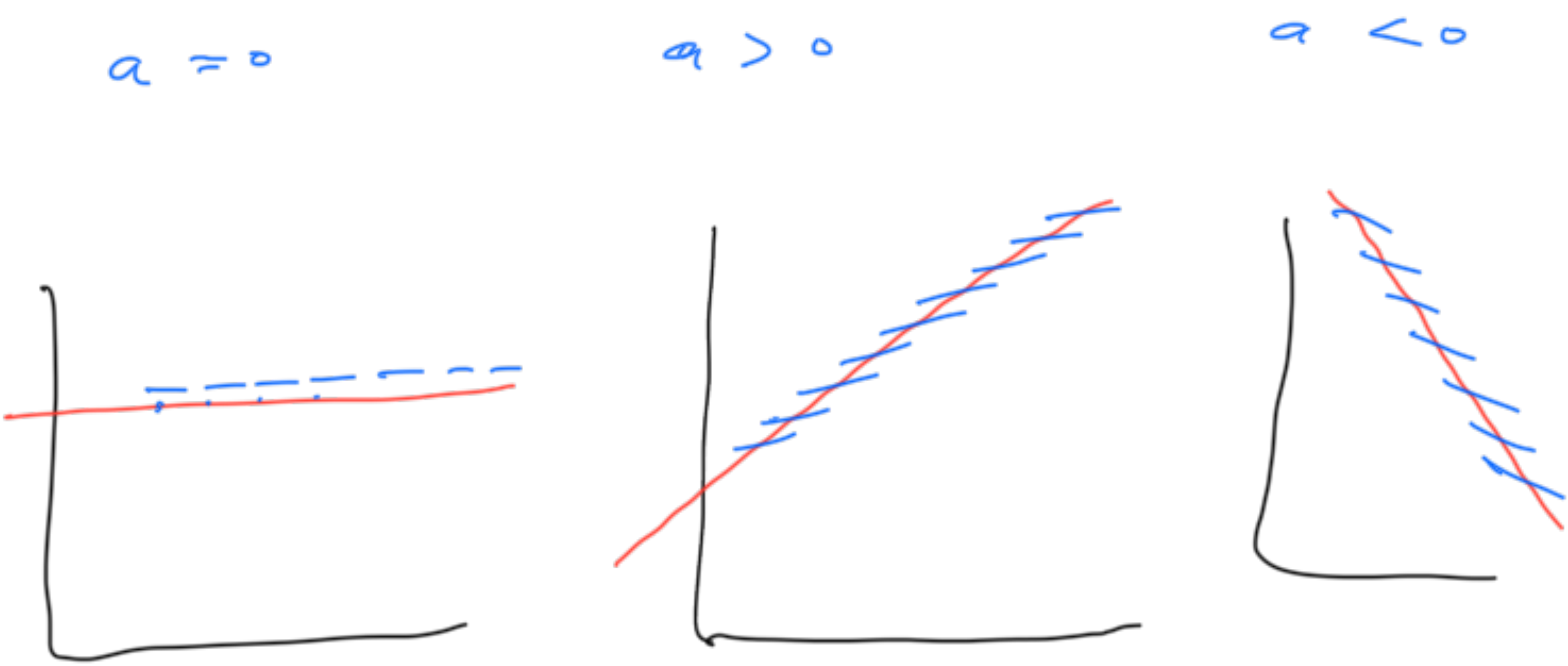
This suggests a nice way to "phrase" the function $f : \mathbb{R}^n \to \mathbb{R}^q$ as its graph Γ_f which is the image of the section $\Gamma : \mathbb{R}^n \longrightarrow \mathbb{R}^n \times \mathbb{R}^q$ of the trivial fibration (= fiber bundle) $\mathbb{R}^n \times \mathbb{R}^q \longrightarrow \mathbb{R}^n$ over $\mathbb{R}^n$. Hence we rather view functions $\mathbb{R}^n \to \mathbb{R}^q$ as sections $\mathbb{R}^n \to \mathbb{R}^n \times \mathbb{R}^q$ of the trivial fibration $\mathbb{R}^n \times \mathbb{R}^q \to \mathbb{R}^n$ over $\mathbb{R}^n$, so that the theory can later be extended to arbitrary fibrations.

For a section $f : \mathbb{R}^n \to \mathbb{R}^n \times \mathbb{R}^q$, the associated section $J_f^r : \mathbb{R}^n \to J^r(\mathbb{R}^n, \mathbb{R}^q)$ of the trivial fibration is called the r-jet of f or the r-jet extension of f.

<u>example</u>: 1-jet of the function

$$f : \mathbb{R} \to \mathbb{R}, \quad f(x) = ax + b$$

$$J_f^1(x) = (ax+b, a)$$



§2 Jets over manifolds

To extend the definition of jets to arbitrary sections of manifolds fibrations, as expected the r-tangenty will be rather defined as an equivalence class of local sections of manifolds fibrations.

Let $X \xrightarrow{p} V$ be a smooth

fibration over V whose fiber is W where $\dim X = n+q$

and $\dim V = n$ and let $v \in V$ and

$\mathcal{O}_p v$ an open (small) neighborhood of v in V.

We say that two local sections

$f, g : \mathcal{O}_p v \longrightarrow X$ of the fibration

$X \longrightarrow V$ are r-tangent at the point v

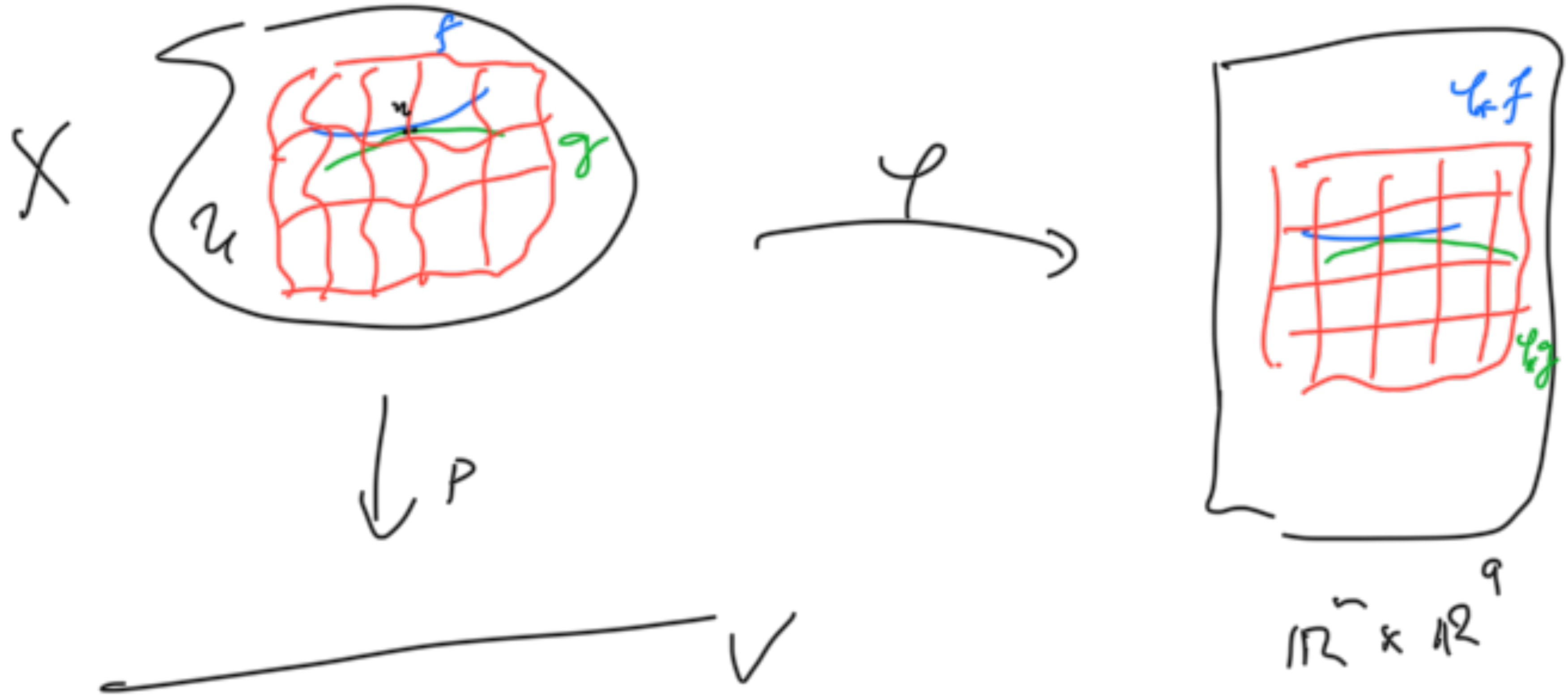
if $f(v) = g(v)$ and

$$J^r_{\varphi \circ f}(\varphi(x)) = J^r_{\varphi \circ g}(\varphi(x))$$

for a local trivialization $\varphi : \mathcal{U} \longrightarrow \mathbb{R}^n \times \mathbb{R}^q$

of X in a neighborhood $\mathcal{U}$ of the

point $x = f(v) = g(v)$.



<u>Remark:</u> The r-tangency condition does not depend on the specific choice of the local trivialization. The r-tangency relation at $v \in V$ is an equivalence relation on the set of local sections on X through v. We call the r-jet of a section $f : \mathcal{U}_p v \longrightarrow X$ at a point $v \in V$, the r-tangency class of f.

Def (Jet bundle of a fibration). The jet bundle $X^{(r)}$ of an arbitrary fibration $p: X \longrightarrow V$ is

$$X^{(r)} = V \times Sec(X)/\sim$$ where $\sim$ is r-th equivalence and $Sec(X)$ space of sections of X. We

$$\{\text{Sections on } X\} \longrightarrow \{\text{sections on } X\}/\sim =: X^{(r)}$$

$$f: U_p V \to X \rightsquigarrow [(f, v)]$$

Note we have the following map:

$$X^{(r)} \xrightarrow{p_0^r} X \quad \text{"set-theoretic fibration"}$$

$$[(f, v)] \longmapsto f(v)$$

We can compose it with $X \xrightarrow{p} V$

to get $p^r = p \circ p_0^r : X^{(r)} \longrightarrow V$.

The extensions

$$\varphi^r : (p_0^r)^{-1}(\mathcal{U}) \longrightarrow J^r(\mathbb{R}^n, \mathbb{R}^q)$$

of the local trivialization $\varphi : \mathcal{U} \longrightarrow \mathbb{R}^n \times \mathbb{R}^q$ which sends the r-tangency classes of local sections of X to the r-tangency classes of their images in $J^r(\mathbb{R}^n, \mathbb{R}^q)$ define a natural smooth structure on $X^{(r)}$ such that $p^r : X^{(r)} \longrightarrow V$ becomes a smooth fibration. This fibration is called the r-jet extension of the fibration

$$p : X \longrightarrow V.$$

The section $J_f^r : V \longrightarrow X^{(r)}, \; v \longmapsto J_f^r(v)$

<u>Remark</u>: The r-tangency of two sections implies their s-tangency for all $0 \leq s < r$, and therefore the chain of projections

$$\cdots \longrightarrow X^{(r)} \longrightarrow \cdots \longrightarrow X^{(2)} \longrightarrow X^{(1)} \longrightarrow X^{(0)} = X.$$

<u>Notation</u>: when $X = V \times W \longrightarrow V$ the trivial fibration over V, denote

$$X^{(r)} := J^r(V, W).$$

<u>example</u>: $J^1(V, \mathbb{R}) = T^*(V) \times \mathbb{R}$ &

$$J^1(\mathbb{R}, W) = \mathbb{R} \times T(W).$$

§3 Holonomic sections

The point view of jets as sections of the jet bundle suggests to consider arbitrary sections of the jet bundle. But note not all sections of the jet bundle are jets of sections $V \longrightarrow X$.

<u>ex</u> (non-example) The section

$$s : \mathbb{R} \longrightarrow J^1(\mathbb{R}, \mathbb{R})$$

$$s(x) = (x, x^2, c) \quad \text{with } c \in \mathbb{R}.$$

This is perfectly a section of $J^1(\mathbb{R}, \mathbb{R})$ over $\mathbb{R}$ since with $p : J^1(\mathbb{R}, \mathbb{R}) \longrightarrow \mathbb{R}$ we have $p \circ s = id_{\mathbb{R}}$.

But s is not a jet of section $\mathbb{R} \to \mathbb{R}$ as there is no real-valued function $f : \mathbb{R} \longrightarrow \mathbb{R}$ such that $f(x) = x^2$ and $f'(x) = c$.

Therefore we need to distinguish between sections that derive from functions and those that do not.

We say that a section $F : V \longrightarrow J^r(V, W)$ is holonomic if there is a section $f : V \longrightarrow V \times W$ such that $J^r_f = F$.

More generally, a section $F : V \longrightarrow X^{(r)}$ is holonomic if $J^r_{bs \circ F} = F$ where

$b_s F$ is the section $b_s F := P_0^r \circ F : V \longrightarrow X$.

Denote by $\mathrm{Hol}\, X^{(r)}$, set of all holonomic sections of $X^{(r)}$.

<u>Remark:</u> There is a 1-1 correspondence

$$J^r : \begin{array}{ccc} \mathrm{Sec}\, X & \longrightarrow & \mathrm{Sec}\, X^{(r)} \\ f & \longmapsto & J^r_f \end{array}.$$

Note $\boxed{J^r(\mathrm{Sec}\, X) \cong \mathrm{Hol}\, X^{(r)} \subset \mathrm{Sec}\, X^{(r)}}$

A homotopy of holonomic sections of $X^{(r)}$ is called a <u>holonomic homotopy</u>.

Functoriality relation

Let $h : X \to Y$ be a fibered map between two fibrations $X \to V$ and $Y \to V$.

Then h induces a map on sections of X

i.e.

$$h_* : \operatorname{Sec} X \longrightarrow \operatorname{Sec} Y$$
$$f \longmapsto h_* f = h \circ f$$

This map descends naturally to

$$h_* : X^{(r)} \longrightarrow Y^{(r)}.$$
$$[(f, v)] \longmapsto [(h \circ f, v)]$$

Remark: The induced map $\operatorname{Sec} X^{(r)} \to \operatorname{Sec} Y^{(r)}$ maps $\operatorname{Hol} X^{(r)}$ to $\operatorname{Hol} Y^{(r)}$.

§ 4 Differential Relations

A differential relation is any collection of ordinary or partial differential equations or inequalities.

More precisely, a differential relation of order r imposed on sections $f : V \to X$ of a fibration $X \to V$ is a subset $\mathcal{R}$ of the jet bundle $X^{(r)}$.

For instance, any system of ordinary $(n=1)$ or partial $(n>1)$ differential equations

$$\Psi(x, f, D^\alpha f) = 0$$

imposed on unknown functions

$$y_j = f_j(x_1, \dots, x_n), j = 1, \dots, q,$$

and their derivatives

$$D^\alpha f_j = \frac{\partial^{|\alpha|} f_j}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}}, \quad \alpha = (\alpha_1, \dots, \alpha_n)$$

$$|\alpha| = \alpha_1 + \dots + \alpha_n \leq r$$

is a differential relation $\mathcal{R}$ in the jet bundle $J^r(\mathbb{R}^n, \mathbb{R}^q)$, defined by the system of "algebraic" equations

$$\Psi(x, y, z_\alpha) = 0$$

where $x = (x_1, \dots, x_n)$

$y = (y_1, \dots, y_q)$ &

are

$z_\alpha = (z_{1,\alpha}, \dots, z_{q,\alpha})$

coordinates in $J^r(\mathbb{R}^n, \mathbb{R}^q)$.

ex 1: Consider the equation $f'(x) = f(x) + x$

on a function $f : \mathbb{R} \to \mathbb{R}$.

This defines a differential relation

$$\mathcal{R} = \{(x, y, z) \in J^1(\mathbb{R}, \mathbb{R}) \mid z = y + x\}$$

$$\subset J^1(\mathbb{R}, \mathbb{R}).$$

ex 2: $y' = y^2$ i.e.

$f'(x) = f(x)^2$ with $f : \mathbb{R} \to \mathbb{R}$

$\{$ corresponds

$$\mathcal{R} = \{(x, y, z) \in J^1(\mathbb{R}, \mathbb{R}) \mid z = y^2\}$$

- $y' \geq y^2$

$$\mathcal{R} = \{(x, y, z) \in J^1(\mathbb{R}, \mathbb{R}) \mid z \geq y^2\}$$

A differential relation $\mathcal{R} \subset X^{(r)}$ is called open or closed if it is open or closed as a subset of the jet space $X^{(r)}$.

Given a differential relation $\mathcal{R} \subset X^{(r)}$, a formal solution of $\mathcal{R}$ is a section $F : V \longrightarrow X^{(r)}$ whose image is in $\mathcal{R}$. i.e $F(V) \subset \mathcal{R}$.

A genuine solution of a differential relation $\mathcal{R} \subset X^{(r)}$ is a holonomic section $f : V \to X^{(r)}$ whose image is in $\mathcal{R}$

i.e $J_f^r(V) \subset \mathcal{R}$.

We will call the holonomic sections $V \to \mathcal{R} \subset X^{(r)}$ r-extended solutions, or just r-solutions when the distinction between the solutions of $\mathcal{R}$ as sections of X or $X^{(r)}$ is not clear from the context.

<u>Résumé</u>:

① $\mathrm{Hol}\, X^{(r)} = \{ F \in \mathrm{Sec}\, X^{(r)} \mid F \text{ is holonomic} \} \xhookrightarrow{1-1} \mathrm{Sec}\, X$

② $\{\text{genuine solutions}\} \xhookrightarrow{1-1} \mathrm{Sec}\,\mathcal{R} \cap \mathrm{Hol}\, X^{(r)}$

$$\mathrm{Sec}\,\mathcal{R} \cap \mathrm{Hol}\, X^{(r)} =: \mathrm{Hol}\,\mathcal{R}$$

• Extension problem

Let $\mathcal{R} \subset X^{(r)}$ be a differential relation and f its solution over $\mathcal{O}_p B$ (i.e. an open neighborhood of B). Given $A \supset B$, we want to extend f as a solution of $\mathcal{R}$ over $\mathcal{O}_p A$.

A formal solution of the extension problem is a section $F : \mathcal{O}_p A \to \mathcal{R}$ which coincides with J^r_f over $\mathcal{O}_p B$.

The important case for us is when $A \simeq D^k$ and $B = \partial D^k \simeq S^{k-1}$.

If $A = V$ and $B = \phi$, we recover the first case ("for global solutions").

§5 h-principle

- h-principle: We say that a differential relation $\mathcal{R}$ satisfies the h-principle or that the h-principle holds for $\mathcal{R}$, if every formal solution of $\mathcal{R}$ is homotopic in $\mathrm{Sec}\,\mathcal{R}$ to a genuine solution of $\mathcal{R}$.

Roughly speaking, h-principle holds for $\mathcal{R}$ if every formal solution can be perturbed in $\operatorname{Sec}\mathcal{R}$ to a genuine solution of $\mathcal{R}$.

- 1-parametric h-principle:

We say $\mathcal{R}$ satisfies the 1-parametric h-principle if every family of formal solutions $\{F_t\}_{t\in I}$ of $\mathcal{R}$ which joins two genuine solutions F_0 and F_1 can be deformed inside $\operatorname{Sec}\mathcal{R}$, keeping F_0 and F_1 fixed into a family $\{\tilde{F}_t\}_{t\in I}$ of genuine solutions of $\mathcal{R}$.

Different flavors of the h-principle

Let $\mathcal{R} \subset X^{(r)}$ be a differential relation.

(i) Parametric h-principle

We say the multi parametric h-principle holds for $\mathcal{R}$ if for every relative spheroid

$$\varphi_0 : (D^k, S^{k-1}) \longrightarrow (\mathrm{Sec}\,\mathcal{R}, \mathrm{Hol}\,\mathcal{R})$$

$k = 0, 1, 2, \ldots$

there exists a homotopy

$$\varphi_t : (D^k, S^{k-1}) \longrightarrow (\mathrm{Sec}\,\mathcal{R}, \mathrm{Hol}\,\mathcal{R}),$$

$t \in [0,1]$,

fixed on S^{k-1} such that $\varphi_1(D^k) \subset \mathrm{Hol}\,\mathcal{R}$.

(ii) C^0-dense h-principle

We say that the C^0-dense h-principle holds for $\mathcal{R}$ if the (usual) h-principle holds for $\mathcal{R}$ and if for every formal solution $F_0 : V \to \mathcal{R}$ and an arbitrarily small neighborhood $\mathcal{U} \subset X$ of the underlying section $f_0 = bs\, F_0$ the homotopy F_t, $t \in [0,1]$, in $\mathcal{R}$ which brings F_0 to a genuine solution F_1 can be chosen in such a way that

$$bs\, F_t(V) \subset \mathcal{U}, \quad t \in [0,1].$$