Talk 4 | Open Diff V- invariant Differential relations

- Natural fibrations
- Diff V- invariant relations
- Local and global h-principle for open Diff V-inv. relations

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1 | Natural fibrations

Definition: ($\mathrm{Diff}_V X$)

Given a fibration $p: X \longrightarrow V$, we say $h_X \in \mathrm{Diff}\, X$ is fiberpreserving or $h_X \in \mathrm{Diff}_V X$ if $\exists\ h_V \in \mathrm{Diff}\, V$ s.t.

$$\begin{tikzcd} X \arrow[r, "h_X"] \arrow[d, "p"'] \arrow[dr, phantom, "\circlearrowleft"] & X \arrow[d, "p"] \\ V \arrow[r, "h_V"] & V \end{tikzcd}$$

Remark: $\mathrm{Diff}_V X \leq \mathrm{Diff}\, X$ (as groups)

Q: We want to know when the fibration $\mathrm{Diff}_V X \to \mathrm{Diff}\, V$, $h_X \mapsto h_V$ has a canonical section.

This would mean that the action of $\mathrm{Diff}\, V$ on V can be canonically lifted to an action of $\mathrm{Diff}_V X$ on X.

[This lifting should also hold for the local diffeomorphism (pseudo-)groups.]

Definition: (natural fibration)

We call $p: X \to V$ natural if $\forall$ diffeomorphisms $h: U \to W$ between open $U, W \subseteq V$: $\exists$ a fiberpreserving $h_*: p^{-1}(U) \to p^{-1}(W)$ s.t. $\forall$ given

$$U \xrightarrow{h} U' \xrightarrow{h'} U''$$

we have

- $(h' \circ h)_* = h'_* \circ h_*$
- $\forall v \in V$ the germ of h_* along $p^{-1}(v)$ depends only on the germ of h at v.

Recall: Two functions are said to be in the same germ if there exist a nbhd where they agree.

Remark: The second property allows us to extend our previous lifting to local diffeos. We will call lifts of local diffeos fiberwise isomorphisms.

examples of natural fiber bundles:

1) trivial bundle $V \times W \to V$ is natural

$\hookrightarrow \quad h_* = h \times id_W$

2) tangent bundle $TV \longrightarrow V$ is natural

↳ $h_* = dh : TU \to TU'$ for $h : U \to U'$

3) cotangent bundle $T^*V \longrightarrow V$ is natural

↳ $h_* = (dh^*)^{-1} : T^*U \longrightarrow T^*U'$ for $h : U \longrightarrow U'$

4) Jet bundles

$X \longrightarrow V$ is natural $\Rightarrow X^{(r)} \longrightarrow V$ is natural

↳ $h_*^{(r)} : (X|_U)^{(r)} \longrightarrow (X|_{U'})^{(r)}$ is given by

$$h_*^{(r)}(s) = J^r_{h_* \circ \bar{s} \circ h^{-1}}(h(v))$$

$h_* \circ \bar{s} \circ h^{-1} : W \to p^{-1}(W)$

with $s \in X^{(r)}$, $v = p^r(s) \in V$, $\bar{s} : \mathcal{O}_p v \longrightarrow X$ local section near v which represents the r-jet s, i.e. $J^r_{\bar{s}}(v) = s$

Proposition: (universal natural principle bundle $P^D(V)$)

For any n-dim. manifold V there is a canonical natural principal $D_0(n)$-bundle $P^D(V)$ such that all natural bundles over V are associated with $P^D(V)$ for some representation of $D_0(n)$ in the group of automorphisms of their standard fibers.

group of germs of diffeos $(\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$ fixing 0

Moreover, any local diffeo $h: V \longrightarrow V'$ lifts to a fiberwise isomorphism

$$P^D(h) := h_* : P^D(V) \longrightarrow P^D(V')$$

such that $P^D(h' \circ h) = P^D(h') \circ P^D(h)$ for any two local diffeos

$$V \xrightarrow{h} V' \xrightarrow{h'} V''$$

Remarks: • P^D is a functor from the category of n-dim. manifolds and local diffeos to natural $D_0(n)$-principal bundles over n-dim. bases and fiberwise isomorphisms.

• in the same sense we can then redefine the notion of natural fibration in two equivalent ways:

- as any fibration over an n-dim. base V with standard fiber W associated with the $D_0(n)$-principal fibration $P^D(V) \longrightarrow V$ wrt. an action of $D_0(n)$ on W

- as a covariant functor $F: \mathcal{M}_n \longrightarrow \mathcal{FM}_n$ from n-mfds & local diffeos $\rightarrow$ smooth fibrations over n-mfds & fiberwise isomorphisms

$$\left(\begin{array}{lcl} \text{i.e. } V, & & p_V: F(V) \longrightarrow V, \\ h: V_1 \to V_2 & \mapsto & F(h): F(V_1) \longrightarrow F(V_2) \end{array} \right)$$

s.t. if $\overset{\text{open}}{U} \overset{i}{\hookrightarrow} V$ inclusion then $p_U: F(U) \to U$

satisfies $p_U = p_V|_{p_V^{-1}(U)} : p_V^{-1}(U) \longrightarrow U$ and $F(i)$ is the inclusion of total spaces $\rightsquigarrow$ $V \xrightarrow{\mathrm{Id}} V$ lifts to $F(V) \xrightarrow{\mathrm{Id}} F(V)$.

proof:

Recall from topology 3:

- For all fiber bundles $p: E \to B$ over connected bases the so called standard fiber is a fixed space F which can be used wlog. for all local trivialisations

$$\begin{tikzcd} E|_U \arrow[rr, "\Phi"] \arrow[dr, "p"'] & & U \times F \arrow[dl, "\mathrm{proj}_1"] \\ & U & \end{tikzcd}$$

this would be a different space for every single $x \in B$ but since all the fibers are homeomorphic we just fix one F

- A principal G-bundle is a fiber bundle $p: P \longrightarrow B$ together with a (smooth) right action $P \times G \longrightarrow P$ such that G preserves the fibers of P and G is a (Lie) group which acting freely and transitively on the fibers.
  ($\forall x,y\ \exists g \in G: g \cdot x = y$)
  (no nontrivial element fixes any point)
  - ↳ in particular: each fiber is (diffeo)morphic to G
  - ↳ the orbits of the G-action are precisely the fibers and $P/G \cong B$

- Every G-bundle $E \to B$ with std. fiber F can be "recovered" (up to G-bundle iso) from a principal bundle $P \to B$. The bundle $p: E \to B$ is then isomorphic to the so called associated bundle $P \times_G F \longrightarrow B$ given by

comes from a left action $G \times F \to F$

$$P \times_G F := (P \times F)/G \quad , \quad g \cdot (p, f) := (pg, g^{-1} f)$$

together with the projection $P \times_G F \to B$ inherited by $P \to B$.

Now we actually start the proof.

For an n-mfd. V consider the space $P^D(V)$ of pairs (v, φ) where $v \in V$ and φ is the germ of a diffeomorphism $(Op_{\mathbb{R}^n} 0, 0) \longrightarrow (Op_V v, v)$.

The group $D_0(n)$ acts freely on $P^D(V)$ by $\delta \cdot (v, \varphi) = (v, \varphi \circ \delta^{-1})$ and the orbits of this action are the fibers of the proj. $(v, \varphi) \longmapsto v$. $\Rightarrow$ $P^D(V) \longrightarrow V$ is a principal $D_0(n)$-bundle ✓

(or alternatively, the fiber over v is the set of germs of diffeos from $\mathbb{R}^n$ to V mapping 0 to v mod $D_0(n)$.

so basically all coordinate charts at v in the fiber)

Any local diffeo $h: V \to V'$ lifts to a fiberwise isomorphism $P^D(h): P^D(V) \longrightarrow P^D(V')$ by

$$P^D(h)(v, \varphi) = (h(v), h \circ \varphi) \quad [\text{observe that: } P^D(h' \circ h) = P^D(h') \circ P^D(h)]$$

Which makes $P^D(V) \to V$ a natural principal fiber bundle with std. fiber $D_0(n)$. ✓

<u>Discussion on the topology of $P^D(V)$:</u> (Whitney topology)

For a domain $U \subseteq \mathbb{R}^n$ identify $P^D(U)$ with $U \times D_0(n)$ via parallel transport. We endow $D_0(n)$ with the C^∞-topology and use this identification to define the product top. on $P^D(U)$. Choose a coord. atlas $\{(U_i, \varphi_i)\}$ for V, where $\varphi_i: U_i \to G_i := \varphi_i(U_i) \subseteq \mathbb{R}^n$ and let $h_{ij} := \varphi_j \circ \varphi_i^{-1}: G_{ij} \longrightarrow G_{ji} = \varphi_j(U_i \cap U_j)$ be the transition maps.

$\rightsquigarrow$ construct $P^D(V)$ by gluing $P^D(G_i)$ and $P^D(G_j)$ with $P^D(h_{ij})$ gluing map.

Ultimately we will show that any natural fiber bundle $p: X \to V$ is associated with the principal bundle $P^D(V) \to V$.

Fix $v_0 \in V$ and denote $W := p^{-1}(v_0) \subseteq X$. By fixing any diffeo $f_0 : (Op_{\mathbb{R}^n} 0, 0) \to (Op_V v_0, v_0)$ we define an action δ_* of $D_0(n)$ on W by $\delta_* w := (f_0 \circ \delta \circ f_0^{-1})_* w \quad , w \in W.$

↑ lift to a fiberpreserving iso

Then define $\Phi : P^D(V) \times W \to X$ as follows. Given $(v, \varphi) \in P^D(V)$ and $w \in W$ consider the germ of $\bar{\varphi} := \varphi \circ f_0^{-1} : (Op_V v_0, v_0) \to (Op_{\mathbb{R}^n} 0, 0) \to (Op_V v, v)$

and define $\Phi((v,\varphi), w) = (\bar{\varphi})_* w \in p^{-1}(v) \subseteq X$.

Observe: $\Phi((v,\varphi),w) = \Phi((v',\varphi'),w') \iff v = v' , \exists \delta \in D_0(n) : \varphi = \varphi' \circ \delta$ and $\delta_* w = w'$.

(transitivity)

Therefore Φ descends to an isomorphism

$$P^D(V) \times_{D_0(n)} W \xrightarrow{\cong} X. \qquad \square$$

Finiteness theorem:

Denote by $L^r(n)$ the group of r-jets at 0 of diffeomorphisms in $D_0(n)$. $L^1(n) = GL(n)$, $L^0(n) = \{1\}$.

Similarly to the construction of the $D_0(n)$-principal bundle we can construct a principal $L^r(n)$-bundle $P^r(V) \to V$. Where $P^r(V)$ consists of pairs (v, φ) $v \in V$ and φ is the r-jet of a diffeo $(Op_{\mathbb{R}^n} 0, 0) \to (Op_V v, v)$ at $0 \in \mathbb{R}^n$.

In particular, $P^1(V)$ is the principal $GL(n)$-bundle of tangent n-frames associated to the tangent bundle TV.
$\forall r \geq 0$ $P^r(V) \to V$ is associated with $P^D(V) \to V$

wrt. the canonical projection $D_0(n) \to L^r(n)$, thus natural.

According to the finiteness theorem (Palais and Terng) we even get the converse statement:

The structure group of a natural fiber bundle $X \to V$ always coincides with $L^r(n)$ for some $r = r(X)$, i.e. for any $v \in V$ the actions of two germs

$$f_1, f_2 : \mathcal{O}_p v \longrightarrow \mathcal{O}_p v' \quad , \quad v' = f_1(v) = f_2(v)$$

coincide if $J^r_{f_1}(v) = J^r_{f_2}(v)$.

Hence, all natural bundles over V with a fiber W are in bijective correspondence with smooth left actions of the jet groups $L^r(n)$ on W. $(r = 0, 1, \dots)$

2 | Diff V-invariant differential relations

Recall that if $X \to V$ is natural so is $X^{(r)} \to V$.

A differential relation $\mathcal{R} \subseteq X^{(r)}$ is called Diff V-invariant if the action $s \mapsto h_*^{(r)}(s)$ for any local diffeo $h: U \to U'$, $U, U' \subseteq V$ leaves $\mathcal{R}$ invariant.

"prolonged action"

$j^r(g) \cdot \mathcal{R} \subseteq \mathcal{R}$

(*)

Remark: A Diff V-invariant $\mathcal{R} \subseteq X^{(r)}$ is a natural subbundle of the natural fiber bundle $X^{(r)}$.

Or alternatively, $\mathcal{R}$ is Diff V-invariant if it can be defined in a V-coordinate $^{(*)}$ free form.

$^{(*)}$ since it looks the same in all coord. systems

The action $s \mapsto h_*^{(r)}(s)$ preserves the set of holonomic sections:

$$h_*^{(r)}\left(J_f^r(v)\right) = J^r_{h_* \circ f \circ h^{-1}}\left(h(v)\right)$$

for $f \in \operatorname{Sec} X$, $h \in \operatorname{Diff} V$.

$\hookrightarrow$ Diff V acts on $\operatorname{Sol} \mathcal{R}$ of a Diff V-inv. $\mathcal{R}$.

Recall from the previous proof that any natural bundle $X \to V$ over a n-mfd. V is determined by an action of $D_0(n)$ on W, call it α. This α lifts to an action α^r of $D_0(n)$ on W^r (standard fiber of $X^{(r)}$), which is the space of r-germs at 0 of maps $\mathbb{R}^n \to W$.

the same as an r-jet "germ up to order r"

A Diff V-invariant relation $\mathcal{R} \subseteq X^{(r)}$ is determined by a subset $R \subseteq W^r$ which is invariant wrt the action on α^r.

examples:

- $\mathcal{R}_{imm}$ and $\mathcal{R}_{subm}$ are Diff V-invariant.
- For any $A \subseteq Gr_n W$, $\mathcal{R}_A$ which defines A-directed maps $V \to W$.

<u>Immersion relation</u>: $R_{imm} \subseteq J^1(V,W)$ over $(v,w) \in V \times W$ consists of monomorphisms $T_vV \to T_wW$, or equivalently
(fib. inj. bund. hom.)
of non-vertical n-planes $P_x \subseteq T_vV \times T_wW$ such that

$$\dim(P_x \cap (T_vV \times 0)) = 0.$$

<u>Submersion relation</u>: $R_{subm} \subseteq J^1(V,W)$ consists of epimorphisms (fiberwise surj. bundle hom.), or non-vertical n-planes s.t.

$$\dim(P_x \cap (T_vV \times 0)) = n-q$$

3 | Local and global h-principle for open Diff V-inv. relations

<u>Theorem</u> (Local h-principle for open Diff V-invariant relations)

Let $X \to V$ be a natural fiber bundle. Then any open Diff V-invariant differential relation $R \overset{\text{open}}{\subseteq} X^{(r)}$ satisfies all forms of the local h-principle near any polyhedron $A \subseteq V$ of positive codimension.

<u>proof 1</u>: at the end

Theorem (Local h-principle implies global for open mfds)

Let V be an open mfd and $X \to V$ a natural fibration. Let $\mathcal{R} \subseteq X^{(r)}$ be Diff V-invariant. Then the parametric local h-principle for $\mathcal{R}$ implies the parametric global one.

proof 2: at the end.

Remark: This will not work for the local C^0-dense h-principle.

From these two theorems we can now harvest:

Gromov h-principle for open Diff V-inv. relations over open mfds

Let V be an open mfd and $X \longrightarrow V$ a natural fiber bundle. Then any open Diff V-invariant differential relation $\mathcal{R} \subseteq X^{(r)}$ satisfies the parametric h-principle. In particular, immersions, submersions, k-mersions and immersions directed by an open set $A \subseteq Gr_n(W)$ satisfy this as long as V is open.

Two more applications:

- maps transverse to a distribution

Let $\tau \subseteq TW$ be a tangent distribution and $\mathcal{R} \subseteq J^1(V,W)$ the diff. rel. consisting of homomorphisms $TV \to TW$ transverse to τ. The relation $\mathcal{R}$ is Diff V-inv. and thus the h-principle holds for $\mathcal{R}$ (i.e. $V \to W$ transverse to τ) if V is open.

- relative version of Gromov

Let $\mathcal{R} \subseteq X^{(r)}$ be an open Diff V-inv. differential relation over an open manifold V. Let $B \subseteq V$ be a closed subset such that each connected component of $V \setminus B$ has an exit to ∞. Then the relative parametric h-principle holds for $\mathcal{R}$ and the pair (V, B).

See chapter 4 open mfds:
V open if $\nexists$ closed connected comp.
↳ $\forall$ path-con. + $\partial V \neq \emptyset \Rightarrow V$ open
$v \in V$ "has exit to ∞" if $\exists$ path which connects v to ∞. i.e. it "escapes every compact set"

proof 1: We will go over the non-parametric case i.e. we will prove that

$$\pi_0(\mathrm{Sec}_{\mathcal{O}p A}\mathcal{R}, \mathrm{Hol}_{\mathcal{O}p A}\mathcal{R}) = 0$$

We need to show that, given a section $F \in \mathrm{Sec}_{\mathcal{O}p A}\mathcal{R}$, $\exists$ a section $G \in \mathrm{Hol}_{\mathcal{O}p A}\mathcal{R}$ which is homotopic in $\mathrm{Sec}_{\mathcal{O}p A}\mathcal{R}$ to F. According to the holonomic approx. theorem(s) $\exists$ an arbitrarily C^0-small diffeotopy $h^\tau : V \to V$, $\tau \in [0,1]$ and a section $\widetilde{F} \in \mathrm{Hol}_{\mathcal{O}p\, h(A)}\mathcal{R}$ s.t. $\widetilde{F}$ is C^0-close to F over $\mathcal{O}p\, h(A)$. In particular, we may assume that the linear homotopy $\widetilde{F}^t$ between $F|_{Op\,h^1(A)} = \widetilde{F}^0$ and $\widetilde{F}^1 = \widetilde{F}$ lies in $\mathcal{R}$. The desired section G can then be defined by the formula $G = H^1(\widetilde{F})$

where $H^\tau = ((h^\tau)^r_*)^{-1} = ((h^\tau)^{-1})^r_* : X^{(r)} \longrightarrow X^{(r)}$ is the induced action of the "straightening" $(h^\tau)^{-1}$ diffeotopy on the natural $X^{(r)} \to V$.

The required homotopy in $\mathcal{R}$ connecting F and G over $Op A$ consists of:
- $H^\tau(F)$ $\tau \in [0,1]$
- $H^1(\widetilde{F}^t)$ $t \in [0,1]$

Note that we get the local C^0-dense h-principle for $\mathcal{R}$ near A in the following way:

The hol. approx. gives us a section $\widetilde{F}$ over $Op\,h(A)$ which is C^0-close in $X^{(r)}$ to F over $Op\,h(A)$ it does not imply the same for G. However for the C^0-approximation it "survives" the straightening.

Now apply the other theorems for parametric, relative and other cases. □

proof 2: Again, only in the non-parametric case since the general case differs only in notation.

Prove that: $\pi_0(\mathrm{Sec}\,\mathcal{R}, \mathrm{Hol}\,\mathcal{R}) = 0$. Given a section $F \in \mathrm{Sec}\,\mathcal{R}$ $\exists$ $G \in \mathrm{Hol}\,\mathcal{R}$ which is homotopic in $\mathrm{Sec}\,\mathcal{R}$ to F. Let $K \subseteq V$ be a core (see open mfd talk) of the manifold V. The local h-principle near K implies the existence of $G_K \in \mathrm{Hol}_{Op K}\mathcal{R}$ which is homotopic in $\mathrm{Sec}_{Op K}\mathcal{R}$ to $F|_{Op K}$.

Let $h^{\tau \in [0,1]} : V \to V$ be an isotopy compressing V into a nbhd $\mathcal{U}$ of K s.t. G_K is defined over $\mathcal{U}$. The desired section $G \in \mathrm{Hol}\,\mathcal{R}$ can now be defined by the formula $G = H^1(G_K)$

where
$$H^\tau = ((h^\tau)^r_*)^{-1} = ((h^\tau)^{-1})^r_* : X^{(r)} \to X^{(r)}$$

"decompressing diffeotopy" on $X^{(r)} \to V$.

Two parts again: $H^\tau(F)$ and then $H^1(G_K^t)$ where G_K^t is the homotopy which connects the local sections $F_{Op K}$ and G_K in $\mathcal{R}$. □