# The Smale–Hirsch Immersion Theorem and other Applications to Closed Manifolds

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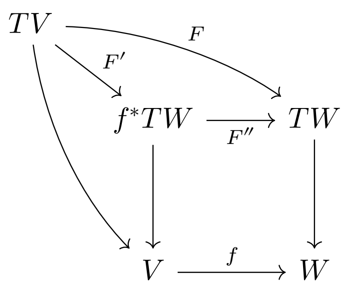
Sommersemester 2025

These are the notes for my talk in this semester's "The $h$-Principle" seminar. The main content of the talk were two proofs, one of the Smale–Hirsch $h$-principle and one of a generalization of it. In presenting this, I mostly follow the material in chapter 9 of the book by Cieliebak, Eliashberg and Mishachev [1]. I reworked the notes to fix some mistakes I made in the talk and hopefully make the contents more accessible.

## 1 The Smale–Hirsch immersion theorem

**Theorem 1** (Smale–Hirsch $h$-principle for immersions)**.** *The relative parametric $C^0$-dense $h$-principle holds for immersions of an $n$-dimensional manifold $V$ into a manifold $W$ of dimension $q > n$.*

*Proof.* Let us only consider the nonparametric case. The let $F\colon \mathrm{T}V \to \mathrm{T}W$ be a formal solution of the differential relation defining immersions $\mathcal{R}_{\mathrm{imm}} \subset J^1(V, W)$. Set $f = \mathrm{bs}\, F$. The pullback bundle $f^*\mathrm{T}W$ is a pullback in the categorical sense, so we get the commuting diagram




where the unnamed maps to $V$ and $W$ are the bundle projections. As $F$ is a formal solution, it acts injectively on tangent spaces, hence $F'$ does too. Thus we can use $F'$ to identify $\mathrm{T}V$ with a subbundle of $f^*\mathrm{T}W$, i.e. $F'(\mathrm{T}V)$. Now let $N$ denote (total space of) the subbundle of $f^*\mathrm{T}W$ that is the orthogonal complement of $F'(\mathrm{T}V)$ and $\pi\colon N \to V$ its bundle projection.

In the presentation there was some confusion about the following construction, so I will write about it in a bit more detail here. It is a standard construction not specfic to the bundle at hand. For a reference on this see e.g. Proposition 15.6.7 (p. 376) in [2]. Additionally, a more concrete explanation of what this means for us will follow after.

I will denote by $\mathrm{T}f$ the tangent map of a map $f$ and by $\mathrm{T}_p f$ the tangent map at a point $p$. For any $n \in N$ we have the short exact sequence of vector spaces

$$0 \longrightarrow N_{\pi(n)} \xrightarrow{\alpha} \mathrm{T}_n N \xrightarrow{\mathrm{T}_n \pi} \mathrm{T}_{\pi(n)} V \longrightarrow 0.$$

Where $N_{\pi(n)} := \pi^{-1}(\pi(n))$ denotes the fiber of $N$ over $\pi(n)$ and $\alpha$ maps $a \in N_{\pi(n)}$ to the derivative of the curve $t \mapsto n + ta$ at $t = 0$. We may split this sequence by choosing an inner product on $\mathrm{T}_n N$ and identifying $T_{\pi(n)}V$ with the orthogonal complement of $\operatorname{im}\alpha$. But we're not just interested in $\mathrm{T}_n N$, we want to decompose the whole bundle $\mathrm{T}N$. To approach this we modify the above into a short exact sequence of vector bundles over $N$

$$0 \longrightarrow \pi^* N \longrightarrow \mathrm{T}N \longrightarrow \pi^*\mathrm{T}V \longrightarrow 0$$

where we had to pull back $N$ and $\mathrm{T}V$ in order to turn them into a bundles over $N$[1]. By choosing any bundle metric on $\mathrm{T}N$ we then get the splitting

$$\mathrm{T}N \cong \pi^* N \oplus \pi^*\mathrm{T}V.$$

Generally, $\pi^* N$ and $\pi^*\mathrm{T}V$ (seen as subbundles of $\mathrm{T}N$) are respectivey referred to as the vertical bundle and horizontal bundle.

To see what this splitting means, consider what data elements of $N$ and $\mathrm{T}N$ contain. Every point $n \in N$ consists of a point $\pi(n) \in V$ and a vector that is normal to $F'(\mathrm{T}_{\pi(n)}V)$. The tangent space $\mathrm{T}_n N$ should then contain the data of the tangent space $\mathrm{T}_{\pi(n)}V$ and some kind of tangent data of the normal vectors, which as the tangent space of a vector space is identified with the space itself. This is precisely what the above split gives us. So for any $x \in \mathrm{T}N$ let $n$ denote its usual projection into the base space $N$ and let $a$ and $b$ respectively denote the corresponding points in $N_n$ and $T_{\pi(n)}V$ given by the splitting.

With this notation, we can explicitly define a lift of $F$ to $\mathrm{T}N$ as

$$\begin{aligned} \tilde{F} \colon \mathrm{T}N &\to \mathrm{T}W \\ x &\mapsto F''(a) + F(b), \end{aligned}$$

where $F''$ is necessary as $N \subset f^*\mathrm{T}W$. This extends $F$ in the sense that we can consider $V$ as a subset $V \subset N$ by lifting it into $N$ as the zero section of $\pi$ (and $\mathrm{T}V \subset \mathrm{T}N$ is e.g.[2] given by the tangent map of the inclusion $V \subset N$). With these identifications we can see that $\tilde{F}$ extends $F$ because $\tilde{F}|_{\mathrm{T}V} = F$.

Now, because $\tilde{F}$ is still formal immersion, we can apply the $C^0$-dense local $h$-principle 8.3.1 near $V \subset N$. The restriction to $V$ yields the sought-after deformation of $F$ to an immersion $V \to W$ proving the $h$-principle for immersions.

It remains to be shown that the relative $h$-principle holds. Assume $F$ is holonomic on $\mathcal{O}\mathrm{p}\, A$ for a closed $A \subset V$. The book does not offer a specific construction to make $\tilde{F}$ holonomic on $\mathcal{O}\mathrm{p}_N\, A$ (open in $N$), but I believe the following is the easiest way.

For all $x \in \mathrm{T}N$ decompose $x$ as we did to define $\tilde{F}$ abvoe. We redefine $\tilde{F}(x)$ as

$$\tilde{F}(x) = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \exp\Big(\varphi(n) \cdot F''(n) + t \cdot (F''(a) + F(b))\Big)$$

where exp is the Riemannian exponential map on $W$. When $n = 0$, this reduces to the previous definition of $\tilde{F}$. Thus it will a formal immersion at least in a neighborhood of $V \subset N$ which is good enough. If $\pi(x) \in \mathcal{O}\mathrm{p}\, A$, then

$$\tilde{F}(x) = \mathrm{T}(\exp \circ F'')(x)$$

[1]To formally see how to find the map "$\mathrm{T}\pi$": $\mathrm{T}N \to \pi^*\mathrm{T}V$ just consider a diagram as the one above but switch the $W$s for $V$s, $V$s for $N$s and $f$ for $\pi$.

[2]One may also define the inlusion of $\mathrm{T}V$ by referring to our hard-earned splitting of $\mathrm{T}N$. In fact, both notions coincide as the splitting is canonical on the zero section $V \subset N$ (see Remark 15.6.8 [2]).

holds[3]. Hence $\tilde{F}$ is holonomic on $\mathcal{O}\mathrm{p}_N\, A$ and we can now apply the relative $C^0$-dense $h$-principle.

□

Note that this proof does not work for submersions because the restriction of a submersion is not, in general, a submersion. In fact, the $h$-principle is false for submersions of closed manifolds. However, one can prove a generalization of the previous theorem.

**Theorem 2.** *Let $\xi$ be a subbundle of* $\mathrm{T}W$*. If* $\dim V < \operatorname{codim} \xi$*, then all forms of the $h$-principle hold for immersions $V \to W$ transverse to $\xi$.*

## 2 Sections transverse to a distribution

To prove the following theorem, we need a modification of the local $h$-principle used above.

**Theorem 3** (Special local $h$-principle)**.** *Let $X \to V \times R$ be a natural fibration and $\mathcal{R} \subset X^{(r)}$ an open differential relation which is invariant with respect to diffeomorphisms of the form*

$$(x,t) \to (x, h(x,t)), \quad x \in V, t \in \mathbb{R}.$$

*Then $\mathcal{R}$ satisfies all forms of the local $h$-principle near $V \times 0$ and the global parametric $h$-principle over $V \times \mathbb{R}$.*

The proof follows from the Holonomic Approximation Theorem 3.1.1 according to the same scheme as the proof of Theorem 8.3.1, with the additional observation that the perturbation $h$ implied by Theorem 3.1.1 has the special form required here.

Given a fibration $X \to V$ we say that a section $f \colon V \to X$ is *transverse to a tangent distribution/subbundle $\tau \subset \mathrm{T}X$* if the composition

$$\mathrm{T}V \xrightarrow{\mathrm{T}f} \mathrm{T}X \longrightarrow \mathrm{T}X/\tau$$

is fiberwise injective when $\operatorname{rank}\tau + \dim V \leq \dim X$ and surjective when $\operatorname{rank}\tau + \dim V \geq \dim X$. Note that this may differ from what you expect transverseness to mean (i.e. requiring $\operatorname{rank}\tau + \dim V = \dim X$).

**Theorem 4** ($h$-principle for sections transverse to a distribution)**.** *Let $X \to V$ be a natural[4] fibration and $\tau$ a subbundle of the tangent bundle* $\mathrm{T}X$*. If*

$$\operatorname{rank}\tau + \dim V < \dim X,$$

*then sections $V \to X$ transverse to $\tau$ satisfy all forms of the $h$-principle.*

There are two things to note about this. Firstly, the respective differential relation $\mathcal{R}$ is *not* $\operatorname{Diff} V$-invariant. This is why we will need the special local $h$-principle.

Secondly, note that for the trivial fibration $V \times W \to V$ and $\tau = \mathrm{T}V \times 0$, this theorem is just the Smale–Hirsch $h$-principle.

[3]To see this note that in the usual trivializations of the pullback and tangent bundles, $F''$ becomes $f \times \mathrm{id}_{\mathbb{R}^n}$ and for our choice of $x$ we have $F(b) = \mathrm{T}f(b)$ as $F$ is holonomic.

[4]The book is missing this modifier but I believe it is necessary.

*Proof.* By choosing a sufficiently small triangulation of $V$ and iterating over the skeleta, we can reduce the problem to the following relative version: $V = D^n$, $X = D^n \times \mathbb{R}^q$ and the section $V \to X$ already transverse to $\tau$ near $\partial V = \partial D^n$.

Let $\mathcal{R} \subset X^{(1)}$ be the differential relation of transversality to $\tau$ and $F\colon V \to \mathcal{R}$ a formal solution which is already holonomic near $\partial V$. We want to perform the inclusion $V = V \times 0 \subset V \times \mathbb{R}$ in order to use the special local $h$-principle. Hence we consider the fibration $X \times \mathbb{R} \to V \times \mathbb{R}$. Now, we need to define $\bar{\mathcal{R}} \subset (X \times \mathbb{R})^{(1)}$ such that solutions of $\bar{\mathcal{R}}$ yield solutions of $\mathcal{R}$. If a vector in $\mathrm{T}(X \times \mathbb{R})$ is transverse to $\tau \times \mathrm{T}\mathbb{R} \subset \mathrm{T}(X \times \mathbb{R})$ then its image in the quotient $\mathrm{T}(X \times \mathbb{R})/\mathrm{T}\mathbb{R} \cong \mathrm{T}X$ will be transverse to $\tau$. Thus we let $\bar{\mathcal{R}}$ be the relation which defines sections transverse to $\tau \times \mathrm{T}\mathbb{R}$.

What is left is for us to do is to extend $F$ to our new bundle as a formal solution of $\bar{\mathcal{R}}$. This means we need to find an appropriate place to map tangent vectors in the new dimension of our base space. Concretely, to preserve transverseness to $\tau \times \mathbb{R}$, the image vector should not be linearly dependent on $\tau \times \mathbb{R}$ and $\xi$, the subbundle of $\mathrm{T}X|_V$ defined by $F$ (as injectivity of the tangent map is required for transverseness). Consider the bundle

$$\nu = \mathrm{T}X|_V/(\tau|_V \oplus \xi).$$

A global non-vanishing section of $\nu$ could be lifted to a section $V \to \mathrm{T}X$ and subsequently $V \times \mathbb{R} \to \mathrm{T}(X \times \mathbb{R})$ to yield precisely what we are looking for. Such a section exists because $\nu$ is a trivial bundle and $\operatorname{rank} \nu > 0$. To see that $\nu$ is trivial just note that its base space $V = D^n$ is contractible (this is the only reason for using the triangulation). To see that $\operatorname{rank} \nu > 0$ note that

$$\begin{aligned}
\operatorname{rank} \nu &= (\dim X + \dim X - \dim V) - (\operatorname{rank} \tau + \dim X - \dim V) - \dim V \\
&= \dim X - \operatorname{rank} \tau - \dim V \\
&> 0
\end{aligned}$$

by assumption. The two $\dim X - \dim V$ terms arise because we are considering $\mathrm{T}X$ and $\tau$ as bundles over $V$ instead of $X$.

The relation $\mathcal{R}$ is open and invariant under diffeomorphisms in the form required by the special local $h$-principle. Thus we can apply it to $\bar{F}$ near $V \times 0 \subset V \times \mathbb{R}$ which by restriction implies the $h$-principle for $\mathcal{R}$.

□

# References


[1] Kai Cieliebak, Yakov Eliashberg, and Nikolai Mishachev. *Introduction to the h-Principle.* American Mathematical Society, second edition, 2024.

[2] Tammo tom Dieck. *Algebraic Topology.* 2008.