This talk is mainly based on [CEM24] chapter 17, with some ideas taken from [MS17].

1 What is symplectic geometry?

Let us first answer another question: What is Riemannian geometry? A Riemannian metric is a way to associate smoothly to each tangent space a scalar product. We would like something similar in symplectic geometry, but for that we need to understand the linear notion of symplectic vector spaces first.

Definition 1.1. Given a real finite-dimensional Vector space V, a **symplectic form** is a bilinear form ω satisfying

- (antisymmetry) $\omega(v, w) = -\omega(w, v)$ for any $v, w \in V$
- (non-degenerateness) $v \mapsto \omega(v, \cdot) : V \to V^*$ is an isomorphism - equivalently for any $v \in V$ there exists some vector $v' \in V$ such that $\omega(v, v') \neq 0$

Example 1.2. On $\mathbb{R}^{2n}$ the standard symplectic form is given by

$$\omega_0 := \sum_{i=1}^{n} dx_i \wedge dy_i,$$

where $x_1, ..., x_n, y_1, ..., y_n$ is the standard basis of $\mathbb{R}^{2n}$. Here the dx_i and dy_i denote the corresponding vectors of the dual basis.

Lemma 1.3. *Every antisymmetric bilinear form ω on a vector space V has a basis*

$$x_1, ..., x_n, y_1, ..., y_n, z_1, ..., z_p$$

such that $\omega(x_i, y_i) = 1$ and on all other pairings of basis vectors ω vanishes. In other words

$$\omega = \sum_{i=1}^{n} dx_i \wedge dy_i$$

In particular if ω is non-degenerate, $p = 0$ and dim$(V) = 2n$ is even and we have an isomorphism $V \to \mathbb{R}^{2n}$ that pushes ω to ω_0. □

Thus in the following (V, ω) will always be a real, $2n-$dimensional vector space with a symplectic form ω.

Definition 1.4. A **linear symplectomorphism** of (V, ω) is a vector space isomorphism $\Psi : V \to V$ that preserves ω, i.e.

$$\Psi^*\omega(v, w) = \omega(\Psi v, \Psi w) = \omega(v, w).$$

These form the subgroup $\mathrm{Sp}(V, \omega) \leq \mathrm{Aut}(V)$ and in the case of $(\mathbb{R}^{2n}, \omega_0)$ we write $\mathrm{Sp}(2n) := \mathrm{Sp}(\mathbb{R}^{2n}, \omega_0)$.

Remark 1.5.
- Since any (V, ω) is isomorphic to $(\mathbb{R}^{2n}, \omega_0)$, it suffices to study $\mathrm{Sp}(2n)$.
- By the above lemma, a symplectic form in some sense "pairs" basis vectors. This restricts what a symplectomorphism can do quite severely. It for example can't only exchange x_1 with x_2. Intuitively, it should also mean that any rescaling of x_i should come with an inverse rescaling of y_i. This is basically what Gromov's non-squeezing theorem tells us, if you have heard of that before.

Similar to the orthogonal complement in the case of scalar products, there is the following notion.

Definition 1.6. The **symplectic complement** of a (linear) subspace $W \subseteq V$ is given by

$$W^\omega := \{v \in V \mid \omega(v, w) = 0, \forall w \in W\}.$$

W is called

- **isotropic** if $W \subseteq W^\omega$,
- **coisotropic** if $W^\omega \subseteq W$,
- **symplectic** if $W \cap W^\omega = \{0\}$,
- **Lagrangian** if $W = W^\omega$.

Lemma 1.7. *For any subspace $W \subseteq V$,*

$$\dim W + \dim W^\omega = \dim V, \qquad (W^\omega)^\omega = W.$$

□

Symplectic geometry is all about the study of these subspaces! Using the lemma here are a few equivalent ways to describe these subspaces:

- W is symplectic $\iff$ W^ω is symplectic $\iff$ ω restricts to a symplectic form on W again.
- W is isotropic $\iff$ W^ω is coisotropic $\iff$ ω vanishes on W
- W is Lagrangian $\iff$ W is isotropic and of dimension n $\iff$ W is coisotropic and of dimension n

Moreover, isotropic subspaces are always of dimension $\leq n$, coisotropic subspaces of dimension $\geq n$ and symplectic subspaces of even dimension.

Example 1.8.
- Any line $G \subset V$ is isotropic with complement G^ω a hyperplane. Every hyperplane is thus coisotropic.
- The span $\langle x_1, y_1 \rangle_V \subset (\mathbb{R}^{2n}, \omega_0)$ is symplectic.
- The span $\langle x_1, ..., x_n \rangle_V \subset (\mathbb{R}^{2n}, \omega_0)$ is Lagrangian.

2 Almost complex structures

Definition 2.1. A **complex structure** on V is an automorphism $J : V \to V$ such that $J^2 = -\operatorname{Id}_V$. Such a structure makes V into an actual complex vector space by

$$(a + ib)v = av + bJv.$$

The group of automorphisms that commute with J is denoted by

$$GL(V, J) := \{\Phi \in Aut(L) \mid \Phi J = J\Phi\}.$$

For any such vector space there exists a basis $x_1, ..., x_n, y_1, ..., y_n$ such that $Jx_i = y_i$. In this basis

$$J = \begin{pmatrix} \mathbf{0} & -\mathbf{1} \\ \mathbf{1} & \mathbf{0} \end{pmatrix} =: J_0.$$

We call J_0 the standard complex structure on $\mathbb{R}^{2n}$. In this case $GL(\mathbb{R}^{2n}, J_0) = GL(n, \mathbb{C})$ where we identify $\mathbb{R}^{2n} \simeq \mathbb{C}^n$. Slogan: "Complex automorphisms are real automorphisms that commute with i".

Definition 2.2. A symplectic form ω and a complex structure J on V are **compatible** if

- J is a symplectomorphism, i.e. $\omega(Jv, Jw) = \omega(v, w)$ for all $v, w \in V$ and
- $\omega(X, JX) > 0$.

In this case one might call (V, ω, J) a **Hermitian** vector space. Then $g(X, Y) := \omega(X, JY)$ defines a scalar product.

In this situation we can find a basis that is symplectic in the sense of Lemma 1.3 and also fulfills $Jx_i = y_i$. Thus any (V, ω, J) looks like $(\mathbb{R}^{2n}, \omega_0, J_0)$.

For completeness sake we will state the following theorem.

Theorem 2.3. *The spaces $\mathcal{S}(V)$ of all symplectic forms and $\mathcal{J}(V)$ of all complex structures on V are Lie group quotients of $GL(2n, \mathbb{R})$ and homotopy equivalent with their topology coming from these quotients.* $\square$

3 Symplectic manifolds

Definition 3.1. A **symplectic form** ω on a real vector bundle $p : X \to V$ is a $2-$form ω such that ω is non-degenerate. Similarly one defines a **complex vector bundle**. We say that (X, ω) is a **symplectic vector bundle**.

One might be inclined to define a manifold to be symplectic when its tangent bundle comes equipped with a symplectic structure. But this turns out not to be enough. We want $(\mathbb{R}^{2n}, \omega_0)$ to be a local model of our symplectic manifolds. In that case one can see the local closedness condition

$$d\omega_0 = 0.$$

is satisfied. This is not true for all nondegenerate $2-$forms, so let's add it to our definition

Definition 3.2. A symplectic structure on a manifold M is a closed nondegenerate $2-$form ω. We call M a **symplectic manifold**. If ω is not closed, we call M an **almost symplectic manifold**. An **almost complex structure** on M is a complex structure on its tangent bundle.

A diffeomorphism $f : M \to M$ is a **symplectomorphism** if $f^*\omega = \omega$.

Notice how $\dim M$ again needs to be even and how every symplectic manifold is orientable. Now the notions of linear subspaces easily carry over from the linear case.

Definition 3.3. Call $N \subseteq M$ **isotropic**, **coisotropic**, **symplectic** or **Lagrangian** if each $T_pN \subseteq T_pM$ satisfies the corresponding linear condition.

Example 3.4. Of course $(\mathbb{R}^{2n}, \omega_0)$ is also a symplectic manifold. Another interesting example is the cotangent bundle T^*M of any manifold M. It is always even dimensional and orientable, so this gives us at least some reason to hope for a symplectic structure.

Given coordinates $x : U \to \mathbb{R}^n$ on M, the dx_i form a basis of the cotangent bundle over U. Thus any $u \in T^*U$ can be written as $(\sum x_i e_i, \sum y_i dx_i)$ in charts. Locally we can now define the **canonical** $1-$**form**

$$\lambda_{\text{can}} := ydx := \sum_{i=1}^{n} y_i dx_i.$$

One can show (see [MS17], page 105) that this is independent of the chosen coordinates. Its differential

$$\omega_{\text{can}} := d\lambda_{\text{can}} = dx \wedge dy = \sum_{i=1}^{n} dx_i \wedge dy_i$$

defines the canonical symplectic structure on T^*M.

Example 3.5. The image of a closed simple loop $\gamma : S^1 \to (\mathbb{R}^2, \omega_0)$ is always Lagrangian: It's tangent space is always half-dimensional and isotropic.

Definition 3.6. A **Lagrangian**, **isotropic** or **coisotropic** embedding into a symplectic manifold is an embedding whose image is Lagrangian, isotropic or coisotropic respectively. Similarly we define different types of immersions to be maps that locally look like the respective embeddings.

A map $f : M \to (N, \omega_N)$ is called **symplectic** if $f^*\omega_N$ is symplectic. If M also comes with a symplectic structure that coincides with $f^*\omega_N$, we call f **isosymplectic**.

4 Symplectic Stability

Symplectic stability describes the idea that there are no nontrivial local invariants on symplectic manifolds. As mentioned before, this is where our assumption of symplectic forms beeing closed is crucial. To get there, let's look at the following situation.

Given a homotopy ω_t of symplectic forms on M, one might ask for a family of diffeomorphisms (**isotopy**) φ_t such that

$$\varphi_t^* \omega_t = \omega_0.$$

Suppose these φ_t exist and are generated by a family of vector fields v_t. Differentiating $\omega_0 = \varphi_t^* \omega_t$ gives

$$0 = \frac{d}{dt} (\varphi_t^* \omega_t) = \varphi_t^* (\mathcal{L}_{v_t} \omega_t + \dot{\omega}_t) \text{ or}$$
$$\mathcal{L}_{v_t} \omega_t = -\dot{\omega}_t \text{ for all } t \in [0, 1].$$

As it turns out such v_t exist under the right homological assumptions.

Theorem 4.1. *Let $\omega_t = \omega_0 + d\alpha_t$ be a smooth family of symplectic forms on a manifold M and v_t the unique vector field such that $\omega_t(v_t, \cdot) = -\dot{\alpha}_t$. Then*

$$\mathcal{L}_{v_t} \omega_t = -\dot{\omega}_t \textit{ for all } t \in [0, 1].$$

Proof. Indeed using Cartans magic formula, $d\omega_t = 0$ and the fact that the time derivative commutes with the exterior derivative we see

$$\mathcal{L}_{v_t} \omega_t = d(\omega_t(v_t, \cdot)) = -d(\dot{\alpha}_t) = -\dot{d\alpha_t} = -\dot{\omega}_t.$$

□

This now gives rise to a bunch of really useful corollaries which are also often called *Symplectic Stability Theorems.*

Theorem 4.2 (**Stability near a compact set**)**.** *Let $A \subseteq M$ be a compact subset. Let $\omega_t = \omega_0 + d\alpha_t$ be a family of symplectic forms on $\mathcal{O}pA \subseteq M$ such that $\alpha_t|_{TM|_A} = 0$. Then there exists an isotopy $\varphi_t : \mathcal{O}pA \to M$ fixed on A such that $\varphi_t^* \omega_t = \omega_0$ for all $t \in [0, 1]$.*

Proof. Choosing $\mathcal{O}pA$ small enough, the v_t Theorem 4.1 are compactly supported. Thus the isotopy generated by the v_t exists on all of $\mathcal{O}pA$. Moreover it is fixed on A by the vanishing of the v_t there. □

Theorem 4.3 (**Darboux's Theorem**)**.** *Any symplectic form is locally equivalent to ω_0 on $\mathbb{R}^{2n}$.*

Proof. Pull back the symplectic form to a symplectic form ω on $\mathbb{R}^{2n}$ using local charts. After some change of coordinates on $\mathbb{R}^{2n}$, ω coincides with ω_0 at the origin, since all linear symplectic forms are equivalent. Moreover

$$\omega_t = (1 - t)\omega_0 + t\omega$$

is a family of symplectic forms in $\mathcal{O}p0$ with $\omega_t = \omega_0 + d\alpha_t$ where $\alpha_t(0) = 0$ for all $t \in [0, 1]$. Now apply stability near $A = \{0\}$ to get $\varphi_1^* \omega = \omega_0$. □

Theorem 4.4 (**(Relative) Moser's Theorem**)**.** *Let $\omega_t = \omega_0 + d\alpha_t$ be a family of symplectic forms on a compact manifold M, maybe with boundary, such that $\omega_t = \omega_0$ on $\mathcal{O}p\partial M$ (this neighborhood might be empty, if ∂M is). Then there exists an isotopy $\varphi_t : M \to M$ fixed on $\mathcal{O}p\partial M$ such that $\varphi_t^* \omega_t = \omega_0$.*

Proof. Since M is compact and α_t is zero on $\mathcal{O}p\partial M$, the flow exists globally again and is fixed on $\mathcal{O}p\partial M$. □

Theorem 4.5 (**Weinstein's Theorem**)**.** *Any isotropic immersion $f : L \to (M, \omega)$ (in particular any Lagrangian one) of a compact manifold L extends to an isosymplectic immersion $\mathcal{O}pL \to M$ where $\mathcal{O}pL$ is a neighborhood of the zero section $L \subset T^*L$ endowed with its canonical symplectic structure.*

Proof. Choosing a $\omega-$compatible almost complex structure J (this always exists, see [MS17], page 81) we get a Riemannian metric $g(X, Y) = \omega(X, JY)$ on M. After pulling it back we can use it to define isomorphisms $\Phi_p : T_p^*L \to T_pL$. Using the exponential map we now define the map $\phi : T^*L \to M$ by

$$\phi(p, q) := \exp_{f(p)}(J_p df_p \Phi_p(q)).$$

Since the immersion is isotropic, $JdfTL$ points away from L, so locally ϕ is an embedding. Moreover one can compute that $\phi^*\omega$ agrees with the canonical symplectic form on T^*L on the zero section $L \subseteq T^*L$. Since L is compact, we can apply stability at L, use the isotopy at $t = 1$ to modify ϕ and make it isosymplectic on a neighborhood. For a more treatment of this, see the "Lagrangian neighbourhood theorem" in [MS17] page 121. □

5 Contact manifolds

Definition 5.1. Given a nonzero $1-$form α on a $(2n + 1)-$dimensional manifold M, $\xi := \ker \alpha$ is a $(2n)-$dimensional tangent distribution. We want to call ξ a contact structure on M, if ξ is as far away from beeing integrable as possible. By Frobenius' integrability theorem ξ is integrable if and only if the sections of ξ are closed under the Lie bracket. A vector field X is a section of ξ if and only if $\alpha(X) = 0$. In view of the identity

$$d\alpha(X,Y) = \mathcal{L}_X\left(\alpha(Y)\right) - \mathcal{L}_Y\left(\alpha(X)\right) + \alpha([X,Y])$$

integrability is equivalent to $d\alpha$ vanishing on ξ, or in other words $\alpha \wedge d\alpha = 0$.

Thus we define a **contact form** α on M to be a $1-$form such that $d\alpha$ is nondegenerate on $\ker\alpha$ or equivalently

$$\alpha \wedge (d\alpha)^n \neq 0.$$

We call $\xi \subset TM$ a **contact structure** on M, if it locally looks like the kernel of contact forms. If there exists a global contact form α such that $\xi = \ker\alpha$, we call ξ coorientable.

Remark 5.2.

- One can show that coorientability of ξ is equivalent to the triviality of the line bundle TM/ξ.
- Different contact forms can give rise to the same contact structure. In fact two contact forms induce the same (coorientable) contact structure if and only if

$$\alpha' = f\alpha$$

  for some smooth $f : M \to \mathbb{R}$.
- A contact form gives rise to a symplectic form $d\alpha|_{\ker\alpha}$ on $\ker\alpha$. For a given contact structure ξ, this is only defined up to a nonzero scaling function by the above remark. We denote this class of symplectic forms by $\mathrm{CS}(\xi)$. Note that the notion of isotropic, coisotropic, lagrangian and symplectic submanifolds does not depend on that scaling function.

Lemma 5.3. *Integral submanifolds of a contact structure ξ are isotropic and in particular must have dimension $\leq n$.*

Proof. Suppose $L \subseteq M$ is integral. Then for $X, Y \in TL \subset TM$ we also have $[X,Y] \in TL$ and any contact form α representing ξ locally vanishes on all of these. Thus

$$d\alpha(X,Y) = \mathcal{L}_X\left(\alpha(Y)\right) - \mathcal{L}_Y\left(\alpha(X)\right) + \alpha([X,Y]) = 0.$$

Therefore T_pL is an isotropic subspace of $(\xi_q, d\alpha_q)$ for every q. □

Definition 5.4. An integral submanifold of (M,ξ) of maximal dimension n is called **Legendrian**.

Example 5.5.

- $J^1(M,\mathbb{R}) = T^*M \times \mathbb{R}$ has a canonical (coorientable) contact structure given by $\alpha = dz - p\,dq$ where z us the coordinate on $\mathbb{R}$ and $p\,dq$ the pullback of the canonical $1-$form on T^*M. In the case of $M = \mathbb{R}^n$ this gives rise to the **standard contact structure** on $\mathbb{R}^{2n+1}$ defined by

$$\alpha_0 = dz - \sum_{i=1}^n y_i dx_i$$

- Let (M,ξ) be a $(2n-1)-$dimensional contact manifold. The $2n-$dimensional manifold $\hat{M} := (TM/\xi)^* \setminus M$ is called the **symplectization** of (M,ξ). Its symplectic structure comes pulling the symplectic structure of T^*M back through the embedding

$$\begin{aligned} \hat{M} &\to T^*M \\ f &\mapsto f \circ \pi \end{aligned}$$

  where $\pi : TM \to TM/\xi$ is the quotient map. This construction gives rise to a correspondence between contact and symplectic structures, e.g. Legendrian manifolds in M correspond to certain Lagrangian submanifolds of $\hat{M}$

Definition 5.6. An immersion $f : N \to (M, \xi)$ is called **isotropic** if $df(TN) \subseteq \xi$. By Lemma 5.3, this implies $\dim N \leq n$. We call an isotropic immersion **Legendrian**, if $\dim N = n$.

If N is odd-dimensional, we call f **contact** if it induces a contact structure on N. Then f is automatically an immersion and transverse to ξ. If N already had a contact structure and this pullback induces that structure, we call f **isocontact**.

Suppose we are given a family of contact $1-$forms α_t. Similar to the case of symplectic stability we can find diffeomorphisms φ_t such that (we don't have any homological conditions this time!)

$$\dot{g}_t \varphi_t^* \alpha_t = \alpha_0$$

for some nonzero scaling functions g_t. The g_t did not appear in the symplectic case. But here they don't change the underlying contact structure and give us enough freedom to always find suitable φ_t.

This gives rise to a similar suite of *Contact Stability Theorems.*

Theorem 5.7 (**Stability near a compact subset**). *Let $\alpha_t, t \in [0,1]$ be a family of contact forms on $\mathcal{O}pA \subseteq M$ of a compact subset $A \subseteq M$ such that $\alpha_t|_{TM|_A} = \alpha_0$. Then there exists $\varphi_t : \mathcal{O}pA \to M$ fixed on A such that $\varphi_t^* \alpha_t = \alpha_0$.* ☐

Theorem 5.8 (**Darboux's theorem**). *Any contact structure (contact form) is locally equivalent to the standard contact structure (contact form) on $\mathbb{R}^{2n+1}$.* ☐

Theorem 5.9 (**(Relative) Gray's theorem**). *Let ξ_t, $t \in [0,1]$ be a family of contact structures on a compact manifold M, maybe with boundary, such that $\xi_t = \xi_0$ on $\mathcal{O}p\partial M$ (this neighborhood can be empty, if ∂M is). Then there exists an isoptopy $\varphi_t : M \to M$ fixed on $\mathcal{O}p\partial M$ such that $\varphi_t^* \xi_t = \xi_0$.* ☐

Theorem 5.10 (**Contact Weinstein's theorem**). *Any isotropic immersion $L \to (M, \xi)$ extends to an isocontact immersion $\mathcal{O}pL \to (M, \xi)$ where $\mathcal{O}pL$ is a neighborhood of the zero section in the $1-$jet space $J^1(L, \mathbb{R})$ with its canonical contact structure.* ☐

I do not think we use them much in the this seminar, but for completeness sake we will define Hamiltonian and Contact vector fields.

Definition 5.11. A vector field X on a symplectic manifold (M, ω) is **symplectic**, if $\mathcal{L}_X \omega = d(\omega(X, \cdot)) = 0$. If $\omega(X, \cdot)$ is exact, i.e. $\omega(X, \cdot) = -dH$ form some $H : M \to \mathbb{R}$, we call X **Hamiltonian**. Conversely any such map H induces a Hamiltonian vector field X_H by the nondegeneracy of ω.

We call a symplectomorphism $\varphi : M \to M$ **Hamiltonian**, if it is homotopic to the identity via a family $\varphi_t : M \to M$ of flows of Hamiltonian vector fields X_{H_t} for $H_t : M \to \mathbb{R}$.

Definition 5.12. Similarly we call a vector field X on a contact manifold (M, ξ) **contact**, if its flow perserves the contact structure ξ.

References

[MS17] Dusa McDuff and Dietmar Salamon. *Introduction to Symplectic Topology.* Third Edition. New York: Oxford University Press, 2017. ISBN: 978-0-198-79489-9.

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