Symplectic and contact structures on open manifolds

Abstract

Given an arbitrary smooth manifold M, we do not (yet) have any way to tell if it admits a symplectic/contact structure and if it does, how many "different" structures there are. This is a hard problem in general, but for open manifolds nice results can be obtained: By applying the h-principle we are going to show that an open manifold admits a symplectic structure if and only if it admits an almost complex structure (the question of existence of almost complex problem can be reduced to statements about algebraic topological invariants), in fact we are going to show that there is a homotopy equivalence between the space of symplectic forms and almost complex structures on a manifold, allowing us to classify symplectic forms up to (symplectic) homotopy.

The content of this talk are mostly based on chapter 18 of (Cieliebak, Eliashberg, Mishachev 2024).

1. Applying the h-principle to differential forms

Let us fix an open manifold V and denote the space of almost symplectic forms on it as $\mathcal{S}_{\text{symp}}$ and the space of symplectic ones as $\mathbb{S}_{\text{symp}}$. Sometimes (e.g. for Mosers approximation theorem) it is important to only look at symplectic forms in some cohomology class, so let us denote with $\mathbb{S}^a_{\text{symp}}$ the symplectic forms in some cohomology class $a \in H^2(V)$.

Clearly the existence of an almost symplectic structure is necessary for the existence of symplectic structures and two symplectic forms can only hope to be homotopic in $\mathbb{S}_{\text{symp}}$ if they are homotopic in $\mathcal{S}_{\text{symp}}$. Given these prerequisites, the h-principles guides us to approach the classification of symplectic structures up to homotopy by studying the homotopy properties of the natural inclusions, with the hope that they are homotopy equivalences and indeed:

Theorem 1.1

$$\mathbb{S}_{\text{symp}} \hookrightarrow \mathcal{S}_{\text{symp}} \text{ and } \mathbb{S}^a_{\text{symp}} \hookrightarrow \mathcal{S}_{\text{symp}}$$

is homotopy equivalence

Remark 1.2 Some people might feel tricked, because the abstract promised a homotopy equivalence to the space of almost complex structures. But these statements are actually equivalent since one can show that there is also a homotopy equivalence between the almost complex and almost symplectic structures on a manifold (and the proof of it isn't too complicated, see e.g. the proof of proposition 4.1.1 in chapter 4 of (McDuff, Salamon 2017) for a readable version)

Proving the above theorem requires some new additions to our h-principle machinery to apply it to differential forms. Looking closely at the theorem, one might already guess that we can take the space of almost symplectic forms as a space of "formal" solutions and the space of symplectic forms as the space of "genuine" ones.

The difference between the these two spaces is of course the closedness of forms, thus we are going to start with a general approximation theorem for closed forms:

Theorem 1.3 *(Approximation by closed forms)* Let $K \subseteq V$ be a polyhedron of codimension ≥ 1. Let ω be a p-form on V and $a \in H^p(V)$ a fixed cohomology class. Then there exists an arbitrarily C^0-small diffeotopy $h^\tau : V \to V$, such that ω can be C^0-approximated near $\tilde{K} = h^1(K)$ by a closed p-form $\tilde{\omega} \in a$.

This feels like it should be an immediate corollary of holonomic extension, if we were dealing with jet bundles. But by shifting our perspective on differential forms, we can make turn them into jet bundles:

Recall that a differential p-form ω is a section of the vector bundle $\Lambda^p V \to V$ with the exterior derivative being a map $\mathrm{d} : \operatorname{Sec} \Lambda^p V \to \operatorname{Sec} \Lambda^{p+1} V$. Sections of the jet bundle include strictly more information, allowing us to express d as a concatenation

$$\operatorname{Sec} \Lambda^p V \xrightarrow{J^1} \operatorname{Sec}(\Lambda^p V)^{(1)} \xrightarrow{\tilde{D}} \operatorname{Sec} \Lambda^{p+1} V$$

with $\tilde{D}$ being induced by a homomorphism of bundles $D : (\Lambda^p V)^{(1)} \to \Lambda^{p+1} V$ which is called the *symbol of* d.[1]

Example 1.4 This technical stuff might be a bit dry, so an example might be helpful for digestion. Lets take a look at $\Lambda^1 \mathbb{R}^2 \to \mathbb{R}^2$. Our favorite choice of coordinates (x, y) on $\mathbb{R}^2$ also induces coordinates on Λ^1, which we are going to denote as (x, y, ξ, ι), representing the linear map $T_{(x,y)}\mathbb{R}^2 \to \mathbb{R}$ given by $\xi \mathrm{d}x + \iota \mathrm{d}y$. Sections of $\Lambda^1 \mathbb{R}^2$ look like $f \mathrm{d}x + g \mathrm{d}y$, leading us to to a choice of coordinates on $\left(\Lambda^1 \mathbb{R}^2\right)^{(1)}$ for which $J^1(f \mathrm{d}x + g \mathrm{d}y)$ is represented as $\left(x, y, f, g, \partial_x f, \partial_y f, \partial_x g, \partial_y g\right)$. Given a point $(x, y, f, g) \in \Lambda^1 \mathbb{R}^2$, we can identify the fiber of $\left(\Lambda^1 \mathbb{R}^2\right)^{(1)}$ with the space of 2×2 matrices and the fiber of $\Lambda^2 \mathbb{R}^2 \to \mathbb{R}^2$ over (x, y) with the space of skew-symmetric 2×2 matrices, allowing us to represent D on the fiber as $D(A) = A - A^T$.

Introducing this additional map turns out to be useful, because it turns out that D is an affine fibration, allowing us to lift any section $\omega : V \to \Lambda^p V$ to a section $F_\omega : V \to \left(\Lambda^{p-1} V\right)^{-1}$ such that $D \circ F_\omega = \omega$ in an (up to homotopy) unique way. This F_ω should be interpreted as a *formal primitive* of ω and while this doesn't give us our approximation by closed forms just yet, it gives us something we can use to proof it:

Theorem 1.5 *(Approximation by exact forms)* Let $K \subseteq V$ be a polyhedron of codimension ≥ 1 and ω a p-form. Then there exists an arbitrarily C^0-small diffeotopy $h^\tau : V \to V$ such that ω can be C^0-approximated near $\tilde{K} = h^1(K)$ by an exact p-form $\tilde{\omega} = \mathrm{d}\tilde{\alpha}$. Moreover, given a $(p-1)$-form α on V, one can choose $\tilde{\alpha}$ to be C^0-close to α near $\tilde{K}$.

Proof Let F_ω be a formal primitive of ω with $\operatorname{bs} F_\omega = \alpha$. By our previous holonomic approximation theorems there exists a C^0-small diffeotopy h^τ such that F_ω has a holonomic approximation $J^1_{\tilde{\alpha}}$ of F_ω along $\tilde{K} = h^1(K)$. Extending $\tilde{\alpha}$ to the whole manifold (and possibly shrinking the neighborhood a bit) produces the desired exact form $\tilde{\omega} := \mathrm{d}\tilde{\alpha}$ □

Using this, we can construct closed approximations:

[1] That such a homomorphism actually exists seems to be a non-trivial statement for which I could not figure out a proper proof. But it seems believable if one think of the exterior derivative as only "looking at" derivatives of sections (in a coordinate independent way).

Proof *(Of Theorem 1.3)* Let $\Omega \in a$ be a closed form. Apply the previous theorem to $\theta \coloneqq \omega - \Omega$ to approximate θ near $\tilde{K}$ by an exact form $\tilde{\theta}$. Then $\tilde{\omega} \coloneqq \tilde{\theta} + \Omega$ is an closed approximation of ω near $\tilde{K}$ that is homologous to Ω i.e. $\tilde{\omega} \in a$ □

Parametric versions of both theorems can be proven analogously, requiring only the additional detail that the space of closed p-forms representing a given cohomology class is convex.

At this point we are basically ready to proof our original theorem, we just have to establish one other technical requirement, which is not only going to be helpful, but is also going to answer a question the reader might have already asked themselves: Why can we proof these statements only for open manifolds?

The reason for that is that open manifolds can always be "retracted" to something lower-dimensional or, more formal, they always admit a (pseudo)-core:

Definition 1.6 Given a manifold V, we call a stratified subset $V_0 \subseteq V$ **pseudo-core** of V if $\operatorname{codim} V_0 \geq 1$ and V is isotopic to $\mathcal{O}p\, V_0$. If there also is such an isotopy that fixes V_0, then it is called a **core** of V.

The existence of cores is what is going to allow us to "globalize" our previous statement about approximation by a closed form. Technically we only require the existence of a pseudo-core and it is only this existence that we are going to proof:

Theorem 1.7 The $(n-1)$-skeleton $V^{n-1} \subseteq V$ of a triangulation of any open manifold V is a pseudo-core

Proof The underlying fact here is the existence of paths to infinity, which are maps $p : [0, \infty) \to V$ that are either proper or for which $\lim_{t\to\infty} p(t) \in \partial V$ holds. On open manifolds these always exists for every point, in particular we can find disjoint paths from the barycenters of the triangulation to infinity, then by moving backwards "from infinity" we can isotop V to something disjoint from the barycenters, which then allows us to just push the content of every n-simplex to its boundary. Note that this doesn't fix V_0 in any way and the final result is usually not homotopy equivalent to V^{n-1} □

Cores also always exists on open manifolds, but we won't proof that since it is much more complicated to do and we don't need proper cores anywhere.

2. This part is actually about symplectic geometry

As it is common in mathematics, our theorem is going to be a corollary of a more general statement:

Theorem 2.8 Let V be an open manifold, $a \in H^p(V)$ a fixed cohomology class, and $\mathcal{R} \subseteq \Lambda^p V$ an open $\operatorname{Diff} V$-invariant subset.[2] Then the inclusion

$$\operatorname{Clo}_a \mathcal{R} \hookrightarrow \operatorname{Sec} \mathcal{R}$$

is a homotopy equivalence. In particular,

[2] Recall that we can only talk about such subsets if the underlying bundle is natural, with the naturality of $\Lambda^p V$ following from the fact that diffeomorphism can be turned into pullbacks/push-forwards of differential forms

- any p-form $\omega : V \to \mathcal{R}$ is homotopic in $\mathcal{R}$ to a closed p-form $\overline{\omega} \in a$
- any homotopy of p-forms $\omega_t : V \to \mathcal{R}$ which connects two closed forms $\omega_0, \omega_1 \in a$ can be deformed in $\mathcal{R}$ into a homotopy of closed forms $\overline{\omega}_t \in a$ connecting ω_0 and $\omega_1 \in a$

Proof We are going to limit this proof to the non-parametric version since both are proven analogously, but the parametric version requires more notational care.

Let K be a pseudo-core of V. By Theorem 1.3 there exists a closed form $\tilde{\omega} \in a$ that is C^0-close to ω over a neighborhood $\tilde{\mathcal{U}}$ of $h^1(K)$ where $h^\tau : V \to V$ is a C^0-small diffeotopy. $\mathcal{R}$ is open, hence by picking $\hat{\omega}$ close enough, we can ensure that it and the linear interpolation ω_t between it and ω fulfill the differential relation on $\tilde{U}$.

Then $\hat{\omega} := (h^1)^* \tilde{\omega}$ fulfills the differential relation a neighborhood of K since h^1 is a diffeomorphism and $\hat{\omega} \in a$ since pullbacks by diffeotopies do not change the cohomology class. Using the fact that K is a core, let $g^t : V \to V$ be an isotopy from V into a neighborhood $\mathcal{U}$ of K on which $\hat{\omega}$ fulfills the differential relation. By the same argument as before $\overline{\omega} := (g^1)^* \hat{\omega}$ fulfills the differential relation on the whole manifold and lies in a (g^1 is not a diffeomorphism any more, but still a local one and that suffices).

This is our closed form, so it only remains to find a homotopy between them, which we are going to describe as a concatenation of several homotopies. First use g^t as a homotopy between $\omega = (g^0)^* \omega$ and $(g^1)^* \omega$ (g^t is a local diffeomorphism, so all of these lie in $\mathcal{R}$), then use our diffeotopy h^t as a homotopy between $(g^1)^* \omega$ and $(g^1)^* (h^1)^* \omega$ (these all lie in $\mathcal{R}$ by the same argument again). We can apply the same reasoning we used to prove that $\overline{\omega}$ fulfills the differential relation everywhere to prove that $(g^1)^* (h^1)^* \omega_t$ fulfills it everywhere, with $\omega_t := (1-t)\omega + t\tilde{\omega}$, giving us our last homotopy between $(g^1)^* (h^1)^* \omega = (g^1)^* (h^1)^* \omega_0$ and $(g^1)^* (h^1)^* \omega_1 = \overline{\omega}$. □

As said, Theorem 1.1 follows directly from that since being non-degenerate is invariant under pull-back by local diffeomorphism and an open condition.

3. Contact Structures

One might also wonder about the classification (up to homotopy) and existence of contact structures on a manifold. Let us recall the notion of a (cooriented) contact structure:

Definition 3.9 A contact structure is a codimension 1 tangent distribution ξ that can be locally be described by the kernel of a 1-form α with $\mathrm{d}\alpha$ being non-degenerate on ξ, or equivalently with $\alpha \wedge (\mathrm{d}\alpha)^k$ non-vanishing.

We call ξ cooriented if there is a global α with these properties.

We did not define the notion of an almost-contact structures yet and it is a bit more involved (one needs bundle-valued 1-forms for that), but cooriented almost-contact structures can be defined a bit easier. As with symplectic forms, we do this by making some exterior derivative related condition less strict:

Definition 3.10 A cooriented almost-contact structure is a codimension 1 tangent distribution ξ which can be globally described as the kernel of some form α together with a 2-form ω that is nondegenerate on ξ or equivalently for which $\alpha \wedge \omega^n$ does not vanish.[3]

[3]technically one should define this for "conformal classes" of 2-forms, which can be understood as 2-forms up to positive scaling

Similar as in the symplectic case, we can now state:

Theorem 3.11 For any open manifold the embedding of coorientable almost contact structures into all contact structures is a homotopy equivalence

To apply our methods, we have to rephrase this in the proper language, so define:

Definition 3.12 For some differential relation $\mathcal{R} \subseteq \Lambda^{p-1} \oplus \Lambda^p V$ denote the subspace of pairs $(\alpha, \mathrm{d}\alpha) \in \mathcal{R}$ as $\operatorname{Exa} \mathcal{R}$.

Then our theorem again follows from a more general statement:

Theorem 3.13 Let $\mathcal{R}$ be an open $\operatorname{Diff} V$-invariant subset of $\Lambda^{p-1} \oplus \Lambda^p V$. Then the inclusion

$$\operatorname{Exa} \mathcal{R} \hookrightarrow \operatorname{Sec} \mathcal{R}$$

is a homotopy equivalence. In particular, any section $(\alpha, \beta) : V \to \mathcal{R}$ is homotopic in $\mathcal{R}$ to a section $(\overline{\alpha}, \mathrm{d}\overline{\alpha}) : V \to \operatorname{Exa} \mathcal{R}$

Proof This is proven exactly as in the proof of Theorem 2.8, only using the approximation by exact forms instead of approximation by closed forms. □

Similar statements can be proven for general almost-contact structures, though it is a bit more involved since one has to work with bundle-valued forms. (which is exactly the reason we won't do it)

Remark 3.14 The problem of existence and classification of contact structures on open manifolds is similar to the case of symplectic structures, but on closed manifolds contact structures behave much better: The h-principle holds existence-wise in all dimensions. The parametric version doesn't in general, but at least for a big class (so called overtwisted) contact structures it does, for details see chapter 19 of (Cieliebak, Eliashberg, Mishachev 2024).

Bibliography

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