11.1.2017 - h-principle seminar: microflexibility & isotropic immersions

some non-open relations:

(1) For $X := \Lambda^p V := \Lambda^p T^* V$, $\mathcal{R}_{clo} \subseteq X^{(1)}$ s.t. Hol $\mathcal{R}_{clo} = \{$closed p-forms on $V\}$

(2) For a symp. mfd (W^{2n}, ω), $\dim V =: k \leq n$, set $X := V \times W$,

case $k = n$: $\mathcal{R}_{Lag} := \{(v, w, \Phi) \in J^1(V, W) \mid \Phi : T_v V \to T_w W$ is inj. & $\omega|_{im\,\Phi} = 0\}$

$\Rightarrow$ Hol $\mathcal{R}_{Lag} = \{$Lagrangian immersions $V \hookrightarrow (W, \omega)\}$

case $k < n$: $\mathcal{R}_{sub\text{-}isot}$:= some defn, "subcritical isotropic"

(3) For a ctct mfd (W^{2n+1}, ξ),

case $k = n$: $\mathcal{R}_{Leg} := \{(v, w, \Phi) \in J^1(V, W) \mid \Phi$ inj & $im\,\Phi \subseteq \xi\}$

$\Rightarrow$ Hol $\mathcal{R}_{Leg} = \{$Legendrian immersions $V \hookrightarrow (W, \xi)\}$

case $k < n$: $\mathcal{R}_{sub\text{-}isot}$:= some defn

Holonomic $\mathcal{R}$-approximation thm: Assume $\mathcal{R} \subseteq X^{(r)}$ satisfies conditions (TBA1) & (TBA2) ("to be announced"), $A \subseteq V$ a polyhedron of codim > 0 & $F : Op A \to \mathcal{R}$ a section. Then $\forall\, \delta, \varepsilon > 0$, $\exists$ a δ-small (in C^0) diffeotopy $\{h^\tau : V \to V\}_{\tau \in [0,1]}$ & a holon. sec. $\tilde{F} : Op\, h^1(A) \to \mathcal{R}$ s.t. $dist(\tilde{F}(v), F(v)) < \varepsilon$ $\forall\, v \in Op\, h^1(A)$.

+ relative & parametric versions.

- difference from original holon. approx thm: explicitly requiring F & $\tilde{F}$ values in given relation $\mathcal{R}$.

cor: If V open & $X \to V$ a natural fiber bndl, then any Diff V-invt $\mathcal{R} \subseteq X^{(r)}$ satisfying (TBA 1+2) satisfies the (parametric) h-princ.

Will show (TBA 1+2) holds for $\mathcal{R}_{sub\text{-}isot}$ & $\mathcal{R}_{Leg}$ on ctct mfds

$\Rightarrow$ cor (via micro-extension trick): The h-princ. holds for subcritical isotropic immersions in ctct mfds.

Recall: Pf of holon. approx. on cubes I^k (assuming already holon. near ∂I^k) for $k < n$: prove by induction on $l = 0, \dots, k$ that can make $F : I^k \to \mathcal{R}$ fiberwise holon. w.r.t. $\pi_{k-l} : I^k \to I^{k-l}$

(after "wiggling" I^k in the nth dimension), without changing it in $Op(\partial I^k)$.

- $l=0$ step: Given $F: I^k \to \mathcal{R}$, holon. on $Op(\partial I^k)$, $\exists$ family of holon. extensions: $\{F_v : Op\, v \xrightarrow{\text{holon.}} \mathcal{R}\}_{v \in I^k}$ s.t. $F_v(v) = F(v)$ $\forall\, v$, $F_v = F|_{Op\, v}$ $\forall\, v \in Op(\partial I^k)$.

$\rightsquigarrow$ condition (TBA1): $\mathcal{R} \subseteq X^{(r)}$ is locally integrable if given $v \in V$ & $F(v) \in \mathcal{R}$, $\exists$ an extension to a holon. section $F: Op\, v \to \mathcal{R}$.

(+ parametric & relative versions)

exs: (1) Any open $\mathcal{R}$

(2) $\mathcal{R}_{Lag}$, $\mathcal{R}_{Leg}$, $\mathcal{R}_{sub\text{-}isot}$: e.g. for $\mathcal{R}_{Lag}$ this means given any $v \in V$, $w \in W$ & $\Phi: T_vV \hookrightarrow T_wW$ w/ $\omega|_{im\,\Phi} = 0$, $\exists$ Lagr. immersion $f: Op\, v \to W$ s.t. $f(v) = w$ & $df(v) = \Phi$.

pf: Use Darboux coords, take any suitable Lagr. subspace. ☒

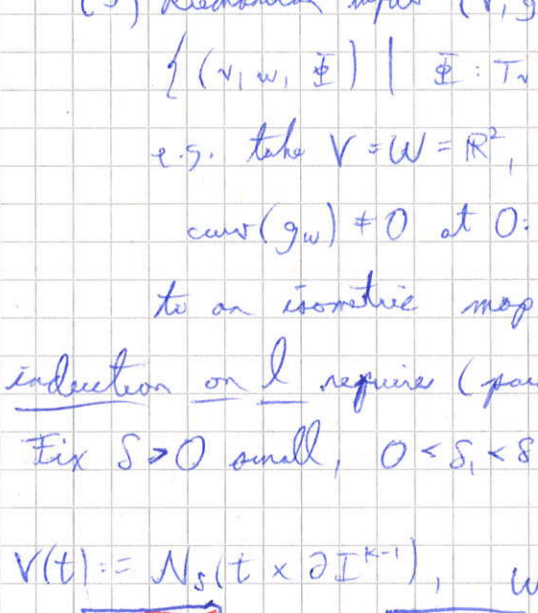
(3) Riemannian mfds (V, g_v), (W, g_w), $J^1(V,W) \supseteq \mathcal{R}_{iso} :=$ $\{(v, w, \Phi) \mid \Phi: T_vV \to T_wW$ is orthogonal$\}$ is not locally integ.

e.g. take $V = W = \mathbb{R}^2$, $g_v = g_w$ at 0 but $curv(g_v) \equiv 0$, $curv(g_w) \neq 0$ at 0: then $Id: T_0V \to T_0W$ does not extend to an isometric map $(Op(0), g_v) \to (Op(0), g_w)$.

- induction on l requires (parametric version of) interpolation property:

Fix $\delta > 0$ small, $0 < \delta_1 < \delta$, let $U(t) := \underset{\text{cubical } \delta\text{-nbhd}}{N_\delta}(t \times I^{k-1})$ for $t \in I$

$V(t) := N_\delta(t \times \partial I^{k-1})$, $W(t) := (U(t) \setminus \overline{U_{\delta_1}(t)}) \cup V(t)$



Given: $F: I^k \to \mathcal{R}$, holon. in $Op(\partial I^k)$ w/ family of holon. extensions $\{F_t : U(t) \to \mathcal{R}\}_{t \in I}$ s.t. $F_t = F$ along $t \times I^{k-1}$ & in $V(t)$, $F_t = F|_{U_t}$ for $t \in Op(\partial I)$.

Then $\forall\, \varepsilon > 0$, $\exists\, \sigma = \frac{1}{N}$ (small!) & a family of holon. sections $\{F_t^\tau : U(t) \to \mathcal{R}\}_{t \in I,\, \tau \in [0,\sigma]}$ s.t.

(a) $F_t^0 = F_t \quad \forall\, t$,

(b) $F_t^\tau\big|_{W(t)} = F_t\big|_{W(t)} \quad \forall\, t,\tau$

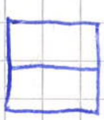
(c) $\| F_t^\tau - F_t \|_{C^0} < \varepsilon \quad \forall\, t,\tau$

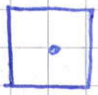
(d) $F_t^\tau\big|_{Op(t\times I^{k-1})} = F_{t+\tau}\big|_{Op(t\times I^{k-1})} \quad \forall\, t,\tau.$

remark 1: This is what is needed for the last step in the induction, going from $l=k-1$ to $l=k$. Earlier steps require the parametric version.

remark 2: It's obvious if we don't worry about $F_t^\tau(v)$ being in R; just extend the homotopy of $f_t^\tau := bs(F_t^\tau)$ & take $F_t^\tau := J^r_{f_t^\tau}$.

⇝ condition (TBA 2): Denote $K^m := [-1,1]^m \subseteq \mathbb{R}^m \times \{0\} \subseteq \mathbb{R}^n$, & for $k<n$, call $B \subseteq A \subseteq V$ a θ_k-pair if $(A,B) \cong (K^n, K^k)$.


$n=2$, $k=1$


$n=2$, $k=0$

(note: Eliashberg-Mishachev says $(K^n, K^k \cup \partial K^n)$ here, but (K^n, K^k) seems to be what they meant.)

$R \subseteq X^{(r)}$ is parametrically k-microflexible if $\forall$ suff small open balls $U \subseteq V$ & families (parametrized by $\rho \in I^m$) of

- θ_k-pairs $(A_\rho, B_\rho) \subseteq U$
- holon. sections $F_\rho^0 : Op\, A_\rho \longrightarrow R$
- holon. homotopies $\{F_\rho^\tau : Op\, B_\rho \longrightarrow R\}_{\tau\in[0,1]}$, τ-indep. on $Op(\partial B_\rho)$ (& everywhere for $\rho \in Op(\partial I^m)$),

$\exists\, \sigma > 0$ & a family of holon. homotopies $\{F_\rho^\tau : Op\, A_\rho \longrightarrow R\}_{\tau\in[0,\sigma]}$ extending them, s.t. τ-indep on $Op(\partial A_\rho)$ (& everywhere for $\rho \in Op(\partial I^m)$).

(TBA 2) := "parametrically k-microflexible $\forall\, k = 0, \dots, n-1$.

remark: I don't see where $k=n-1$ is needed in the pf. of holon. R-approx — does k-microflexibility for $k=0,\dots,n-2$ suffice?

exs: (1) Any open R

(2) $R_{cl} \subseteq (\Lambda^p V)^{(1)}$ is k-flexible (meaning can take $\sigma = 1$) $\forall\, k \neq p$, but not p-microflexible.

pf: (non-parametric case) Given closed p-form ω^0 on $\mathcal{O}p(K^k)$ & homotopy of closed p-forms ω^τ on $\mathcal{O}p(K^k)$ matching ω^0 on $\mathcal{O}p(\partial K^k)$, $\exists$? closed extension ω^τ on $\mathcal{O}p(K^k)$ matching ω^0 on $\mathcal{O}p(\partial K^k)$?

Poincaré lemma: WLOG $\omega^\tau = d\alpha^\tau$ for a homot. of $(p-1)$-forms α^τ s.t. $d\alpha^\tau - d\alpha^0 = 0$ on $\mathcal{O}p(\partial K^k)$, so

$$[\alpha^\tau - \alpha^0] \in H^{p-1}_{dR}(\underset{\substack{\text{\rotatebox{90}{$\cong$}} \\ S^{k-1}}}{\partial K^k}) = \begin{cases} 0 & \text{if } p \neq k \ \& \ p > 1 \\ \mathbb{R} & \text{if } p = k \text{ or } p = 1 \end{cases}$$

In case $p=1$, if $k \neq 1$, S^{k-1} connected $\Rightarrow$ $\alpha^\tau - \alpha^0$ = a const fn on $\mathcal{O}p(\partial K^k)$
$\therefore$ can adjust α^0 by const s.t. $\alpha^\tau = \alpha^0$ on $\mathcal{O}p(\partial K^k)$ WLOG.

If $p > 1$ & $p \neq k$, $\alpha^\tau = \alpha^0 + d\beta^\tau$ for some family $\beta^\tau \in \Omega^{p-2}(\mathcal{O}p(\partial K^k))$,
then can replace α^τ by $\alpha^0 + d((\text{cutoff fn}) \cdot \beta^\tau)$ s.t. WLOG
$\alpha^\tau = \alpha^0$ on $\mathcal{O}p(\partial K^k)$.

Then extend α^τ to $\mathcal{O}p(K^k)$ matching α^0 on $\mathcal{O}p(\partial K^k)$ & set $\omega^\tau := d\alpha^\tau$.

If $p = k$: can construct examples w/ $[\alpha^\tau - \alpha^0] \neq 0 \in H^{k-1}_{dR}(\partial K^k)$ & $\nexists$ extension. ☒

(3) $\mathcal{R}_{Lag}$, $\mathcal{R}_{sub\text{-}isot} \subseteq J^1(V, W)$ on a sympl. mfd (W, ω) are k-microflexible for $k \neq 1$. sketch of pf: By Weinstein nbhd thm, can consider WLOG families of ~~Hopf~~ Lagrangian embeddings near 0-section in T^*V — up to parametrization, these are sections of T^*V, i.e. 1-forms, Lagrangian $\Leftrightarrow$ closed.
$\therefore$ Reduces to (2).

~~pf idea~~ $\xrightarrow{\text{(some effort)}}$ cor: h-principle holds for subcritical isotropic immersions in (W, ω) w/ additional cohomological condition on formal sols: $f: V \to W$ s.t. $[f^*\omega] = 0 \in H^2_{dR}(V)$.

STOPPED HERE, but would have mentioned this if there'd been time:
(4) On cont (W, ξ), $\mathcal{R}_{Leg}$ & $\mathcal{R}_{sub\text{-}isot}$ are microflexible. pf: Nbhd thm $\Rightarrow$ WLOG can consider embeddings near 0-section in $T^*V \times \mathbb{R} = J^1(V, \mathbb{R})$, so Legendrian $\Leftrightarrow$ holonomic.