SOBOLEV SPACES

Consider fns. $\mathbb{R}^n \overset{\text{open}}{\supseteq} \mathcal{U} \xrightarrow{f} \mathbb{C}^m$, let $C_0^\infty(\mathcal{U}) := \{C^\infty$ w/ cpt support in $\mathcal{U}\}$ (usually w/ values in $\mathbb{R}$)

For $f, g \in L^1_{loc}(\mathcal{U})$ & any multi-index α, say

$\partial^\alpha f = g$ weakly if $\forall\, \varphi \in C_0^\infty(\mathcal{U})$, $\int_{\mathcal{U}} g\varphi = (-1)^{|\alpha|}\int_{\mathcal{U}} f\,\partial^\alpha \varphi$.

ex: For $\mathcal{U} = \mathbb{R}$, $\frac{d}{dx}|x| = \operatorname{sgn}(x)$ weakly.

Fix integer $k \geq 0$, $1 \leq p \leq \infty$.

$W^{k,p}(\mathcal{U}) := \{ f \in L^p(\mathcal{U}) \mid \exists$ weak derivs. $\partial^\alpha f \in L^p(\mathcal{U})\ \forall\, |\alpha| \leq k\}$

$\|f\|_{W^{k,p}} := \sum_{|\alpha|\leq k} \|\partial^\alpha f\|_{L^p}$. Case $p=2$: $H^k := W^{k,2}$.
(note: Roe call this "W^k".)

$\rightsquigarrow$ inner product $\langle f, g\rangle_{H^k} := \sum_{|\alpha|\leq k} \langle \partial^\alpha f, \partial^\alpha g\rangle_{L^2}$.

prop: $W^{k,p}(\mathcal{U})$ are Banach spaces, $H^k(\mathcal{U})$ are Hilbert.

e.g. $W^{1,p} \cong \{(f_0, f_1, \dots, f_n) \in \bigoplus_{j=0}^n L^p(\mathcal{U}) \mid \partial_j f_0 = f_j$ weakly $\forall\, j = 1, \dots, n\}$
$=$ closed subspace of $(L^p(\mathcal{U}))^{\oplus(n+1)}$. ☒

prop: $C^\infty(\mathcal{U})$ is dense in $W^{k,p}(\mathcal{U})$. ☒

defn: $W_0^{k,p}(\mathcal{U}) := (W^{k,p}$-closure of $C_0^\infty(\mathcal{U})) \overset{\text{closed subspace}}{\subseteq} W^{k,p}(\mathcal{U})$. Sim: $H_0^k \subseteq H^k$.

sections of bundles: Fix closed n-mfd M & vec. bndl $\pi: E \to M$.
$\mathcal{A}(\pi)$ = "bndl atlas": each $\alpha \in \mathcal{A}(\pi)$ consists of an open $\mathcal{U}_\alpha \subseteq M$, smooth chart $\varphi_\alpha : \mathcal{U}_\alpha \xrightarrow{\cong} \Omega_\alpha \subseteq \mathbb{R}^n$ & smooth local triv. $\Phi_\alpha : E|_{\mathcal{U}_\alpha} \xrightarrow{\cong} \mathcal{U}_\alpha \times \mathbb{C}^m$, $\therefore$ α associates to each section $\eta : M \to E$ its "coord. / local representative" $\eta^\alpha := pr_2 \circ \Phi_\alpha \circ \eta \circ \varphi_\alpha^{-1} : \Omega_\alpha \to \mathbb{C}^m$.

defn: $W^{k,p}(E) := \{$sections $\eta : M \to E \mid \eta^\alpha \in W^{k,p}_{loc}(\Omega_\alpha)\ \forall\, \alpha \in \mathcal{A}(\pi)\}$
i.e. in $W^{k,p}$ on all open subsets w/ cpt closure in Ω_α.

Say $\alpha \in \mathcal{A}(\pi)$ is precompact if
$\exists\, \alpha' \in \mathcal{A}(\pi)$ & a cpt set $K \subseteq M$ s.t. $\mathcal{U}_\alpha \subseteq K \subseteq \mathcal{U}_{\alpha'}$ &
$\varphi_\alpha, \Phi_\alpha$ are restrictions of $\varphi_{\alpha'}, \Phi_{\alpha'}$.

M cpt $\Rightarrow$ can choose finite collection $I \subseteq \mathcal{A}(\pi)$ of precompact charts s.t. $M = \bigcup_{\alpha\in I} \mathcal{U}_\alpha$. Then can define a norm on $W^{k,p}(E)$ by
$\|\eta\|_{W^{k,p}} := \sum_{\alpha \in I} \|\eta^\alpha\|_{W^{k,p}(\Omega_\alpha)}$.

lemma: For any other precpt chart $\beta \in \mathcal{A}(\pi)$, $\exists\, c > 0$ s.t.
$\|\eta^\beta\|_{W^{k,p}(\Omega_\beta)} \leq c \sum_{\alpha\in I} \|\eta^\alpha\|_{W^{k,p}(\Omega_\alpha)}$. ☒

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$\Rightarrow$ prop: $W^{k,p}(E)$ is a Banach space & its norm is indep. of choices up to equivalence. Can similarly defn. $H^k(E)$ as a Hilbert space. $\boxtimes$

prop: Smooth sections $\Gamma(E)$ are dense in $W^{k,p}(E)$. $\boxtimes$

rk: A partial diff. op. $D: \Gamma(E) \to \Gamma(F)$ of order k over a closed mfd M extend to a bdd linear op. $W^{m+k,p}(E) \to W^{m,p}(F)$ $\forall$ $m \geq 0$, $p \in [1,\infty]$.

Sobolev embedding thm: If $k \in \mathbb{N}$ & $p \in [1,\infty)$ satisfy $kp > n$, then all $\eta \in W^{k,p}(E)$ are continuous & the inclusion $W^{k,p}(E) \hookrightarrow C^0(E)$ is a bdd. & cpt lin. operator.

(Recall: X, Y = Banach spaces, $\mathcal{L}(X,Y) := \{$bdd lin. ops. $X \to Y\} \ni T$ is called compact if $\forall$ bdd seqs. $x_n \in X$, $Tx_n \in Y$ has a conv. subseq.)

Rellich-Kondrachov cptness thm: For $k \in \mathbb{N}$ & $p \in [1,\infty)$, the natural inclusion $W^{k,p}(E) \hookrightarrow W^{k-1,p}(E)$ is cpt.

Fourier description of $H^k(\mathbb{R}^n)$: $d^n x :=$ Lebesgue measure for fns. of $x \in \mathbb{R}^n$.

Fourier transform $\mathcal{F}: L^1(\mathbb{R}^n) \to C^0(\mathbb{R}^n)$

$$\mathcal{F}f(p) := \hat{f}(p) := \int_{\mathbb{R}^n} f(x)\, e^{-2\pi i x \cdot p}\, d^n x$$

$$\mathcal{F}^* f(x) := \check{f}(x) := \int_{\mathbb{R}^n} f(p)\, e^{2\pi i p \cdot x}\, d^n p$$

Plancherel's thm: $\mathcal{F}$ & $\mathcal{F}^*$ extend to isometries $L^2(\mathbb{R}^n) \circlearrowleft$ & are inverses. $\boxtimes$

For suff. smooth fns, $f(x) = \int_{\mathbb{R}^n} \hat{f}(p)\, e^{2\pi i p \cdot x}\, d^n p \Longrightarrow$

$$\partial_j f(x) = \int_{\mathbb{R}^n} 2\pi i p_j\, \hat{f}(p)\, e^{2\pi i p\cdot x}\, d^n p \implies \widehat{\partial_j f}(p) = 2\pi i p_j\, \hat{f}(p).$$

(Plancherel)

$$\Longrightarrow \text{ e.g. } \|f\|_{H^1(\mathbb{R}^n)} = \|\hat{f}\|_{L^2} + c\sum_{j=1}^{n} \| |p_j| \cdot \hat{f} \|_{L^2} \text{ etc} \ldots$$

$$\Rightarrow H^k(\mathbb{R}^n) = \Big\{ f \in L^2(\mathbb{R}^n) \ \Big|\ \underbrace{\int_{\mathbb{R}^n} (1+|p|^2)^k\, |\hat{f}(p)|^2\, d^n p}_{\text{equivalent defn. of } \|f\|^2_{H^k}} < \infty \Big\}$$

Pf of the Sobolev embedding $H^k(\mathbb{R}^n) \hookrightarrow C^0(\mathbb{R}^n)$: Given $2k > n$, need to prove $\|\hat{f}\|_{L^1} \leq c\,\|f\|_{H^k}$ (sufficient since $\mathcal{F}^*: L^1 \to C^0$ is bdd).

Observe: $2k > n \iff \int_{\mathbb{R}^n} \frac{d^n p}{(1+|p|^2)^k} < \infty$, so use Cauchy-Schwartz:

$$\|\hat{f}\|_{L^1} = \int_{\mathbb{R}^n} \frac{1}{(1+|p|^2)^{k/2}} \left[(1+|p|^2)^{k/2} |\hat{f}(p)|\right] d^n p \leq \underbrace{\left(\int_{\mathbb{R}^n} \frac{d^n p}{(1+|p|^2)^k}\right)^{1/2}}_{= \text{const} < \infty} \cdot \|f\|_{H^k}.$$

$\boxtimes$

Pf of Rellich-Kondrachov cptness for $H^k(\mathbb{T}^n) \hookrightarrow H^{k-1}(\mathbb{T}^n)$ (requires a cpct domain)

Fourier series: for $f : \underset{\mathbb{R}^n/\mathbb{Z}^n}{\overset{\shortparallel}{\mathbb{T}^n}} \to \mathbb{C}$, $f(x) = \sum_{p \in \mathbb{Z}^n} \hat{f}_p e^{2\pi i p \cdot x}$ where

$\hat{f}_p = \int_{\mathbb{T}^n} f(x) e^{-2\pi i x \cdot p} d^n x$, so $\partial_j f(x) = \sum_p 2\pi i p_j \hat{f}_p e^{2\pi i p \cdot x}$

Parseval: $\langle f, g \rangle_{L^2} = \sum_p \langle \hat{f}_p, \hat{g}_p \rangle \Rightarrow$ can also equivalently defn H^k-norm as $\|f\|^2_{H^k(\mathbb{T}^n)} := \sum_{p \in \mathbb{Z}^n} (1 + |p|^2)^k |\hat{f}_p|^2$.

Need to show: $B := \{ f \in H^k \mid \|f\|_{H^k} \le 1 \}$ has cpct closure in $H^{k-1}(\mathbb{T}^n)$.

follows from claim: $\forall \varepsilon > 0$, $\exists$ finite set of ε-balls in H^{k-1} that cover B.

Choose $N > 0$ & let $Z_N := \{ f \in H^k(\mathbb{T}^n) \mid \hat{f}_p = 0 \ \forall \ |p| \le N \}$

$Y_N := \{ f \in H^k(\mathbb{T}^n) \mid \hat{f}_p = 0 \ \forall \ |p| > N \}$.

Then $\dim Y_N < \infty \Rightarrow Y_N \cap B$ cpct.

For $f \in Z_N \cap B$, $\|f\|^2_{H^{k-1}} = \sum_{|p| > N} (1 + |p|^2)^{k-1} |\hat{f}_p|^2 < \frac{1}{1+N^2} \sum_{|p|>N} (1+|p|^2)^k |\hat{f}_p| \le \frac{1}{1+N^2}$

$\therefore$ Can cover $Y_N \cap B$ w/ fin.-many ε-balls, & if $N \gg 1$, these also cover B (since $B \cap Z_N$ is arbitrarily H^{k-1}-small). $\boxtimes$

general case for $p = 2$ on a bndl $E \to M$: Choose P.O.U. $\{ \rho_\alpha : U_\alpha \to [0,1] \}_{\alpha \in I}$ & apply $\mathbb{T}^n$ result to the local reps. $(\rho_\alpha \eta)^\alpha : \Omega_\alpha \to \mathbb{C}^m$ after embedding $\Omega_\alpha \hookrightarrow \mathbb{T}^n$.

Fredholm operators: X, Y Banach, $T \in \mathcal{L}(X,Y)$ is Fredholm if $\ker T$ & $\operatorname{coker} T := Y/\operatorname{im} T$ are fin.-dim. $\operatorname{ind}(T) := \dim \ker T - \dim \operatorname{coker} T$.

Lemma: If $T : X \to Y$ is Fredholm & $K : X \to Y$ is cpct, then $T + K$ is also Fredholm. ($\Rightarrow$ so is $T + sK$ $\forall s \in \mathbb{R}$ & $\operatorname{ind}(T + sK) = \operatorname{ind}(T)$.)

Cor: If $D : H^{m+k}(E) \to H^m(F)$ is a Fredholm differential op. of order k & A is a diff. op. of order $< k$ from E to F, then

$D + A : H^{m+k}(E) \to H^m(F)$ is also Fredholm w/ same index as D.

Pf: A has order $l < k \Rightarrow H^{m+k} \xrightarrow{\text{cpt}} H^{m+l} \xrightarrow[\text{bdd}]{A} H^m \Rightarrow H^{m+k} \xrightarrow{A} H^m$ is cpt. $\boxtimes$

ex: If $\operatorname{ind}(D) = 0$, let A = multiplication by $-\lambda \in \mathbb{C}$, then $D : H^k \to L^2$ perturbs to $D - \lambda : H^k \to L^2$ & still has index 0, $\therefore$ inj $\Leftrightarrow$ surj. Regard D as an unbdd lin. op. on L^2 w/ dense domain $H^k \subseteq L^2$; its spectrum is $\operatorname{spec}(D) = \{ \lambda \in \mathbb{C} \mid D - \lambda$ is not surj. w/ bdd inverse $(D-\lambda)^{-1} : L^2 \to L^2 \}$.

This proves: $\operatorname{spec}(D) = \{ \lambda \in \mathbb{C} \mid \ker(D - \lambda) \neq 0 \} = \{$eigenvalues of $D\}$.