Traces and eigenvalue asymptotics

Let M be closed, oriented, Riemannian with Laplacian Δ on $L^2(M)$.

Last talk: $e^{-t\Delta} : L^2(M) \longrightarrow L^2(M)$ is a smoothing operator:

$$\forall f \in L^2(M): \; e^{-t\Delta} f(p) = \int_M k_t(p,q)\, f(q) \, \mathrm{vol}_q$$

with k_t smooth in p,q (heat kernel)

Today: Study the Spectrum of Δ: $0 \le \lambda_1 \le \lambda_2 \le \dots$

use $\mathrm{tr}(e^{-t\Delta})$; will show: $\mathrm{tr}(e^{-t\Delta}) = \sum_i e^{-t\lambda_i}$ & $\mathrm{tr}(e^{-t\Delta}) = \int_M k_t(p,p)\, \mathrm{vol}_p$

$\Rightarrow$ can relate the Spectrum of Δ to the heat kernel k_t

Remark: Most results can be generalized for D^2 with D = Dirac-operator on Clifford bundle

1. Trace class operators

Let H, H' be separable, $\dim = \infty$ Hilbert space with ONBs (e_i) & (e_j')

$A: H \to H'$ linear bounded $\rightsquigarrow c_{ij}(A) := \langle Ae_i, e_j' \rangle$ "infinite matrix of coefficients"

Def: $\|A\|_{HS}^2 := \sum_{i,j} |c_{ij}(A)|^2 \in [0;\infty]$ is the Hilbert-Schmidt norm of A.

Prop: $\|A\|_{HS}$ is independent of (e_i) & (e_j').

Proof: Parseval: $\|A\|_{HS}^2 = \sum_{i,j} |\langle Ae_i, e_j'\rangle|^2 = \sum_i \|Ae_i\|^2$

$\Rightarrow$ independent of (e_j')

use $c_{ij}(A) = \overline{c_{ji}}(A^*)$, hence $\|A\|_{HS} = \|A^*\|_{HS} \Rightarrow$ independent of (e_i) $\square$

Def: A is a Hilbert-Schmidt operator $:\Leftrightarrow \|A\|_{HS} < \infty$

Prop: Properties of Hilbert-Schmidt operators

(i) The inner product $\langle A, B\rangle_{HS} := \sum_{i,j} \overline{c_{ij}}(A)\, c_{ij}(B)$ induces $\|\cdot\|_{HS}$

(ii) The Hilbert-Schmidt operators form a Hilbert space.

(iii) $\|A\|_{\mathcal{L}(H,H')} \le \|A\|_{HS}$ $\quad$ ($\|\cdot\|_{\mathcal{L}(H,H')}$ = operator norm)

(iv) Hilbert-Schmidt $\Rightarrow$ compact

(v) A bounded, B Hilbert-Schmidt $\Rightarrow AB, BA$ Hilbert-Schmidt

Proof: (iii) $\|Av\| = \|\sum_i \langle v, e_i\rangle Ae_i\| \le \sum_i |\langle v, e_i\rangle| \cdot \|Ae_i\| \overset{CSI}{\le} \|v\| \cdot \|A\|_{HS}$

(ii) follows from (iii): every $\|\cdot\|_{HS}$-Cauchy sequence is a $\|\cdot\|_{\mathcal{L}}$-Cauchy sequence, hence converges

(iv) $A_n : H \to H'$, $v \mapsto \sum_{i=1}^{n} \langle v, e_i \rangle A e_i$ $\dim \operatorname{im} A_n = n < \infty$

Then $\|A - A_n\|_{\mathcal{L}} \le \|A - A_n\|_{HS} = \left(\sum_{i>n} \|Ae_i\|^2 \right)^{\frac{1}{2}} \longrightarrow 0$

$\Rightarrow A_n \xrightarrow{n\to\infty} A$ in $\mathcal{L}(H, H')$, i.e. A is compact

(v) $\|BA\|_{HS}^2 = \sum_i \|BAe_i\|^2 \le \|B\|_{\mathcal{L}}^2 \|A\|_{HS}^2 < \infty$ and $\|AB\|_{HS} = \|(AB)^*\|_{HS} = \|B^* A^*\|_{HS} < \infty$ □

Def: $T : H \to H$ is of trace class $:\Leftrightarrow T = AB$ with $A, B : H \to H$ HS-operators.

$\operatorname{Tr}(T) := \langle A^* ; B \rangle_{HS}$

Independent of choice of A, B:

$\operatorname{Tr}(T) = \sum_{i,j} \bar{c}_{ij}(A^*) c_{ij}(B) = \sum_{i,j} c_{ji}(A) c_{ij}(B) = \sum_j c_{jj}(T)$ (*)

Remark: Trace class $\subseteq$ Hilbert-Schmidt $\subseteq$ compact $\subseteq$ bounded operators

$\hat{=}$ $\quad \ell^1 \subseteq \ell^2 \subseteq c_0 \subseteq \ell^\infty$ (sequence spaces)

(see also Simon, Trace ideals and ~~related~~ their applications)

Prop: T trace class and self-adjoint $\Rightarrow \operatorname{Tr}(T) = \sum_{\lambda \in \operatorname{Spec}(T)} \lambda$

Proof: T compact and self-adjoint $\Rightarrow \exists$ ONB of eigenvectors, use formula * □

Remark: Still holds if T is not self-adjoint; "Lidskii's theorem"

Prop: T trace class, $B : H \to H$ bounded $\Rightarrow BT, TB$ trace class

and $\operatorname{Tr}(TB) = \operatorname{Tr}(BT)$.

Proof: product of HS and bounded operator is again HS $\rightsquigarrow$ 1st statement

$$\operatorname{Tr}(TB) \overset{*}{=} \sum_j \langle TBe_j, e_j \rangle = \sum_j \langle Be_j, T^* e_j \rangle = \underbrace{\sum_j \sum_i \bar{c}_{ji}(B) \underset{\bar{c}_{ji}(T)}{c_{ij}(T^*)}}_{\text{symmetric in } B, T} \Rightarrow = \operatorname{Tr}(BT) \; □$$

Integral operators on $L^2(M)$

Prop: For $k \in C^0(M \times M)$ define $A : L^2(M) \to L^2(M)$, $Au(p) = \int_M k(p,q) u(q) \operatorname{vol} q$.

A is a Hilbert-Schmidt operator with $\|A\|_{HS}^2 = \int_M \int_M |k(p,q)|^2 \operatorname{vol} q \operatorname{vol} p$

Proof: $\|A\|_{HS}^2 = \sum_i \|Ae_i\|^2 = \sum_i \int_M |Ae_i(p)|^2 \operatorname{vol} p$

$$= \sum_i \int_M \underbrace{\left| \int_M k(p,q) e_i(q) \operatorname{vol} q \right|}_{|\langle k(p,\cdot), e_i \rangle|^2} \operatorname{vol} p$$

(Parseval)

$$= \int_M \|k(p,\cdot)\|_{L^2}^2 \operatorname{vol} p = \int_M \int_M |k(p,q)|^2 \operatorname{vol} q \operatorname{vol} p < \infty$$

Theorem: Now let $k \in C^\infty(M \times M)$, A as before.

$\Rightarrow A$ is trace class with $Tr(A) = \int_M k(p,p) \, vol\, p$.

Proof: 1.) $A = BC$ with (cont.) integral operators B, C on $L^2(M)$:

Take $f(x) = (1+x^2)^{-N}$, for N suff. big: $f(D) = (1+\Delta)^{-N}$ has a continuous kernel (with $D^2 = \Delta$), see talk 8.1 (functional calculus).

$\Rightarrow$ Take $B = (1+\Delta)^{-N}$ and $C = (1+\Delta)^N A$ (smoothing operator $\Rightarrow$ HS)

2.) Compute $Tr(A)$: $A = BC \rightarrow k_A(p,r) = \int_M k_B(p,q) k_C(q,r) \, vol\, q$

$$Tr(A) = \langle B^*; C \rangle_{HS} = \int_M \int_M k_B(p,q) k_C(q,p) \, vol\, q \, vol\, p = \int_M k_A(p,p) \, vol\, p \quad \square$$

Theorem: Let S be a Clifford bundle, $k \in \Gamma(S \boxtimes S^*)$ (i.e. $k(p,q) \in Hom(S_q, S_p)$)

$A: L^2(S) \rightarrow L^2(S)$, $As(p) = \int_M k(p,q) s(q) \, vol\, q$

is trace class with $Tr(A) = \int_M tr\, k(p,p) \, vol\, p$

[with $tr: Hom(S_p, S_p) \rightarrow \mathbb{C}$ = trace of endomorphisms on S_p].

Proof: trivializations & partition of unity & last theorem $\square$

2. Weyl's asymptotic formula

goal: relate Spectrum of Δ to geometry of M [can replace Δ by D^2 for a Dirac operator on $S \rightarrow M$]

$$Tr(e^{-t\Delta}) = \sum \text{eigenvalues of } e^{-t\Delta} = \sum_i e^{-t\lambda_i}$$
$$= \int_M k_t(p,p) \, vol\, p$$

asymptotic expansion of k_t ($\rightarrow$ talk 8)

$$h_t(p,q) := (4\pi t)^{-\frac{n}{2}} \exp\left(\frac{d(p,q)^2}{4t}\right)$$

$$k_t(p,q) \sim h_t(p,q) \sum_{j=1}^{\infty} \Theta_j(p,q) t^j \quad \text{with } \Theta_j \in C^\infty(M \times M) \quad [\Theta_j \in \Gamma(S \boxtimes S^*)]$$

$$\Rightarrow Tr(e^{-t\Delta}) = \sum_i e^{-t\lambda_i} \sim \int_M h_t(p,p) \sum_{j=1}^{\infty} \Theta_j(p,p) t^j \, vol\, p$$
$$= (4\pi t)^{-\frac{n}{2}} \sum_{j=1}^{\infty} t^j \underbrace{\int_M \Theta_j(p,p) \, vol\, p}_{a_j}$$

- $\Theta_0(p,p) = 1 \Rightarrow a_0 = vol\, M$ [for D: $\Theta_0(p,p) = \underline{1} \in Hom(S_p, S_p) \rightarrow a_0 = rk S \cdot vol\, M$]
- $\Theta_1(p,p) = \frac{1}{6} K(p) \Rightarrow a_1 = \frac{1}{6} \int K(p) \, vol\, p = \frac{1}{6} \cdot$ total scalar curvature

Prop: (Spectrum $\rightsquigarrow$ Geometry)

$Spec(\Delta)$ determines $n = \dim M$, $vol\, M$ and $\int_M \mathbb{R}$ = total scalar curvature.

If $n=2$, this determines the topology of M.

Converse direction: Geometry $\rightsquigarrow$ Spectrum

Let $\mathcal{N}(\lambda) := \#$ Eigenvalues $\leq \lambda$

goal: asymptotic (for $\lambda \to \infty$) estimate of $\mathcal{N}(\lambda)$.

Karamata's theorem

$$t^\alpha \sum_j e^{-t\lambda_j} \xrightarrow{t \to 0} A \quad \text{with } \alpha, A > 0 \quad \Rightarrow \mathcal{N}(\lambda) \sim A\lambda^\alpha \frac{1}{\Gamma(\alpha+1)} \text{ as } \lambda \to \infty$$

(holds for any counting function $\mathcal{N}(\lambda)$ of a sequence $0 \leq \lambda_1 \leq \lambda_2 \leq \dots$)

Proof: To show: $\lim\limits_{\lambda \to \infty} \frac{\mathcal{N}(\lambda)}{\lambda^\alpha} = \frac{A}{\Gamma(\alpha+1)}$.

Step1: For continuous $f: [0;1] \to \mathbb{C}$ let $\varphi_f(t) := \sum_j f(e^{-t\lambda_j}) e^{-t\lambda_j}$.

Then $\lim\limits_{t \to 0} t^\alpha \varphi_f(t) = \frac{A}{\Gamma(\alpha)} \int_0^\infty f(e^{-s}) s^{\alpha-1} e^{-s} \, ds$.

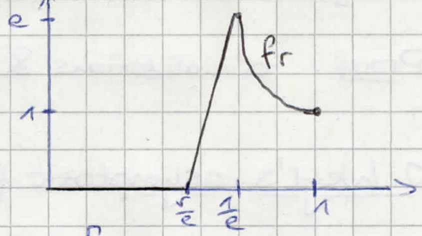
Suffices to check for $f = t^n$ (polynomials are dense in $C^0([0;1])$).

$$t^\alpha \varphi_f(t) = t^\alpha \sum_j e^{-n\lambda_j t} e^{-\lambda_j t} = t^\alpha \sum_j e^{-(n+1)t\lambda_j} = (n+1)^{-\alpha} ((n+1)t)^\alpha \sum_j e^{-(n+1)t\lambda_j} \longrightarrow (n+1)^{-\alpha} A$$

$$\frac{A}{\Gamma(\alpha)} \int_0^\infty f(e^{-s}) s^{\alpha-1} e^{-s} \, ds = \frac{A}{\Gamma(\alpha)} \int_0^\infty e^{-(n+1)s} s^{\alpha-1} \, ds \overset{t=(n+1)s}{=} \frac{A}{\Gamma(\alpha)} \int_0^\infty e^{-t} (n+1)^{-(\alpha-1)} t^{\alpha-1} (n+1)^{-1} \, dt$$

$$= \frac{A}{\Gamma(\alpha)} (n+1)^{-\alpha} \Gamma(\alpha) = A(n+1)^{-\alpha}$$

Step2: $(r<1)$ Take
$$f_r(x) = \begin{cases} 0 & x \in [0; \frac{r}{e}] \\ \text{linear interpolation} & x \in [\frac{r}{e}; \frac{1}{e}] \\ \frac{1}{x} & x \in [\frac{1}{e}; 1] \end{cases}$$



$$f_r(e^{-\frac{\lambda_j}{\lambda}}) = \begin{cases} \geq 0 & , e^{-\frac{\lambda_j}{\lambda}} < \frac{1}{e} \Leftrightarrow \lambda_j > \lambda \\ e^{\frac{\lambda_j}{\lambda}} & , e^{-\frac{\lambda_j}{\lambda}} \geq \frac{1}{e} \Leftrightarrow \lambda_j \leq \lambda \end{cases} \Rightarrow f_r(e^{-\frac{\lambda_j}{\lambda}}) e^{-\frac{\lambda_j}{\lambda}} = \begin{cases} \geq 0 & , \lambda_j > \lambda \\ 1 & , \lambda_j \leq \lambda \end{cases}$$

$$\Rightarrow \varphi_{f_r}(\tfrac{1}{\lambda}) \geq \mathcal{N}(\lambda) \text{ and } \lambda^\alpha \varphi_{f_r}(\tfrac{1}{\lambda}) \xrightarrow[\text{Step1}]{\lambda \to \infty} \frac{A}{\Gamma(\alpha)} \int_0^\infty f_r(e^{-s}) s^{\alpha-1} e^{-s} \, ds$$
$$\leq \frac{A}{\Gamma(\alpha)} \int_0^1 e^s s^{\alpha-1} e^{-s} \, ds = \frac{A}{\Gamma(\alpha) \cdot \alpha} = \frac{A}{\Gamma(\alpha+1)}$$

$$\Rightarrow \limsup_{\lambda \to \infty} \lambda^{-\alpha} \mathcal{N}(\lambda) \leq \frac{A}{\Gamma(\alpha+1)}$$

- similarly: $\varphi_{f_r}(\frac{1}{r\lambda}) \leq \mathcal{N}(\lambda)$ and $\lim\limits_{\lambda \to \infty} \lambda^{-\alpha} \varphi_{f_r}(\frac{1}{r\lambda}) \geq \frac{Ar^\alpha}{\Gamma(\alpha+1)}$

$$\Rightarrow \liminf \lambda^{-\alpha} \mathcal{N}(\lambda) \geq \frac{Ar^\alpha}{\Gamma(\alpha+1)}$$

$r < 1$ arbitrary $\Rightarrow$ $\lim\limits_{\lambda \to \infty} \lambda^{-\alpha} \mathcal{N}(\lambda) = \frac{A}{\Gamma(\alpha+1)}$

Theorem: ($\mathcal{N}$ is now counting function of Spec(Δ))

$$\mathcal{N}(\lambda) \sim \frac{1}{(4\pi)^{\frac{n}{2}} \Gamma(\frac{n}{2}+1)} \operatorname{vol}(M) \lambda^{\frac{n}{2}} \text{ as } \lambda \to \infty$$

Proof: asymptotic expansion: $t^{\frac{n}{2}} \sum_j e^{-t\lambda_j} \xrightarrow{t \to 0} \frac{1}{(4\pi)^{\frac{n}{2}}} \overset{\operatorname{vol} M}{\overset{\shortparallel}{a_0}}$

$\Rightarrow \alpha = \frac{n}{2}$, $A = \frac{1}{(4\pi)^{\frac{n}{2}}} \operatorname{vol} M$