Fourier series

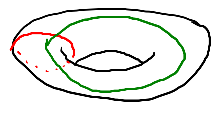
We consider (fully) periodic fns. on $\mathbb{R}^n$:

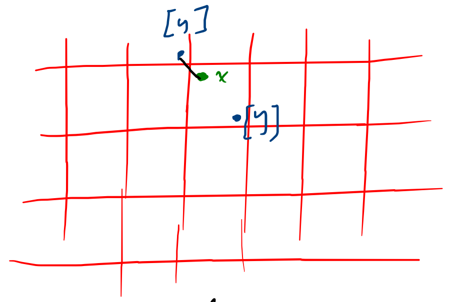
$$f(x_1,\dots,x_j,\dots,x_n) = f(x_1,\dots,x_j+1,\dots,x_n)$$

equivalently: f is def'd on the n-torus $\mathbb{T}^n := \mathbb{R}^n/\mathbb{Z}^n = (\underbrace{\mathbb{R}/\mathbb{Z}}_{=: S^1 \text{ "circle"}})^n$

$\mathbb{T}^n$ is a compact metric space with

$$d([x],[y]) := \inf\{|x'-y'| \,;\, x'\in[x],\ y'\in[y]\}$$

$\mathbb{T}^2$ 



$\mathbb{R}^2/\mathbb{Z}^2$

compact since it is the image of $[0,1]^n$ under projection $\mathbb{R}^n \longrightarrow \mathbb{R}^n/\mathbb{Z}^n : x \longmapsto [x]$.

It also is a finite measure space where $A\subseteq \mathbb{T}^n$ has measure

$$m(A) := m\big(\underbrace{\pi^{-1}(A)\cap[0,1]^n}_{\subseteq\mathbb{R}^n}\big), \quad \text{so } m(\mathbb{T}^n) = m([0,1]^n) = 1.$$

$\uparrow$ Leb. measure on $\mathbb{R}^n$

exs of fully per. fns: $x\longmapsto \sin(2\pi k x_j)$ for $k\in\mathbb{N}$, $j\in\{1,\dots,n\}$

$x\longmapsto\cos(2\pi k x_j)$ for integers $k\geq 0$,

all products of these

Q: Can per. fn. on $\mathbb{R}^n$ be written as convergent infinite linear combis. of products of these sin & cos. fns?

computational shortcut: sin & cos fns are cpx lin. combins. of $e^{2\pi i k x_j}$ for $k\in\mathbb{Z}$.

notice: product $e^{2\pi i k_1 x_1}\cdots e^{2\pi i k_n x_n} = e^{2\pi i k\cdot x}$

for $k=(k_1,\dots,k_n)\in\mathbb{Z}^n$, $k\cdot x := \sum_j k_j x_j$ (Euclidean inner product)

observation: For $\mathbb{C}$-val'd fns. on $\mathbb{T}^n$, defn. $\langle f,g\rangle_{L^2} := \int_{\mathbb{T}^n} \overline{f(x)}\, g(x)\,dx$

$\rightsquigarrow$ $L^2(\mathbb{T}^n)$ is a Hilbert space.

The fns. $\varphi_k(x) := e^{2\pi i k\cdot x}$ for $k\in\mathbb{Z}^n$ are an orthonormal set:

$$\langle\varphi_k,\varphi_{k'}\rangle_{L^2} = \begin{cases}1 & \text{if } k=k'\\ 0 & \text{if } k\neq k'\end{cases}.$$

thm: $\{\varphi_k\}_{k\in\mathbb{Z}^n}$ is an O-N basis of $L^2(\mathbb{T}^n)$.

cor: Every $f\in L^2(\mathbb{T}^n)$ can be written as an L^2-convergent series

$f(x) = \sum_{k\in\mathbb{Z}^n} e^{2\pi i k\cdot x}\hat f_k$ (the Fourier series of f), with Fourier coefficients

$$\hat f_k = \langle\varphi_k, f\rangle_{L^2} = \int_{\mathbb{T}^n} e^{-2\pi i k\cdot x} f(x)\,dx.$$

function spaces:

$C^0(\mathbb{T}^n) \supseteq C^1(\mathbb{T}^n) \supseteq C^2(\mathbb{T}^n) \supseteq \dots \supseteq C^\infty(\mathbb{T}^n)$

$C^k(\mathbb{T}^n) := \{$fully periodic C^k-fns. on $\mathbb{R}^n\}$

$\Rightarrow$ for $f \in C^k(\mathbb{T}^n)$ & $|\alpha| \leq k$, $\partial^\alpha f$ is a contin. per. fn., i.e.

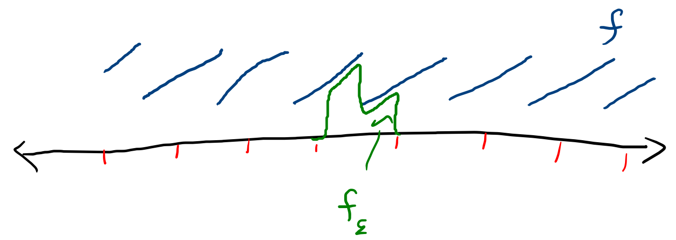
a contin. fn. on $\mathbb{T}^n$ $\Rightarrow$ always bdd (since $\mathbb{T}^n$ cpt), $\|f\|_{C^k} < \infty$ always!

prop: $C^\infty(\mathbb{T}^n)$ is dense in $L^p(\mathbb{T}^n)$ $\forall\ p \in [1, \infty)$.

pf: Given $f \in L^p(\mathbb{T}^n)$, i.e. f is periodic on $\mathbb{R}^n$ s.t. $\displaystyle\int_{[0,1]^n} |f|^p\, dm < \infty$.

Def $\tilde{f} \in L^p(\mathbb{R}^n)$ s.t. $\tilde{f} = \begin{cases} f & \text{on } (0,1)^n \\ 0 & \text{everywhere else.} \end{cases}$

Given $\varepsilon > 0$, choose $f_\varepsilon \in C_0^\infty((0,1)^n)$ s.t. $\|\tilde{f} - f_\varepsilon\|_{L^p(\mathbb{R}^n)} < \varepsilon$



f_ε has a unique extension to a per. C^∞-fn on $\mathbb{R}^n$ $\Rightarrow$ can regard as living in $C^\infty(\mathbb{T}^n)$, $\|f - f_\varepsilon\|_{L^p(\mathbb{T}^n)} < \varepsilon$.

spaces of coefficients (i.e. fns. $g : \mathbb{Z}^n \longrightarrow \mathbb{C}$ or $g : \mathbb{Z}^n \longrightarrow V$) $\boxed{E}$

Fix a fin-dim. cpx inner product space $(V, \langle\ ,\ \rangle)$.

Let $\nu :=$ counting measure on $\mathbb{Z}^n$, so

$\ell^p(\mathbb{Z}^n) := L^p(\mathbb{Z}^n, \nu) = \{ f : \mathbb{Z}^n \longrightarrow V \mid \sum_{k \in \mathbb{Z}^n} |f(k)|^p < \infty \}$ $\quad p < \infty$

$\ell^\infty(\mathbb{Z}^n) = \{$ bdd fns. $f : \mathbb{Z}^n \longrightarrow V \}$

$\ell^2(\mathbb{Z}^n)$ is a Hilbert space with $\langle f, g \rangle_{\ell^2} := \sum_{k \in \mathbb{Z}^n} \langle f(k), g(k) \rangle$

defn: $\mathcal{S}(\mathbb{Z}^n) := \{ f : \mathbb{Z}^n \longrightarrow V \mid \forall$ polynomial fns. $P : \mathbb{R}^n \longrightarrow \mathbb{R}$, the fn. $\mathbb{Z}^n \longrightarrow V : k \longmapsto P(k) f(k)$ is bdd

$\Leftrightarrow \forall\ m \in \mathbb{N},\ \exists\ C > 0$ (dep. on m) s.t. $|f(k)| \leq \dfrac{C}{|k|^m}$ $\forall\ k \in \mathbb{Z}^n$ $\}$

We say $f \in \mathcal{S}(\mathbb{Z}^n)$ is *rapidly decreasing*.

ex: $f \in \mathcal{S}(\mathbb{Z}^n)$ whenever f has bdd support.

ex: $f(k) = e^{-|k|}$ is in $\mathcal{S}(\mathbb{Z}^n)$.

EX: $\mathcal{S}(\mathbb{Z}^n) \subseteq \ell^p(\mathbb{Z}^n)$ $\forall\ p \in [1, \infty]$.

EX: $\mathcal{S}(\mathbb{Z}^n)$ is dense in $\ell^p(\mathbb{Z}^n)$ $\forall\ p \in [1, \infty)$.

For fns. $f : \mathbb{T}^n \longrightarrow V$, defn. $\mathcal{F} f = \hat{f} : \mathbb{Z}^n \longrightarrow V$ by

$$\hat{f}_k := \hat{f}(k) = \int_{\mathbb{T}^n} e^{-2\pi i k \cdot x} f(x)\, dx.$$

This is def'd $\forall\ f \in L^1(\mathbb{T}^n)$

& $|\hat{f}_k| \le \|f\|_{L^1} \implies \mathcal{F}$ is a bdd linear op. $L^1(\mathbb{T}^n) \longrightarrow \ell^\infty(\mathbb{Z}^n)$.

For $g : \mathbb{Z}^n \longrightarrow V$, defn. $\mathcal{F}^* g = \check{g} : \mathbb{T}^n \longrightarrow V$ by

$$\check{g}(x) := \sum_{k \in \mathbb{Z}^n} e^{2\pi i k \cdot x} g_k \qquad (g_k := g(k)).$$

This is well-def'd if

(1) $g \in \ell^2(\mathbb{Z}^n) \implies \sum_k |g_k|^2 < \infty$, orthonormality $\implies$ sum converges in L^2 to a fn. $\check{g} \in L^2(\mathbb{T}^n)$.

(2) $g \in \ell^1(\mathbb{Z}^n) \implies$ sum converges absolutely & unif. $\implies \check{g}$ is in $C^0(\mathbb{T}^n)$ s.t.

$$|\check{g}(x)| \le \sum_k |g_k| = \|g\|_{\ell^1}$$

$\implies \mathcal{F}^*$ is a bdd lin. op. $\ell^1(\mathbb{Z}^n) \longrightarrow C^0(\mathbb{T}^n)$.

thm 1: $\mathcal{F}$ & $\mathcal{F}^*$ define bijection $C^\infty(\mathbb{T}^n) \underset{\mathcal{F}^*}{\overset{\mathcal{F}}{\rightleftarrows}} \mathcal{S}(\mathbb{Z}^n)$ inverse to each other.

Moreover, $\forall\ f \in C^\infty(\mathbb{T}^n)$, the Fourier series $\sum_k e^{2\pi i k \cdot x} \hat{f}_k$ converges absolutely & unif. w/ all derivs. to the fn. f (i.e. it converges in C^∞).

thm 2 (Parseval's identity): $\forall\ f, g \in C^\infty(\mathbb{T}^n)$, $\langle \hat{f}, \hat{g} \rangle_{\ell^2} = \langle f, g \rangle_{L^2}$.

cor (by density): $\mathcal{F}$ & $\mathcal{F}^*$ are inverse unitary isomorphisms

$$L^2(\mathbb{T}^n) \underset{\mathcal{F}^*}{\overset{\mathcal{F}}{\rightleftarrows}} \ell^2(\mathbb{Z}^n).$$

Since $f \in L^2(\mathbb{T}^n)$ satisfies $f = \mathcal{F}^* \hat{f}$, this proves the O-N set $\{e^{2\pi i k \cdot x}\}_{k \in \mathbb{Z}^n}$ is a basis of $L^2(\mathbb{T}^n)$.

Caution: For $f \in L^2(\mathbb{T}^n) \setminus C^\infty(\mathbb{T}^n)$, we do not claim $\sum_k e^{2\pi i k \cdot x} \hat{f}_k$ converges $\forall\ x$ (unless e.g. $\hat{f} \in \ell^1(\mathbb{Z}^n)$).

derivatives:

$$\hat{f}_k = \int_{\mathbb{T}^n} e^{-2\pi i k\cdot x} f(x)\, dx \qquad \check{g}(x) = \sum_{k\in\mathbb{Z}^n} e^{2\pi i k\cdot x} g_k$$

If $f \in C^1(\mathbb{T}^n)$,

$$(\widehat{\partial_j f})_k = \int_{\mathbb{T}^n} e^{-2\pi i k\cdot x} \partial_j f(x)\, dx$$

(by parts)

$$= -\int_{\mathbb{T}^n} \frac{\partial}{\partial x_j}\left(e^{-2\pi i k\cdot x}\right) f(x)\, dx$$

$$= 2\pi i k_j \hat{f}_k$$

$$\partial_j \check{g}(x) = \sum_{k\in\mathbb{Z}^n} \frac{\partial}{\partial x_j} e^{2\pi i k\cdot x} g_k$$

$$= \sum_k 2\pi i k_j e^{2\pi i k\cdot x} g_k$$

$$= \sum_k e^{2\pi i k\cdot x}(2\pi i k_j g_k)$$

$$=: 2\pi i \check{g_j}(x) \text{ where}$$

$$g_j(k) := k_j g(k).$$

EX: That formula is correct if $g \in \ell^1(\mathbb{Z}^n)$ & $g_j \in \ell^1(\mathbb{Z}^n)$.

True in particular if $g \in \mathcal{S}(\mathbb{Z}^n)$, since then $g_j \in \mathcal{S}(\mathbb{Z}^n)$ too.

$\Rightarrow$ prop: If $f \in C^\infty(\mathbb{T}^n)$, $(\widehat{\partial^\alpha f})_k = (2\pi i k)^\alpha \hat{f}_k$ $\forall$ multi-indices α,

where for $z = (z_1, \dots, z_n) \in \mathbb{C}^n$, $z^\alpha := z_1^{\alpha_1} \dots z_n^{\alpha_n}$.

If $g \in \mathcal{S}(\mathbb{Z}^n)$, $\partial^\alpha \check{g}(x) = (2\pi i)^{|\alpha|} \check{g_\alpha}(x)$ where $g_\alpha(k) := k^\alpha g(k)$.

cor: $\mathcal{F}^*(\mathcal{S}(\mathbb{Z}^n)) \subseteq C^\infty(\mathbb{T}^n)$.

Similarly $\mathcal{F}(C^\infty(\mathbb{T}^n)) \subseteq \mathcal{S}(\mathbb{Z}^n)$:

Pf: $f \in C^\infty(\mathbb{T}^n) \Rightarrow k^\alpha \hat{f}_k = \dfrac{k^\alpha}{(2\pi i)^{|\alpha|} k^\alpha} (\widehat{\partial^\alpha f})_k = \dfrac{1}{(2\pi i)^{|\alpha|}} (\widehat{\partial^\alpha f})_k$

Since $\partial^\alpha f \in C^0(\mathbb{T}^n)$ & $\mathbb{T}^n$ has finite measure & is cpct,

$\partial^\alpha f \in L^1(\mathbb{T}^n) \Rightarrow \widehat{\partial^\alpha f} \in \ell^\infty(\mathbb{Z}^n) \Rightarrow k^\alpha \hat{f}_k$ is bdd in k.

$\Rightarrow \hat{f} \in \mathcal{S}(\mathbb{Z}^n)$. ☐