X a vec. sp., family of seminorms $\{\|\cdot\|_\alpha : X \to [0,\infty)\}_{\alpha \in I}$

$\rightsquigarrow$ <u>locally</u> <u>convex</u> <u>space</u> (LCS): topology is smallest containing all

"balls" $B^\alpha_R(x_0) := \{x \in X \mid \|x - x_0\|_\alpha < R\}$.

<u>prop</u>: (1) X is a TVS. (2) $x_n \to x$ in $X \iff \|x - x_n\|_\alpha \xrightarrow{n\to\infty} 0 \;\; \forall \alpha \in I$.

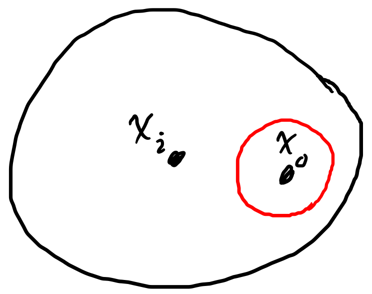
(3) $U \subseteq X$ open $\iff \forall x_0 \in U$, $\exists$ finite subset $I_0 \subseteq I$ & $\{\varepsilon_i > 0\}_{i \in I_0}$

s.t. $\bigcap_{i \in I_0} B^{\alpha_i}_{\varepsilon_i}(x_0) \subseteq U$.

<u>pf of (3)</u>: U open $\iff$ U is a union of fin. intersections of balls

$\implies$ given $x_0 \in U$, some fin. int. of balls in U contains x_0,

$x_0 \in \bigcap_{i \in I_0} B^{\alpha_i}_{R_i}(x_i) \subseteq U$



Δ-ineq. $\implies$ for $\varepsilon_i > 0$ small,

$B^{\alpha_i}_{\varepsilon_i}(x_0) \subseteq B^{\alpha_i}_{R_i}(x_i)$.

<u>pf of (2)</u>: If $x_n \to x$, then since $B^\alpha_\varepsilon(x)$ is a nbhd of x $\forall \varepsilon > 0, \alpha \in I$,

$x_n \in B^\alpha_\varepsilon(x)$ $\forall n$ suff large $\implies \|x - x_n\|_\alpha \to 0$ as $n \to \infty$.

<u>Converse</u>: if $\|x - x_n\|_\alpha \to 0$, given a nbhd $U \subseteq X$ of x,

(2) $\exists$ fin. set of balls $x \in \bigcap_{i \in I_0} B^{\alpha_i}_{\varepsilon_i}(x) \subseteq U$. then $\forall n$ large,

$x_n \in B^{\alpha_i}_{\varepsilon_i}(x)$ $\forall i \in I_0 \implies x_n \in U \implies x_n \to x$. $\boxtimes$

<u>rk</u>: We usually assume $\forall x \neq 0$, $\|x\|_\alpha \neq 0$ for <u>some</u> $\alpha \in I$.

$\iff \forall$ conv. seqs., $x_n \to x$ & $x_n \to y \implies x = y$

$\iff X$ is a <u>Hausdorff</u> space.

exs (3): $C_b^\infty(\Omega)$ is a LCS with norms $\{\|\cdot\|_{C^m}\}_{m \geq 0}$, so

$f_k \xrightarrow{C^\infty} f \iff \|f_k - f\|_{C^m} \to 0 \ \forall m \geq 0 \iff f_k \xrightarrow{C^m} f \ \forall m.$

cor: $U \subseteq C_b^\infty(\Omega)$ is open $\iff \forall f_0 \in U, \exists m \geq 0, \varepsilon > 0$ s.t.

$\{f \in C_b^\infty(\Omega) \mid \|f - f_0\|_{C^m} < \varepsilon\} \subseteq U.$

(4) $C^m_{loc}(\Omega)$ is a LCS w/ seminorms $\{\|\cdot\|_{C^j(K)}\}_{\substack{0 \leq j \leq m \\ K \subseteq \Omega \text{ cpct}}}$

rk: There may exist multiple distinct families of seminorms that generate same top. on an LCS.

e.g. in (4): If $\Omega = \bigcup_{j=1}^\infty \Omega_j$ s.t. $\Omega_1 \overset{\text{open}}{\subseteq} \Omega_2 \overset{\text{open}}{\subseteq} \Omega_3 \subseteq \dots$ & $K_j := \overline{\Omega_j}$ cpct

$\Rightarrow$ any cpct $K \subseteq \Omega$ is contained in Ω_j for j suff large

$\Rightarrow$ the countable fam. of seminorms $\{\|\cdot\|_{C^j(K_i)}\}_{\substack{0 \leq j \leq m \\ i = 1,2,3,\dots}}$ defns same loc. convex top. on $C^m_{loc}(\Omega)$.

Recall: Norms $\|\cdot\|_0$ & $\|\cdot\|_1$ on X are equivalent if $\exists c > 0$ s.t.

$\frac{1}{c}\|f\|_0 \leq \|f\|_1 \leq c\|f\|_0 \quad \forall f \in X.$

EX (PSET 1): $\iff \|\cdot\|_0$ & $\|\cdot\|_1$ generate the same top. on X.

ex: $\|\cdot\|_{C^m}$ is equivalent to $\|f\| := \max_{|\alpha| \leq m} \|\partial^\alpha f\|_{C^0}$.

thm: A LCS is metrizable (i.e. its top. is generated by open balls w.r.t. a metric)

$\iff \exists$ a countable fam. of seminorms generating its top.

pf of $\Leftarrow$: If X has top. gen. by $\{\|\cdot\|_n\}_{n=1}^\infty$, can defn metric

on X by $d(x,y) := \sum_{n=1}^\infty \frac{1}{2^n} \frac{\|x-y\|_n}{1 + \|x-y\|_n}$.

check: d defns a metric & the same notion of open sets as the seminorms.

defn: A metrizable LCS is a Fréchet space if it is complete w.r.t. this metric.

ex: $C_b^\infty(\Omega)$ & $C^m_{loc}(\Omega)$ are Fréchet spaces ($\Leftarrow$ completeness of $C_b^m(\Omega)$ $\forall$ finite m, $K \subseteq \Omega$ cpct or $C^m(K)$)

ex (not metrizable): $C^0_c(\Omega) := \{ f : \Omega \xrightarrow{C^0} V \mid \underbrace{\overline{\{ x \in \Omega \mid f(x) \neq 0 \}}}_{\text{support of } f} \text{ is cpct} \}$.

Seminorms $\{ \| \cdot \|_\varphi \}_{\varphi \in I}$ where $I := \{ \varphi : \Omega \to [0, \infty) \text{ contin.} \}$

$\| f \|_\varphi := \| \varphi f \|_{C^0}$, i.e. $\| f \|_\varphi < \varepsilon$ means $|f(x)| < \dfrac{\varepsilon}{\varphi(x)}$ $\forall x \in \Omega$.

claim (PSET 1): (a) $f_k \to f_\infty$ in $C^0_c(\Omega)$ iff $\exists$ a cpct set $K \subseteq \Omega$ s.t. f_k has support in K $\forall k$ & $f_k \to f_\infty$ unif on K.

(b) $\nexists$ metric d s.t. every open set in $C^0_c(\Omega)$ is a union of balls $\{ f \mid d(f, f_0) < \varepsilon \}$.

ex (not a TVS): $\| \cdot \|_\varphi$ is not always finite on $f \in C^0(\Omega)$ without cpct support but can defn. a top. on $C^0(\Omega)$ gen. by balls $\{ f \mid \| f - f_0 \|_\varphi < R \}$.

Then $f_k \to f_\infty$ iff $\exists$ cpct $K \subseteq \Omega$ s.t. $f_k \equiv f_\infty$ on $\Omega \setminus K$ $\forall k$, & $f_k \xrightarrow{\text{unif}} f_\infty$ on K.

claim: $\mathbb{R} \times C^0(\Omega) \to C^0(\Omega) : (\lambda, f) \mapsto \lambda f$ is not contin.

pf: $f \in C^0(\Omega)$ without cpct supp., $\lambda_k \in \mathbb{R}$, $\lambda_k \neq 0$ but $\lambda_k \to 0$,

$\lambda_k f \not\to 0$ since $\lambda_k f \neq 0$ on $\Omega \setminus K$ for any $K \subseteq \Omega$ cpct.

operators on Banach spaces

<u>very useful lemma</u>: X, Y normed vec. spaces, Y complete, $V \subseteq X$ dense subspace. Then every $A \in \mathcal{L}(V,Y)$ has a unique contin. extension $A \in \mathcal{L}(X,Y)$.

$$\begin{tikzcd} X \arrow[r, dotted] & Y \\ V \arrow[u, phantom, "\rotatebox{90}{$\subseteq$}"] \arrow[ur, "A"'] & \end{tikzcd}$$

<u>pf</u>: $\forall x \in X$, density $\Rightarrow \exists$ seq $x_n \in V$ s.t. $x_n \to x$.

Then $\|Ax_n - Ax_m\| \le \|A\| \cdot \|x_n - x_m\| \Rightarrow Ax_n$ is Cauchy in Y

$\Rightarrow$ can defn. $Ax := \lim_{n\to\infty} Ax_n \in Y$.

<u>check</u>: $A : X \to Y$ is linear & bdd. ☒

<u>defn</u>: For X a TVS over $\mathbb{K}$, the <u>dual space</u> of X is

$X^* := \mathcal{L}(X, \mathbb{K}) = \{A : X \to \mathbb{K}$ contin. linear operators$\}$

"contin./bdd linear <u>functionals</u>".

<u>rk</u>: (1) $\nexists$ guarantee that $X^* \neq \{0\}$.

(2) If X is a LCS, Hahn-Banach thm (later) $\Rightarrow \forall\ x \neq 0 \in X$, $\exists\ \Lambda \in X^*$ s.t. $\Lambda(x) = 1$.

(3) X a normed vec. sp $\Rightarrow X^*$ is a Banach sp. (since $\mathbb{K}$ is complete)

<u>ex</u>: (1) Any finite measure μ on $\Omega \subseteq \mathbb{R}^n$, def. $\Lambda_\mu \in (C_b^0(\Omega))^*$ by

$\Lambda_\mu(f) := \int_\Omega f\, d\mu$, $\quad |\Lambda_\mu(f)| \le \mu(\Omega) \cdot \|f\|_{C^0} \Rightarrow \|\Lambda_\mu\| \le \mu(\Omega)$.

(2) For $1 \le p, q \le \infty$ w/ $\frac{1}{p} + \frac{1}{q} = 1$, any $g \in L^q(\Omega)$ defn

$\Lambda_g \in (L^p(\Omega))^*$ by $\Lambda_g(f) := \int_\Omega \langle g(x), f(x) \rangle\, dm(x)$.

Lebesgue measure

$|\Lambda_g(f)| \overset{\text{(Hölder)}}{\le} \|g\|_{L^q} \cdot \|f\|_{L^p} \Rightarrow \|\Lambda_g\| \le \|g\|_{L^q}$.

<u>defn</u>: The <u>transpose</u> of $A \in \mathcal{L}(X,Y)$ is $A^* \in \mathcal{L}(Y^*, X^*)$ def'd by

$(A^*\lambda)(x) := \lambda(Ax)$ for $\lambda \in Y^*$, $x \in X$. <u>check</u>: $\|A^*\| \le \|A\|$.

<u>prop</u>: $\exists$ canonical bdd lin. map $J : X \to X^{**}$ given by

$(Jx)\lambda := \lambda(x)$

<u>pf</u>: $\|(Jx)\lambda\| \le \|\lambda\| \cdot \|x\| \Rightarrow \|Jx\| \le \|x\| \Rightarrow \|J\| \le 1$ ☒

<u>thm for later</u> (cor. of Hahn-Banach): J is injective & is an isometry, i.e. $\|Jx\| = \|x\|$.

<u>defn</u>: a Banach space X is <u>reflexive</u> if $J : X \to X^{**}$ is an iso.

<u>We'll see</u>: L^p are reflexive for $1 < p < \infty$.