Midterm

(1)

(a) X separable real Banach space

$V \subset X$ subspace $\quad \lambda : V \to \mathbb{R}$ is a bd. lin. functional

To show $V = V_0 \subset V_1 \subset V_2 \subset \cdots \subset X$ s.t.

$\dim\left(V_k / V \right) < \infty \quad \forall\, k \geq 0$

and $\bigcup_{k=0}^{\infty} V_k$ is dense in X.

Suppose $\{x_1, x_2, \ldots\} \subseteq X$ is dense.

$V_0 = V$

Let $j_1 = \min\{ j \in \mathbb{N} \mid x_j \notin V_0 \}$ set $V_1 = V_0 + \langle x_{j_1} \rangle$

$\dim(V_1 / V) = 1$

$j_2 = \min\{ j \in \mathbb{N} \mid x_j \notin V_1 \}$, set $V_2 = V_1 + \langle x_{j_2} \rangle$

Construct V_k inductively.

if this set we're min. is ever empty

$\Rightarrow$ V_i is dense $\Rightarrow$ choose $V_j = V_i \quad \forall\, j > i$

$V_0 = V \subset V_1 \subset V_2 \subset \cdots \subset X$

$\dim\left(V_{i+1}/V_i\right) = 1 \quad \forall i \Rightarrow \dim\left(V_i/V\right) < \infty$

$\{x_1, x_2, x_3, \dots\} \subset \bigcup_{k=0}^{\infty} V_k \Rightarrow \bigcup_{k=0}^{\infty} V_k$ dense in X.

(b) construct an extension Λ of λ

$$\Lambda : X \to \mathbb{R}.$$

Suppose $V_i \subseteq V_{i+1} \subseteq X$ & $\lambda : V_i \to \mathbb{R}$ bd s.t. $\dim\left(V_{i+1}/V_i\right) = 1$.

$\exists$ a deterministic alg. (depends on a choice of a vector $y \in V_{i+1} \setminus V_i$) for extending λ to $\Lambda : V_{i+1} \to \mathbb{R}$ s.t. $\|\Lambda\| = \|\lambda\|$.

set $\Lambda(y) = a \in \mathbb{R}$ s.t.

$$a \in \left[\sup_{\substack{\alpha > 0 \\ x \in V_i}} \frac{1}{\alpha}\left(\lambda(x) - \|x - \alpha y\|\right), \inf_{\substack{\alpha > 0 \\ v_i \in X}} \frac{1}{\alpha}\left(\|x + \alpha y\| - \lambda(x)\right) \right]$$

Choose

$$a = \sup_{\substack{\alpha > 0 \\ x_i \in V_i}} \frac{1}{\alpha}\left(\lambda(x) - \|x - \alpha y\|\right)$$

Repeating this procedure gives us an extension of $\lambda : V \to \mathbb{R}$ to a $\Lambda : \bigcup_{k=0}^{\infty} V_k \to \mathbb{R}$

s.t. $\|\Lambda\| = \|\lambda\|$

dense in X

$\Rightarrow \exists!$ extension

$\Lambda : X \to \mathbb{R}$ s.t. $\|\Lambda\| = \|\lambda\|$.

$\boxtimes$

2

$$f(x) = \int_2^{\infty} \frac{e^{2\pi i p x}}{p \ln p} \, dp = \lim_{N \to \infty} \int_2^{N} \frac{e^{2\pi i p x}}{p \ln p} \, dp \quad \text{——— } (1)$$

(a) T.S: $\exists\ g \in L^2(\mathbb{R})$ s.t

$$\hat{g}(p) = \begin{cases} \frac{1}{p \ln p} & \text{if } p \geq 2 \\ 0 & \text{if } p < 2 \end{cases}$$

$g \in H^s(\mathbb{R})$, $s \in [0, 1/2]$ but not $s > 1/2$.

$\hat{g} \in L^2(\mathbb{R})$. $\quad \mathcal{F}, \mathcal{F}^* : L^2(\mathbb{R}) \to L^2(\mathbb{R})$

$\Rightarrow \exists\ g \in L^2(\mathbb{R})$ s.t

$\mathcal{F}^* \hat{g} = g$.

$$\int_2^\infty \frac{1}{p^2(\ln p)^2}\,dp \underset{\text{Want}}{<} \infty.$$

$$u = \ln p \Rightarrow du = \frac{dp}{p}$$

$$p = e^u$$

$$= \int_{\ln 2}^\infty e^{-u}\frac{1}{u^2}\,du \le \int_{\ln 2}^\infty e^{-u}\,du < \infty$$

$$\Rightarrow \hat{g} \in L^2(\mathbb{R}).$$

Want:-

$g \in H^s(\mathbb{R})$ for $s \in [0, 1/2]$

$\partial^\alpha g$

need to check $\| (1+|p|^2)^{s/2}\hat{g}\|_{L^2}^2 = \int_2^\infty (1+p^2)^s \frac{1}{p^2(\ln p)^2}\,dp$ converges

$$\Longleftrightarrow \int_2^\infty p^{2k}\frac{1}{p^2(\ln p)^2}\,dp < \infty \quad \forall\ k \in [0, s]$$

Substitute $u = \ln p$

$$= \int_{\ln 2}^\infty e^{2ku-u}\frac{1}{u^2}\,du = I$$

i) $k < \frac{1}{2}$, $2ku - u < 0 \implies I < \infty$ converges

ii) $k = 1/2$ $\displaystyle\int_{\ln 2}^{\infty} \frac{du}{u^2} < \infty$ converges

iii) $k > 1/2$, $2ku - u > 0$ $I \to \infty$
$\implies$ diverges.

$\therefore$ $g \in H^s(\mathbb{R})$ $\forall$ $s \le 1/2$ but not $s > 1/2$.

$\blacksquare$

(b) $$f_N(x) = \int_2^N \frac{e^{2\pi i p x}}{p \ln p}\, dp \xrightarrow[N \to \infty]{L^2} g.$$

$f_N = \mathcal{F}^* \hat{f}_N$ where

$$\hat{f}_N = \chi_{[2,N]}(p) \frac{1}{p \ln p}$$

$\hat{f}_N \xrightarrow{L^2} \hat{g}$ as $N \to \infty$.

$$\| \hat{g} - \hat{f}_N \|_{L^2}^2 = \int_N^{\infty} \frac{dp}{p^2 (\ln p)^2} \xrightarrow{N \to \infty} 0$$

$\mathcal{F}^* : L^2(\mathbb{R}) \to L^2(\mathbb{R})$

$\implies$ $f_N \xrightarrow{L^2} g$. $\blacksquare$

(c) $M \geq 2$

$$\int_M^\infty \frac{e^{2\pi i p x}}{p \ln p}\, dp = \lim_{N \to \infty} \int_M^N \frac{e^{2\pi i p x}}{p \ln p}\, dp \quad \text{exists}$$

$$\left| \int_M^\infty \frac{e^{2\pi i p x}}{p \ln p}\, dp \right| \leq \frac{1}{\pi |x| . M \ln M}$$

$2 \leq M \leq N$, $x \neq 0$

$$\int_M^N \frac{e^{2\pi i p x}}{p \ln p}\, dx = \int_M^N \frac{1}{p \ln p} \frac{d}{dp} \left(\frac{1}{2\pi i x} e^{2\pi i p x} \right) dp$$

$$\underset{\text{IBP}}{=} \frac{1}{2\pi i x} \left[\underbrace{\left. \frac{e^{2\pi i p x}}{p \ln p} \right|_M^N}_{A_N} - \underbrace{\int_M^N \frac{e^{2\pi i p x}}{p \ln p}\, dp}_{B_N} \right]$$

$$A_N = \frac{1}{2\pi i x} \left. \frac{e^{2\pi i p x}}{p \ln p} \right|_M^N \xrightarrow{N \to \infty} -\frac{1}{2\pi i x} \frac{e^{2\pi i M x}}{M \ln M}$$

$$B_N = -\frac{1}{2\pi i x} \int_M^N e^{2\pi i p x} \frac{d}{dp}\left(\frac{1}{p \ln p}\right) dp$$

$\downarrow$ decreasing $p \geq 2$

$$\int_M^\infty \left| e^{2\pi i p x} \frac{d}{dp}\left(\frac{1}{p \ln p}\right) \right| dp$$

$$= -\int_M^\infty \frac{d}{dp}\left(\frac{1}{p \ln p}\right) dp = \frac{1}{M \ln M}$$

B_N is Lebesgue integrable on $[M, \infty]$

$$\lim_{N \to \infty} B_N = -\frac{1}{2\pi i x} \int_M^N e^{2\pi i p x} \frac{d}{dp}\left(\frac{1}{p \ln p}\right) dp$$

exists

$$\left|\lim_{N \to \infty} B_N\right| \leq \frac{1}{2\pi |x|} \frac{1}{M \ln M}$$

$\Rightarrow$ original limit $= \lim_{N \to \infty} (A_N + B_N)$

exists and is bounded by

$$\frac{1}{2\pi |x|} \frac{1}{M \ln M} + \frac{1}{2\pi |x|} \cdot \frac{1}{M \ln} = \frac{1}{\pi |x| M \ln M}$$

Want $g(x) = f(x) = \lim_{N\to\infty} \int_2^N \frac{e^{2\pi i p x}}{p \ln p}\, dp$

$f_N(x) \longrightarrow f$ pointwise on $\mathbb{R} \setminus \{0\}$

$f_N \xrightarrow{L^2} g$ Hint $\Rightarrow$

some subsequence $f_{N_j} \longrightarrow g$ a.e. pointwise

$\Rightarrow f = g$ a.e. $\boxed{}$

(d) $\lim_{x\to\infty} |f(x)| = \infty$

$\epsilon > 0$, $x \in \mathbb{R}$ w/ $0 < |x| < \epsilon/2$

$\Rightarrow \frac{\epsilon}{|x|} > 2$. $\quad 2 \leq p \leq \frac{\epsilon}{|x|}$

$$f(x) = \underbrace{\int_2^{\epsilon/|x|} \frac{e^{2\pi i p x}}{p \ln p}\, dp}_{A} + \underbrace{\lim_{N\to\infty} \int_{\epsilon/|x|}^{N} \frac{e^{2\pi i p x}}{p \ln p}\, dp}_{B}$$

part (c) $\Rightarrow |B| \leq \frac{1}{\pi |x| \cdot \frac{\epsilon}{|x|} \ln\left(\frac{\epsilon}{|x|}\right)}$

$$= \frac{1}{\pi \epsilon \ln\left(\frac{\epsilon}{|x|}\right)} \longrightarrow 0 \quad \text{as } |x| \to 0$$

$B \longrightarrow 0$ as $|x| \to 0$.

pick $\epsilon > 0$ small enough

$$A \geq C \int_2^{\epsilon/|x|} \frac{dp}{p \ln p} \longrightarrow \infty \quad \text{as } |x| \to 0$$

$$\left(\int_{\ln 2}^{\infty} \frac{dy}{y} \quad , \; y = \ln p \right)$$

$\Rightarrow$ $A \longrightarrow \infty$ as $|x| \to 0$

$|f(x)| \longrightarrow \infty$ as $|x| \to 0$.

3

(a) $f \in L^1_{loc}(\mathbb{R})$

$u(t, x) = f(t \pm x)$

Want:- $u \in L^1_{loc}(\mathbb{R}^2)$

weak solution to $\partial_t^2 u - \partial_x^2 u = 0.$

$f \in L^1_{loc}(\mathbb{R})$

suppose $K \subseteq \mathbb{R}^2$ is some compact set

$$K \subseteq [-N, N] \times [-N \times N] \subseteq \mathbb{R}^2$$

$$\Rightarrow \quad \| f|_{[-2N, 2N]} \|_{L^1} < \infty$$

$$\int_K |u| \, dm \leq \int_{[-N,N] \times [-N,N]} |u| \, dm$$

$$= \int_{-[N,N]} \int_{[-N,N]} |u(t,x)| \, dt \, dx$$

$$= \int_{-[N,N]} \int_{[-N,N]} |f(t \pm x)| \, dt \, dx$$

(Fubini's theorem)

$$\leq 2N \|f\|_{L^1_{[-2N, 2N]}}$$

$$< \infty.$$

$\Rightarrow u \in L^2_{loc}(\mathbb{R}^2)$.

<u>Want</u> :- $\partial_t^2 u - \partial_x^2 u = 0$ weakly.

$\Rightarrow$ $\Lambda_u(\varphi) = \int_{\mathbb{R}^2} \varphi u \, dm$ satisfy

$$(\partial_t^2 \Lambda_u - \partial_x^2 \Lambda_u, \varphi)$$
$$= (\Lambda_u, \partial_t^2 \varphi) - (\Lambda_u, \partial_x^2 \varphi)$$
$$= \int_{\mathbb{R}^2} (\partial_t^2 \varphi - \partial_x^2 \varphi) u \, dm = 0.$$

Use change of variable $\begin{cases} s = t \pm x \\ y = x \end{cases}$

$$\partial_t \varphi = \partial_s \varphi \cdot \frac{\partial s}{\partial t} + \partial_y \varphi \cdot \frac{\partial y}{\partial t}$$
$$= \partial_s \varphi$$
$$\partial_t^2 \varphi = \partial_s \partial_s \varphi \cdot \frac{\partial s}{\partial t} + \partial_y \partial_s \varphi \frac{\partial y}{\partial t}$$
$$= \partial_s^2 \varphi$$

$$\partial_x \varphi = \partial_s \varphi \cdot \frac{\partial s}{\partial x} + \partial_y \varphi \cdot \frac{\partial y}{\partial x}$$
$$= \pm \partial_s \varphi + \partial_y \varphi$$

$$\Rightarrow \quad \partial_x^2 \varphi = \partial_s^2 \varphi \pm \partial_s \partial_y \varphi + \partial_y^2 \varphi$$

$$\partial_t^2 \varphi - \partial_x^2 \varphi = -\partial_y^2 \varphi \mp \partial_s \partial_y \varphi$$

$$= -\partial_y \left(\partial_y \varphi \pm \partial_s \varphi \right).$$

$$\therefore \int_{\mathbb{R}^2} [\partial_t^2 \varphi(t,x) - \partial_x^2 \varphi(t,x)] \, u(t,x) \, dt \, dx$$

$$= - \int_{\mathbb{R}^2} \partial_y [\partial_y \varphi(s,y) \pm \partial_s \varphi(s,y)] f(s) \, ds \, dy$$

$$= - \int_{\mathbb{R}^2} \partial_y \left[(\partial_y \varphi(s,y) \pm \partial_s \varphi(s,y)) f(s) \right] ds \, dy$$

$$= - \int_{-\infty}^{\infty} \underbrace{\left(\int_{-\infty}^{\infty} \partial_y [\partial_y \varphi(s,y) \pm \partial_s \varphi(s,y)] f(s) \, dy \right)}_{\substack{= 0 \text{ due to FTC} \\ \text{and } \varphi \in C_0^\infty(\mathbb{R}^2)}} ds$$

$$= 0 \quad \Rightarrow \quad \partial_t^2 u - \partial_x^2 u = 0$$

in the weak sense. $\boxtimes$

(b). $LK = \delta$, K - fundamental solution.

$f \longmapsto u$ $\quad u = K * f$

$\Rightarrow Lu = f.$

$K \in L^1_{loc}(\mathbb{R}^n) \Rightarrow K * f$ is smooth.

$\Rightarrow u = K * f$ is smooth

$\partial^\alpha u = \partial^\alpha K * f$ $\forall$ multi-index α.

$LK = \delta \Rightarrow$

$$Lu = \sum c_\alpha \partial^\alpha u = \sum c_\alpha (\partial^\alpha (K * f))$$
$$= \sum c_\alpha (\partial^\alpha K * f)$$
$$= \left(\sum (c_\alpha \partial^\alpha K)\right) * f$$
$$= LK * f$$
$$= \delta * f = f.$$

$\boxtimes$

(c) $L = \partial_x^2$ on $\mathbb{R}$.

find a fundamental sol. K.

u to $u'' = f$ $\quad \forall f \in C_0^\infty(\mathbb{R})$.

from PSET 9, $f(x) = \frac{1}{2}|x|$

$$f'(x) = \begin{cases} 1/2 & \text{if } x > 0 \\ -1/2 & \text{if } x < 0 \end{cases}$$

$f''(x) = \delta$ in the sense of distribution.

set $K(x) = \frac{1}{2}|x|$ $\rightsquigarrow$ fundamental sol.

$\Rightarrow$ $LK = K'' = \delta$.

$u'' = f$.

$$u(x) = \int_{-\infty}^{\infty} K(x-y) f(y)\,dy$$

$$= \frac{1}{2}\int_{-\infty}^{\infty} |x-y| f(y)\,dy$$

$$= \frac{1}{2}\int_{-\infty}^{x} (x-y) f(y)\,dy - \frac{1}{2}\int_{x}^{\infty} (x-y) f(y)\,dy$$

$$= \frac{1}{2}\int_{-\infty}^{x} (x-y) f(y)\,dy + \frac{1}{2}\int_{\infty}^{x} (x-y) f(y)\,dy$$

Check $\quad u''(x) = f(x)$

$$u'(x) = \frac{1}{2}\int_{-\infty}^{x} f(y)\,dy + \frac{1}{2}(x-x)f(x)$$

$$+ \frac{1}{2}\int_{\infty}^{x} f(y)\,dy + \frac{1}{2}(x-x)f(x)$$

$$= \frac{1}{2}\int_{-\infty}^{x} f(y)\,dy + \frac{1}{2}\int_{\infty}^{x} f(y)\,dy$$

$$u''(x) = \frac{1}{2}f(x) + \frac{1}{2}f(x)$$

$$= f(x)$$

Indeed $u'' = f$. $\square$

Sketch of 4

$$I_1(\mathcal{H}) = I_0(\mathcal{H}, \mathcal{H})$$

$\operatorname{im} A \subseteq \mathcal{H}$, complement is $(\operatorname{im} A)^{\perp}$

$A \in I_1(\mathcal{H})$ by part c) $A \in I_0(\mathcal{H}, \mathcal{H})$

$\mathcal{I}_1(\mathcal{H})$ is open.

$\mathcal{I}_2(\mathcal{H}) = \{ A \in \mathcal{I}(\mathcal{H}) \mid A \text{ is inj } \}$

$x \in \ell^\infty$ st $\inf_{n \in \mathbb{N}} |x_n| = 0$.

$\Rightarrow \Phi(x) \in \mathcal{I}_2(\mathcal{H})$

$\epsilon > 0$, $\exists\ n \in \mathbb{N}$ s.t

$|x_n| < \epsilon$

define $y \in \ell^\infty$ st. $y_m = \begin{cases} x_m, & m \neq n \\ 0, & \text{for } m = n \end{cases}$

$\|y - x\|_{\ell^\infty} < \epsilon$

by part b) $\|\Phi(y) - \Phi(x)\| = \|y - x\|_{\ell^\infty}$

$< \epsilon$

$\Phi(y)$ is not inj

B/C $\Phi(y) e_n = 0 \Rightarrow$

$\Rightarrow \Phi(y) \notin \mathcal{I}_2(\mathcal{H})$

Problem 4 (added later)

a) $A \in \mathcal{L}(X, Y)$ is injective w/ closed range

$\Longleftrightarrow$

$\exists$ $c>0$ s.t. $\|Ax\| \geq c\|x\|$ $\forall x \in X$.

$\Rightarrow$ $\because$ im A is closed in X $\Rightarrow$ im A is a Banach space

and $A : X \longrightarrow \text{im } A$ is a bounded linear bijection

$\Rightarrow$ by IMT $\exists$ a bounded inverse

$A^{-1} : \text{im } A \longrightarrow X$ w/ $\|A^{-1}y\| \leq C\|y\|$ $\forall y \in \text{im } A$

for some $c>0$ independent of y.

$\Rightarrow$ $\|Ax\| \geq c\|x\|$ $\forall x \in X$.

$\Leftarrow$ $\|Ax\| \geq c\|x\|$, $c>0$ $\Rightarrow$ if $Ax=0$ then $x=0$

$\Rightarrow$ A is injective. Let $y_n \in \text{im } A$ & $y_n \longrightarrow y$ in Y.

Want to show that $y \in \text{im } A$.

Let $y_n = Ax_n$. Then

$$\|x_n - x_m\| \leq \frac{1}{c}\|Ax_n - Ax_m\|$$

$\therefore$ if Ax_n is Cauchy $\Rightarrow$ x_n is Cauchy $\Rightarrow$ $x_n \longrightarrow x \in X$.

Then $Ax_n \longrightarrow Ax$ $\Rightarrow$ $Ax = y$ $\Rightarrow$ $y \in \text{im} A$ $\Rightarrow$ im A is closed. $\boxtimes$

b) Want to show that

$\Phi : \ell^\infty \longrightarrow \mathcal{L}(\mathcal{H})$ satisfies $\|\Phi(x)\| = \|x\|_{\ell^\infty}$ $\forall\, x \in \ell^\infty$.

Let $v = \sum_{n\in\mathbb{N}} v_n e_n \in \mathcal{H}$. Then $\|v\|^2 = \sum_{n\in\mathbb{N}} |v_n|^2$

Also, $\Phi(x)v = \sum x_n v_n e_n$

$$\Rightarrow \|\Phi(x)v\|^2 = \sum_{n\in\mathbb{N}} |x_n v_n|^2 = \sum_{n\in\mathbb{N}} |x_n|^2 |v_n|^2 \leq \|x\|^2_{\ell^\infty} \cdot \sum |v_n|^2$$
$$= \|x\|^2_{\ell^\infty} \|v\|^2$$

$\therefore \|\Phi(x)\| \leq \|x\|_{\ell^\infty}$. For the other inequality, let x_{n_j} be a subsequence of $(x_1, x_2, x_3, \dots) = x \in \ell^\infty$ s.t.

$|x_{n_j}| \longrightarrow \|x\|_{\ell^\infty}$ as $j \to \infty$. Then

$$\frac{\|\Phi(x)e_{n_j}\|}{\|e_{n_j}\|} = \|x_{n_j} e_{n_j}\| = |x_{n_j}|.$$

Thus $$\|\Phi(x)\| = \sup_{\|v\|=1} \frac{\|\Phi(x)v\|}{\|v\|} \geq \sup_{j\in\mathbb{N}} |x_{n_j}| = \|x\|_{\ell^\infty}$$

$$\Rightarrow \quad \|\Phi(x)\| = \|x\|_{\ell^\infty}$$

$\Rightarrow$ from part a) $\operatorname{im} \Phi$ is a closed subspace of $\mathcal{L}(\mathcal{H})$.

c) To prove:- $A \in \mathcal{L}(X,Y)$ injective admits a bounded left-inverse $\iff$ $\operatorname{im} A$ is closed and $Y = \operatorname{im} A \oplus W$ for W closed.

$\Rightarrow$ Suppose $B \in \mathcal{L}(Y,X)$ s.t. $BA = id_X$.

Let $W = \ker(AB) \subseteq Y$. $\because$ W is a kernel of a bounded linear map $\Rightarrow$ W is closed

Note that $(Id_Y - AB)y = 0 \iff y = A(By) \Rightarrow y \in \operatorname{im} A$.

& conversely, if $y \in \operatorname{im} A \Rightarrow y = Ax \Rightarrow (Id_Y - AB)y$

$= y - A(By) = Ax - A(BAx) = Ax - Ax = 0$.

$\Rightarrow$ $\operatorname{im} A = \ker(Id_Y - AB)$ $\Rightarrow$ $\operatorname{im} A$ is also closed.

Now, $\because$ $BA = Id_X$ $\Rightarrow$ B is surjective $\Rightarrow$ $\operatorname{im}(AB) = \operatorname{im} A$

and $y = Ax \in \operatorname{im} A$ $\Rightarrow$ $ABy = ABAx = Ax = y$

$\Rightarrow$ AB is the projection to $\operatorname{im} A$ along its kernel and $\ker(AB) = W$.

Thus $Y = \operatorname{im} A \oplus W$.

$\Leftarrow$ Suppose $Y = \operatorname{im} A \oplus W$, $\operatorname{im} A, W \subseteq Y$ closed.

Let $\pi \in \mathcal{L}(Y)$ denote the projection to $\operatorname{im} A$ along W and define $B \in \mathcal{L}(X,Y)$ by $B = A^{-1}\pi : Y \longrightarrow X$.

Note that $\because$ $\operatorname{im}\Pi = \operatorname{im} A$ and $\operatorname{im} A \xrightarrow{A^{-1}} X$ is a bounded linear map by IMT, $A^{-1}\Pi$ is well-defined.

Thus $BA = A^{-1}\Pi A = Id_X$ as $\Pi|_{\operatorname{im} A} = Id_{\operatorname{im} A}$. ▫

d) Want to prove

$$I_0(X,Y) = \{A \in \mathcal{L}(X,Y) \mid A \text{ admits a bounded left-inverse}\}$$

is open in $\mathcal{L}(X,Y)$.

Let $A \in I_0(X,Y) \Rightarrow \exists\ B \in \mathcal{L}(Y,X)$ s.t. $BA = Id_X$.

By a corollary of the IMT, the set of invertible bounded linear maps $X \longrightarrow X$ is open. Moreover, $\forall\ C \in \mathcal{L}(X)$ w/ $\|C\| < 1$, $Id_X - C$ is invertible.

Then any $A' \in \mathcal{L}(X,Y)$ w/ $\|A' - A\| < \frac{1}{\|B\|}$ satisfies

$\|BA' - Id_X\| = \|B(A' - A)\| \le \|B\|\,\|A' - A\| < 1$

$\Rightarrow$ BA' is also invertible.

now $B' = (BA')^{-1}B \in \mathcal{L}(Y,X)$ satisfies

$B'A' = (BA')^{-1}BA' = Id_X \Rightarrow A' \in I_0(X,Y)$.

e) Want to prove that :- $\Phi(x) \in \mathcal{L}(\mathcal{H})$ is injective w/ closed image $\iff \inf_{n\in\mathbb{N}} |x_n| > 0$.

$\Rightarrow$ Let $\Phi(x)$ be injective. Then $x_n \neq 0 \ \forall n \in \mathbb{N}$

Otherwise $\Phi(x) e_n = x_n e_n = 0$ for some $n \in \mathbb{N}$.

Suppose $\inf_{n\in\mathbb{N}} |x_n| = 0 \Rightarrow \exists$ a subsequence x_{n_j} of $(x_1, x_2 \dots) \in \ell^\infty$

s.t. $|x_{n_j}| \to 0$ as $j \to \infty$.

Then
$$\frac{\|\Phi(x) e_{n_j}\|}{\|e_{n_j}\|} = \|x_{n_j} e_{n_j}\| = |x_{n_j}| \to 0$$

$$\Rightarrow \inf_{v \in \mathcal{H} \setminus \{0\}} \frac{\|\Phi(x) v\|}{\|v\|} = 0$$

$\Rightarrow \nexists \ c > 0$ s.t. $\|\Phi(x) v\| \geq c \|v\| \ \forall v \in \mathcal{H}$

$\Rightarrow$ by part a), $\operatorname{im} \Phi(x)$ is not closed which is a contradiction.

$\Leftarrow$ Let $c = \inf_{n\in\mathbb{N}} |x_n| > 0 \Rightarrow \forall \ v = \sum v_n e_n \in \mathcal{H}$

$$\|\Phi(x) v\|^2 = \Big\|\sum_{n\in\mathbb{N}} x_n v_n e_n\Big\|^2 = \sum |x_n v_n|^2 \geq \inf_{n\in\mathbb{N}} |x_n|^2 \cdot \sum |v_n|^2$$
$$= c^2 \|v\|^2$$

$\Rightarrow$ by part a) $\Phi(x)$ is injective w/ closed image.

f) $I_1(\mathcal{H}) = \{ A \in \mathcal{L}(\mathcal{H}) \mid A$ in injective w/ closed image $\}$

$\because$ $\mathcal{H}$ is a Hilbert space $\Rightarrow$ every closed subspace has a complement (its orthogonal complement)

$\Rightarrow$ $\mathcal{H} = \operatorname{im} A \oplus W$ where $W = (\operatorname{im} A)^{\perp}$

$\Rightarrow$ from part c) and d), $I_1(\mathcal{H}) = I_0(\mathcal{H}, \mathcal{H})$

$\Rightarrow$ $I_1(\mathcal{H})$ is open in $\mathcal{L}(\mathcal{H})$.

$I_2(\mathcal{H}) = \{ A \in \mathcal{L}(\mathcal{H}) \mid A$ is injective $\}$ is <u>NOT</u> open.

pick $x \in \ell^\infty$ s.t. $\inf\limits_{n \in \mathbb{N}} |x_n| = 0 \Rightarrow \Phi(x) \in I_2(\mathcal{H})$

But $\forall\ \epsilon > 0$, $\exists\ n \in \mathbb{N}$ s.t $|x_n| < \epsilon$ so we define $y \in \ell^\infty$ by

$$y_m = \begin{cases} x_m, & m \neq n \\ 0, & m = n \end{cases} \Rightarrow$$

$\|y - x\|_{\ell^\infty} < \epsilon \Rightarrow$ by part b) $\|\Phi(y) - \Phi(x)\| = \|y - x\|_{\ell^\infty} < \epsilon$.

But $\Phi(y) e_n = 0$ w/ $e_n \neq 0 \Rightarrow \Phi(y)$ is NOT injective.

$\Rightarrow$ $\Phi(y) \notin I_2(\mathcal{H})$.

$\blacksquare$

——— × ——— × ———