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Funktionalanalysis
WiSe 2020–21

Problem Set 10

Due: Thursday, 11.02.2021 (15pts + 5 for free)[1]

Problems marked with $(*)$ will be graded. Solutions may be written up in German or English and should be submitted electronically via the moodle before the Übung on the due date. For problems without $(*)$, you do not need to write up your solutions, but it is highly recommended that you think through them before the next Tuesday lecture. You may also use the results of those problems in your written solutions to the graded problems.

Problem 1

For this problem, we consider functions valued in a fixed finite-dimensional complex inner product space $(V, \langle\ ,\ \rangle)$. Recall that for $s \in \mathbb{R}$, the Hilbert space $H^s(\mathbb{R}^n)$ is defined to consist of all tempered distributions $f \in \mathscr{S}'(\mathbb{R}^n)$ whose Fourier transforms $\widehat{f} \in \mathscr{S}'(\mathbb{R}^n)$ are represented by functions of the form $\widehat{f}(p) = (1 + |p|^2)^{-s/2} g(p)$ for some $g \in L^2(\mathbb{R}^n)$. The inner product on $H^s(\mathbb{R}^n)$ is given by

$$\langle f, g\rangle_{H^s} := \left\langle (1 + |p|^2)^{s/2} \widehat{f}, (1 + |p|^2)^{s/2} \widehat{g} \right\rangle_{L^2}.$$

If a distribution $f \in H^s(\mathbb{R}^n)$ is representable by a locally integrable function, we generally identify it with this function; note that this is always possible when $s \geqslant 0$, but not when $s < 0$. For an open subset $\Omega \subset \mathbb{R}^n$, the closure in $H^s(\mathbb{R}^n)$ of the space $C_0^\infty(\Omega)$ of smooth functions on $\mathbb{R}^n$ with compact support in Ω defines a closed subspace $\widetilde{H}^s(\Omega) \subset H^s(\mathbb{R}^n)$, and the quotient of $H^s(\mathbb{R}^n)$ by the closed subspace of distributions that vanish on test functions supported in Ω is denoted by $H^s(\Omega)$.

(a) $(*)$ Given $n \in \mathbb{N}$, for which $s \in \mathbb{R}$ is the Dirac δ-distribution in $H^s(\mathbb{R}^n)$? [3pts]

(b) Prove that $\mathscr{S}(\mathbb{R}^n)$ is dense in $H^s(\mathbb{R}^n)$ for every $s \in \mathbb{R}$.

(c) Prove that the pairing $\mathscr{S}(\mathbb{R}^n) \times \mathscr{S}(\mathbb{R}^n) \to \mathbb{C} : (\varphi, \psi) \mapsto \langle \varphi, \psi\rangle_{L^2}$ extends to a continuous real-bilinear pairing

$$H^{-s}(\mathbb{R}^n) \times H^s(\mathbb{R}^n) \to \mathbb{C} : (f, g) \mapsto \langle f, g\rangle := \langle (1 + |p|^2)^{-s/2} \widehat{f}, (1 + |p|^2)^{s/2} \widehat{g}\rangle_{L^2},$$

such that the real-linear map $f \mapsto \langle f, \cdot\rangle$ sends $H^{-s}(\mathbb{R}^n)$ isomorphically to the dual space of $H^s(\mathbb{R}^n)$.

(d) Given an open subset Ω with compact closure in $(0,1)^n$, associate to each $f \in C_0^\infty(\Omega)$ the unique function $F \in C^\infty(\mathbb{T}^n)$ such that $f(x) = F(x)$ for $x \in (0,1)^n$. Show that the map $C_0^\infty(\Omega) \to C^\infty(\mathbb{T}^n) : f \mapsto F$ extends to bounded linear injections

$$L^2(\Omega) \cong \widetilde{H}^0(\Omega) \hookrightarrow L^2(\mathbb{T}^n) \qquad \text{and} \qquad \widetilde{H}^1(\Omega) \hookrightarrow H^1(\mathbb{T}^n)$$

whose images are closed.
Hint: Avoid Fourier analysis here by replacing the usual H^1-norm with the equivalent norm $\|u\| := \sum_{|\alpha| \leqslant 1} \|\partial^\alpha u\|_{L^2}$. This works equally well on $\mathbb{R}^n$ or $\mathbb{T}^n$.

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[1] This version of the problem set has been revised to correct some errors that invalidated the original version of Problem 1(j) (worth 5 points).

(e) Deduce from the compactness of the inclusion $H^1(\mathbb{T}^n) \hookrightarrow L^2(\mathbb{T}^n)$ that the map $\widetilde{H}^s(\Omega) \to H^{-s}(\Omega) : f \mapsto [f]$ is compact for every $s \geqslant 1$ and every bounded open set $\Omega \subset \mathbb{R}^n$.

(f) Let $\Delta := \sum_{j=1}^n \partial_j^2$ denote the Laplace operator. Show that the linear map

$$\Phi : \mathscr{S}(\mathbb{R}^n) \to \mathscr{S}(\mathbb{R}^n) : u \mapsto u - \frac{1}{4\pi^2}\Delta u$$

has a unique extension to a unitary isomorphism $\Phi : H^1(\mathbb{R}^n) \to H^{-1}(\mathbb{R}^n)$.

(g) Let $I : H^{-1}(\mathbb{R}^n) \to (H^1(\mathbb{R}^n))^*$ denote the real-linear isomorphism from part (c). Show that the map $\widetilde{H}^1(\Omega) \to \left(\widetilde{H}^1(\Omega)\right)^* : u \mapsto I\Phi(u)\big|_{\widetilde{H}^1(\Omega)}$ is an isometric real-linear isomorphism, and deduce that $\widetilde{H}^1(\Omega) \to H^{-1}(\Omega) : u \mapsto [\Phi(u)]$ is an isomorphism.
Hint: Write down an explicit formula for $I\Phi(u)f$ for $u, f \in \widetilde{H}^1(\Omega)$.

(h) Deduce that $\widetilde{H}^1(\Omega) \to H^{-1}(\Omega) : u \mapsto [\Delta u]$ is a Fredholm operator of index 0.

(i) Show that the equation $\Delta u = 0$ has no nontrivial solutions $u \in C_0^\infty(\mathbb{R}^n)$.
Hint: What does integration by parts tell you about $\int_{\mathbb{R}^n} \langle u, \Delta u\rangle \, dm$?

(j) Prove that $\widetilde{H}^1(\Omega) \to H^{-1}(\Omega) : u \mapsto [\Delta u]$ is an isomorphism.
Hint: Extend the formula for $\int_{\mathbb{R}^n} \langle u, \Delta u\rangle \, dm$ in part (i) to all $u \in \widetilde{H}^1(\Omega)$, and use this to prove injectivity.

Problem 2
Assume X is a complex Banach space and $T \in \mathscr{L}(X)$. We say that $\lambda \in \mathbb{C}$ is an *approximate eigenvalue* of T if there exists a sequence $x_n \in X$ with $\|x_n\| = 1$ for all n such that $(\lambda - T)x_n \to 0$. Prove:

(a) Every approximate eigenvalue of T belongs to the spectrum $\sigma(T)$.

(b) $(*)$ If $\lambda \in \sigma(T)$ is neither an eigenvalue nor belongs to the residual spectrum of T, then it is an approximate eigenvalue of T. [4pts]

(c) $(*)$ For the operator $T : \ell^1 \to \ell^1 : (x_1, x_2, x_3, \ldots) \mapsto (x_2, x_3, x_4, \ldots)$, 1 is not an eigenvalue but is an approximate eigenvalue. [4pts]

Problem 3
Given a complex Banach space X and $T \in \mathscr{L}(X)$, let $T' \in \mathscr{L}(X^*)$ denote the transpose, also known as the dual operator of T.[2] Prove:

(a) If $\lambda \in \sigma(T)$ is in the residual spectrum of T then it is an eigenvalue of T'.

(b) $(*)$ If $\lambda \in \sigma(T')$ is an eigenvalue of T', then it is either an eigenvalue of T or belongs to the residual spectrum of T. [4pts]

Now suppose X is a complex Hilbert space $\mathcal{H}$, and $T^* : \mathcal{H} \to \mathcal{H}$ denotes the adjoint operator, defined via the condition $\langle x, Ty\rangle = \langle T^*x, y\rangle$ for all $x, y \in \mathcal{H}$. Prove:

(c) $\sigma(T^*) = \left\{\bar{\lambda} \in \mathbb{C} \;\middle|\; \lambda \in \sigma(T')\right\}$

(d) $\sigma(T) = \sigma(T')$

Hint: $T^* : \mathcal{H} \to \mathcal{H}$ and $T' : \mathcal{H}^* \to \mathcal{H}^*$ are closely related via the complex-antilinear isomorphism $\mathcal{H} \to \mathcal{H}^* : x \mapsto \langle x, \cdot\rangle$.

[2] We have sometimes denoted T' in the past by T^*, but will now be reserving the latter notation for the adjoint of an operator on a complex Hilbert space.

Problem 2

Assume X is a complex Banach space and $T \in \mathscr{L}(X)$. We say that $\lambda \in \mathbb{C}$ is an *approximate eigenvalue* of T if there exists a sequence $x_n \in X$ with $\|x_n\| = 1$ for all n such that $(\lambda - T)x_n \to 0$. Prove:

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(c) $(*)$ For the operator $T : \ell^1 \to \ell^1 : (x_1, x_2, x_3, \ldots) \mapsto (x_2, x_3, x_4, \ldots)$, 1 is not an eigenvalue but is an approximate eigenvalue. [4pts]

Problem 3

$\lambda \in \mathbb{C}$ is a.e.v. if $\exists\, x_n \in X$ w/ $\|x_n\| = 1$ $\forall n$

s.t. $(\lambda I - T)x_n \longrightarrow 0$ as $n \to \infty$.

a)

Want:- every approximate e.v. belongs to the $\sigma(T)$.

Suppose λ is an a.e.v. and $\lambda \notin \sigma(T)$

$\Rightarrow$ $\lambda - T$ is invertible $\Rightarrow$ $(\lambda - T)^{-1} \in \mathscr{L}(X)$

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} (\lambda - T)^{-1}(\lambda - T)x_n$$

$$= (\lambda - T)^{-1} \lim_{n\to\infty} (\lambda - T)x_n$$

$$= (\lambda - T)^{-1}\, 0 = 0$$

$\Rightarrow$ $\lim_{n\to\infty} x_n = 0$ contradiction as $\|x_n\| = 1$ $\forall n$

$\Rightarrow$ $\lambda \in \sigma(T)$. ∎

b) $\lambda \in \sigma(T)$ neither an e.v nor in the residual spectrum.

Want :- λ is an approx. e.v.

$\Rightarrow$ $\ker(\lambda - T) = \{0\}$, $\lambda - T$ is injective.

$\Rightarrow$ $\text{im}(\lambda - T)$ is dense in X.

But $\text{im}(\lambda - T) \neq X$ as if it were to be equal to X then by the inverse mapping theorem we'd get that $\lambda - T$ is invertible contradicting $\lambda \in \sigma(T)$.

Recall Prob. 4(a) from the midterm

$\nexists$ any $c > 0$ s.t. $\|(\lambda - T)x\| \geq c\|x\|$ $\forall x$.

This means that $\forall n$, $\exists$ $x_n \in X$

s.t. $\|(\lambda - T)x_n\| < \frac{1}{n}\|x_n\|$

$\Rightarrow$ $\|x_n\| > 0$ $\forall n$ $\Rightarrow$ choose $x_n' = \dfrac{x_n}{\|x_n\|}$ $\forall n$

$\Rightarrow$ $\|x_n'\| = 1$

$\|(\lambda - T)x_n'\| < \frac{1}{n}$ $\Rightarrow$ $\lim\limits_{n \to \infty} \|(\lambda - T)x_n'\| \to 0$

$\Rightarrow$ λ is an approx. e.v. of T. $\boxtimes$

c) $T: \ell^1 \to \ell^1$, $T((x_1, x_2, x_3, \dots)) = (x_2, x_3, x_4, \dots)$

1 is NOT an eigenvalue of T.

If 1 were an e.v. $\Rightarrow$ $\exists\ x \neq 0 \in \ell^1$

s.t. $Tx = 1 \cdot x = x$

$\Rightarrow (x_2, x_3, \dots) = (x_1, x_2, x_3, \dots)$

$\Rightarrow x_1 = x_2 = x_3 = \dots$

$\Rightarrow x = (x_1, x_1, x_1, \dots)$

Then $\|x\|_{\ell^1} = \sum_n |x_1| < \infty \iff x_1 = 0$
$\iff x = 0.$

contradiction $\Rightarrow$ 1 is NOT an eigenvalue of T.

<u>1 is an approx. eigenvalue of T.</u>

Consider the sequence $(x^m)_{m \in \mathbb{N}}$ in ℓ^1

$$x^1 = (1, 0, 0, 0, \dots)$$
$$x^2 = (1/2, 1/2, 0, 0, \dots)$$
$$x^3 = (1/3, 1/3, 1/3, 0, 0, \dots)$$
$$\vdots$$
$$x^n = (\underbrace{1/n, 1/n, 1/n, \dots, 1/n}_{n\text{-times}}, 0, 0, \dots)$$
$$\vdots$$

$$\Rightarrow \quad \|x^m\| = 1 \quad \forall\, m \in \mathbb{N}.$$

$$\|(1-T)x^m\|_{\ell^1} \longrightarrow 0 \quad \text{as } m \to \infty$$

$$\|$$

$$\Big\| \big(\underbrace{\tfrac{1}{m}, \tfrac{1}{m}, \tfrac{1}{m}, \dots, \tfrac{1}{m}}_{m\text{-times}}, 0, 0, \dots\big) - \big(\underbrace{\tfrac{1}{m}, \tfrac{1}{m}, \dots, \tfrac{1}{m}}_{m-1\text{-times}}, 0, 0, \dots\big) \Big\|_{\ell^1}$$

$$= \big\| \big(0, 0, \dots, 0, \tfrac{1}{m}, 0, \dots 0\big) \big\|_{\ell^1} = \frac{1}{m}$$

$$\longrightarrow 0 \quad \text{as } m \to \infty$$

$\Rightarrow$ 1 is an approx. eigenvalue of T.

$\boxtimes$

eigenvalue but is an approximate eigenvalue. [2pts]

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Problem 3
Given a complex Banach space X and $T \in \mathscr{L}(X)$, let $T' \in \mathscr{L}(X^*)$ denote the transpose, also known as the dual operator of T.[2] Prove:

(a) If $\lambda \in \sigma(T)$ is in the residual spectrum of T then it is an eigenvalue of T'.

(b) $(*)$ If $\lambda \in \sigma(T')$ is an eigenvalue of T', then it is either an eigenvalue of T or belongs to the residual spectrum of T. [4pts]

Now suppose X is a complex Hilbert space $\mathcal{H}$, and $T^* : \mathcal{H} \to \mathcal{H}$ denotes the adjoint operator, defined via the condition $\langle x, Ty \rangle = \langle T^* x, y \rangle$ for all $x, y \in \mathcal{H}$. Prove:

(c) $\sigma(T^*) = \{\bar{\lambda} \in \mathbb{C} \mid \lambda \in \sigma(T')\}$

(d) $\sigma(T) = \sigma(T')$

Hint: $T^* : \mathcal{H} \to \mathcal{H}$ and $T' : \mathcal{H}^* \to \mathcal{H}^*$ are closely related via the complex-antilinear isomorphism $\mathcal{H} \to \mathcal{H}^* : x \mapsto \langle x, \cdot \rangle$.

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a) $\lambda \in \sigma(T)$ is in the residual spectrum of T.

Want:- λ is an e.v. of T'.

$T' : X^* \longrightarrow X^*$ is defined as $f : X \to \mathbb{C}$

$T'(f) = f \circ T$

$\Rightarrow$ we want a nonzero $f \in X^*$ w.r.t

$T'(f) = \lambda f \Rightarrow f \circ T = \lambda f$

$\Rightarrow \operatorname{im}(\lambda - T) \subseteq \ker(f).$

$$f|_{\operatorname{im}(\lambda - T)} = 0$$

follows from one of the consequences of the Hahn-Banach theorem.

$\operatorname{im}(\lambda - T)$ is NOT dense in X.

$\Rightarrow f \in X^*$ is non-zero.

$\Rightarrow \lambda$ is an eigenvalue of T'. ☒

b) $\lambda \in \sigma(T')$ and is an eigenvalue of T'.

either λ is an e.v. of T or is in the residual spectrum.

$\Rightarrow \exists$ some $f \in X^*$ s.t.

$T'(f) = \lambda f$, $f \neq 0$.

$\Rightarrow f \circ T = \lambda f$

$\Rightarrow f((\lambda - T)x) = 0 \quad \forall x \in X.$

Suppose $\mathrm{im}(\lambda - T)$ is dense.

$\Rightarrow \quad f_{|\mathrm{im}(\lambda-T)} = 0$

$\Rightarrow \quad f_{|X} = 0 \quad \Rightarrow \quad f = 0$ contradiction

$\Rightarrow$ $\mathrm{im}(\lambda - T)$ is NOT dense in X

$\Rightarrow$ λ is in the residual spectrum of X.

$\boxtimes$

For c) and d) X is a complex Hilbert space

$\Rightarrow$ if $Y \subseteq X$ is a closed subspace

$\Rightarrow$ Y has an orthogonal complement $Y^\perp$

Want:- $\sigma(T^*) = \{ \bar{\lambda} \in \mathbb{C} \mid \lambda \in \sigma(T') \}$

If λ is in the residual spectrum of T

$\Rightarrow$ $\bar{\lambda}$ is an e.v. of T^*

$\mathrm{im}(\lambda - T)$ is a closed subspace of X

$\mathrm{im}(\lambda - T) \neq X$. as λ is in residual spectrum of T.

$\Rightarrow \quad \exists \quad x_0 \neq 0 \in \mathrm{im}(\lambda - T)^\perp$

as X is a Hilbert space.

$\Rightarrow$ for any $x \in X$

$$0 = \langle (\lambda - T)x, x_0 \rangle = \langle x, (\bar{\lambda} - T^*) x_0 \rangle$$

$\Rightarrow$ $\bar{\lambda}$ is an eigenvalue of T^* $\boxtimes$

part d). In the same way. $\boxtimes$

(1) $\delta \in H^s(\mathbb{R}^n)$ for what s?

$\delta \in \mathcal{S}'(\mathbb{R}^n)$ $\quad \mathcal{F}(\delta) = \mathcal{F}^{-1}(\delta) = 1$

$\| (1+|p|^2)^{s/2} \|_{L^2} < \infty \rightarrow$ want to check.

$$\Rightarrow \int_{\mathbb{R}^n} (1+|p|^2)^s \, dp < \infty$$

For $s \geq 0$ $\quad \int_{\mathbb{R}^n} (1+|p|^2)^s \, dp \geq \int_{\mathbb{R}} 1 \, dp = \infty$

$\therefore$ $s < 0$.

So suppose $s < 0$

integrate $\int_{\mathbb{R}^n} (1+|p|^2) \, dp$ using polar coordinates

in $\mathbb{R}^n$.

$$\int_{\mathbb{R}^n} (1+|p|^2)^s \, dp = \operatorname{vol}(S^{n-1}) \int_0^\infty r^{n-1} (1+r^2)^s \, dr$$

$$= \operatorname{vol}(S^{n-1}) \left[\int_0^1 r^{n-1} (1+r^2)^s \, dr + \underbrace{\int_1^\infty r^{n-1} (1+r^2)^s \, dr}_{I_2} \right.$$

$\because \quad s < 0 \Rightarrow \quad (1+r^2)^s \geq (2r^2)^s = 2^s r^{2s}$

$$I_2 \geq \int_1^\infty r^{n-1+2s} \, dr < \infty$$

$\Leftarrow \quad n + 2s < 0 \Rightarrow s < -\frac{n}{2}.$

$\therefore \quad \delta \in H^s(\mathbb{R}^n) \Rightarrow s < -n/2.$

$\boxtimes$

(b) $\mathcal{S}(\mathbb{R}^n)$ is dense in $H^s(\mathbb{R}^n)$ $\forall s \in \mathbb{R}$.
follows from the fact $C_0^\infty(\mathbb{R}^n)$ is dense in $H^s(\mathbb{R}^n)$.

c) $\mathcal{S}(\mathbb{R}^n) \times \mathcal{S}(\mathbb{R}^n) \to \mathbb{C}$

$(\varphi, \psi) \longmapsto \langle \varphi, \psi \rangle_{L^2}$ —— 1

extends to a real bilinear pairing

$$H^{-s}(\mathbb{R}^n) \times H^s(\mathbb{R}^n) \longrightarrow \mathbb{C}$$

$$(f, g) = \langle (1+|p|^2)^{-s/2} \hat{f}, (1+|p|^2)^{s/2} \hat{g} \rangle_{L^2} \quad \text{—— } 2$$

$$f \longmapsto \langle f, \cdot \rangle$$

$$H^{-s}(\mathbb{R}^n) \mapsto (H^s(\mathbb{R}^n))^*$$

To check 2 is indeed an extension of 1

$\varphi, \psi \in \mathcal{S}(\mathbb{R}^n)$

$$\langle (1+|p|^2)^{-s/2} \hat{\varphi}, (1+|p|^2)^{s/2} \hat{\psi} \rangle_{L^2}$$

$$= \int_{\mathbb{R}^n} (1+|p|^2)^{-s/2} |\hat{\varphi}(p)| (1+|p|^2)^{s/2} |\hat{\psi}(p)| \, dp$$

$$= \int_{\mathbb{R}^n} |\hat{\varphi}(p)| |\hat{\psi}(p)| \, dp \underset{\text{Plancheral}}{=} \int |\varphi(p)| |\psi(p)| \, dp$$

$$= \langle \varphi, \psi \rangle_{L^2}$$

$$H^{-s}(\mathbb{R}^n) \longmapsto (H^s(\mathbb{R}^n))^*$$

$$\alpha : f \longmapsto \langle f, \cdot \rangle$$

$$\alpha(f) = 0 \implies f = 0$$

$\Rightarrow \quad \langle f, g \rangle = 0 \quad \forall \; g \in \mathcal{H}^s(\mathbb{R}^n).$

$$\left\langle (1+|p|^2)^{-s/2} \hat{f} , \; (1+|p|^2)^{s/2} \hat{g} \right\rangle_{L^2} = 0$$

$\Rightarrow \quad (1+|p|^2)^{-s/2} \hat{f} = 0$

$\Rightarrow \quad f = 0$

$\Rightarrow \quad \alpha$ is injective.

Hint:- Riesz representation theorem.

for any arbitrary $g \in L^2(\mathbb{R}^n)$

we can define L^1_{loc} $\tilde{g} = (1+|p|^2)^{-s/2} g$.

$\Lambda : L^2(\mathbb{R}^n) \to \mathbb{C}$ by

$$\Lambda(g) = \beta\left(\mathcal{F}^*\left((1+|p|^2)^{-s/2} g\right)\right)$$

$\beta \in [H^s(\mathbb{R}^n)]^*$

$\Rightarrow \quad \exists! \; h \in L^2(\mathbb{R}^n)$ s.t.

$\langle h, g \rangle_{L^2} = \Lambda(g) \quad \forall \; g \in L^2.$

To get surjectivity, define

$$f = \mathcal{F}^*\left((1+|p|^2)^{-s/2} h\right)$$

$$\hat{H}^{-s}(\mathbb{R}^n)$$

$$\alpha(f) = \beta.$$

$\boxtimes$

(i) $\Delta u = 0$, $u \in C_0^\infty(\mathbb{R}^n)$,

Want $u \equiv 0$.

Suppose Ω a bounded open set which contains the supp(u).

Ω is connected

$$\int_{\mathbb{R}^n} \langle u, \underline{\Delta u} \rangle \, dm \underset{\text{IBP}}{=} -\int |\nabla u|^2 dm = 0$$

$$\int_{\mathbb{R}^n} u \nabla \nabla u \, dm = -\int \nabla u \nabla u \, dm$$
$$= -\int |\nabla u|^2 dm$$

$\Rightarrow |\nabla u|^2 = 0 \Rightarrow \nabla u = 0 \Rightarrow u$ is constant.

u is constant on $\mathbb{R}^n$

$u = 0$ outside supp(u).

$\Rightarrow$ the constant must be 0.

$\Rightarrow u \equiv 0$. $\boxtimes$