# Geodesics

$M$ a mfld w/ conn. $\nabla$,

<u>defn</u>: $\gamma : (a,b) \longrightarrow M$ is a <u>geodesic</u> if

$\nabla_t \dot\gamma \equiv 0$ (i.e. $\dot\gamma$ is a <u>parallel</u> vec. fld. along $\gamma$).

In local coords., $\gamma = (\gamma^1, \dots, \gamma^n)$,

$$\boxed{\ddot\gamma^i + \Gamma^i_{jk}(\gamma)\dot\gamma^j\dot\gamma^k = 0}$$

<u>Theory of ODEs</u>:

$\forall$ $p \in M$ & $X \in T_pM$, $\exists$ a maximal interval $(a,b) \subseteq \mathbb{R}$ & ! sol. $\gamma_X : (a,b) \longrightarrow M$ to the geod. eqn. s.t. $\gamma_X(0) = p$, $\dot\gamma_X(0) = X$; $\gamma_X(t)$ deps. smoothly on $t \in \mathbb{R}$ & $X \in TM$.

<u>observation</u>: If $\gamma$ a geod. & $\hat\gamma(t) = \gamma(ct)$ for some $c \in \mathbb{R}$, then $\nabla_t \partial_t \hat\gamma(t) = \nabla_t (c\dot\gamma(ct))$
$= c\nabla_t \dot\gamma(ct) \equiv 0 \implies \hat\gamma$ also a geod.

$\implies$ <u>cor</u>: $\gamma_X(t) =: \exp(tX)$ deps. only on $tX \in TM$.

$\implies$ <u>prop</u>: $\exists$ a maximal open set $\mathcal{O} \subseteq TM$ & smooth map $\exp : \mathcal{O} \longrightarrow M$ s.t. $\forall$ $X \in \mathcal{O}$, $I := \{t \in \mathbb{R} \mid tX \in \mathcal{O}\}$ is an open interval around $0$ & $\gamma : I \longrightarrow M : t \longmapsto \exp(tX)$ is the maximal geod. w/ $\dot\gamma(0) = X$. "<u>exponential map</u>" $\boxtimes$

<u>notation</u>: $p \in M$, $\exp_p := \exp|_{\mathcal{O} \cap T_pM}$. $\exp_p(0) = p$

The deriv. of $\exp_p$ at $0 \in T_pM$ as $\mathrm{Id} : T_pM \longrightarrow T_pM$

$\overset{\text{IFT}}{\implies}$ $\exp_p$ maps a nbhd of $0 \in T_pM$ to a nbhd of $p \in M$ diffeomorphically!

Assume $(M, g)$ a pseudo-Riemannian mfd.,
$g =: \langle \, , \, \rangle$. $\Rightarrow$ $TM$ has a $O(k,l)$-structure for
some $k + l = n = \dim N$.

<u>EX</u>: On a real V.B. $E \to M$ w/ bndl metric $\langle \, , \, \rangle =: g$
(i.e. $O(k,l)$-str.), a conn. is $O(k,l)$-compat.

$$\Leftrightarrow \partial_X \langle \eta, \xi \rangle = \langle \nabla_X \eta, \xi \rangle + \langle \eta, \nabla_X \xi \rangle$$

$$\Leftrightarrow \quad g \in \Gamma(E^0_2) \text{ satisfies } \nabla g \equiv 0.$$

We say $\nabla$ is a "<u>metric conn.</u>", $\Leftrightarrow$ it is <u>compatible</u> w/ $\langle \, , \, \rangle$.

<u>Fundamental</u> <u>thm</u> <u>of</u> <u>Riem.</u> <u>geometry</u>: $\forall$ ps.-Riem. mfds
$(M, g)$, $\exists !$ symmetric conn. on $TM$ compat. w/ $g$.

pf: If such $\nabla$ exists, then $\forall\ X, Y, Z \in \mathfrak{X}(M)$,

$$\partial_X \langle Y, Z\rangle = \langle \nabla_X Y, Z\rangle + \langle Y, \nabla_X Z\rangle \quad ①$$

$$\partial_Y \langle Z, X\rangle = \langle \nabla_Y Z, X\rangle + \langle Z, \nabla_Y X\rangle \quad ②$$

$$\partial_Z \langle X, Y\rangle = \langle \nabla_Z X, Y\rangle + \langle X, \nabla_Z Y\rangle \quad ③$$

$$① + ② - ③ = \langle \underbrace{\nabla_X Y + \nabla_Y X}_{2\nabla_X Y - [X,Y] \text{ (symmetry!)}}, Z\rangle + \langle \nabla_X Z - \nabla_Z X, Y\rangle + \langle \underbrace{\nabla_Y Z - \nabla_Z Y}_{[Y,Z] \text{ (symmetry!)}}, X\rangle$$

$$\Rightarrow 2\langle \nabla_X Y, Z\rangle = \langle [X,Y], Z\rangle - \langle [X,Z], Y\rangle - \langle [Y,Z], X\rangle + \partial_X\langle Y, Z\rangle + \partial_Y\langle Z, X\rangle - \partial_Z\langle X, Y\rangle.$$

claim (EX): RHS is $C^\infty$-linear w.r.t. $X$ & $Z$

$\Rightarrow$ can rewrite it as $S_Y(X, Z)$ for a tensor field $S_Y \in \Gamma(T^0_2 M)$ that deps. linearly on $Y \in \mathfrak{X}(M)$

$\Rightarrow$ We can defn $\nabla_X Y$ by $\langle \nabla_X Y, \cdot\rangle := \frac{1}{2} S_Y(X, \cdot)$

(determines $\nabla_X Y$ uniquely since $\langle\ ,\ \rangle$ is nondeg.!)

Now check: $\nabla_X Y$ satisfies Leibniz rule $\Rightarrow$ defn. a conn.,

& is symmetric & compat. w/ $\langle\ ,\ \rangle$. $\square$

## Local coords

Sps $E$ a real V.B. w/ bndl metric $\langle \, , \, \rangle$.

$\exists$ an iso, $\flat : E \longrightarrow E^* : v \longmapsto v_\flat := \langle v, \cdot \rangle$

$\# := \flat^{-1} : E^* \longrightarrow E : \lambda \longmapsto \lambda^\#$ (musical isomorphism)

Choose frame $e_1, \dots, e_m$ for $E$ over $U \overset{\text{open}}{\subseteq} M$,

$\rightsquigarrow$ dual frame $e^1_*, \dots, e^m_*$ for $E^*$, $v = v^i e_i \in E|_U$

$\lambda = \lambda_i e^i_* \in E^*|_U$.

Write $g_{ij} := \langle e_i, e_j \rangle$, $\Longrightarrow \langle v, w \rangle = g_{ij} v^i w^j$

Defn natural bndl metric on $E^*$ by $\langle \lambda, \mu \rangle := \langle \lambda^\#, \mu^\# \rangle$.

$g^{ij} := \langle e^i_*, e^j_* \rangle \implies \langle \lambda, \mu \rangle = g^{ij} \lambda_i \mu_j$.

notational convention:

$v = v^i e_i \iff v_\flat = v_i e^i_*$, $\quad \lambda = \lambda_i e^i_* \iff \lambda^\# = \lambda^i e_i$.

Now $\langle v, w \rangle = v_\flat(w) = w_\flat(v) = g_{ij} v^i w^j = v_i w^i = v^i w_i$

$\implies \boxed{v_i = g_{ij} v^j}$, sim. $\boxed{\lambda^i = g^{ij} \lambda_j}$

$\implies$ the metrics $(g_{ij})$ & $(g^{ij})$ are inverses!

$$\boxed{g^{ik} g_{kj} = \delta^i_j}.$$

On $(M, g)$, the ! symm. conn. $\nabla$ compat. w/ $g$ is called the Levi-Civitá connection.

Choose local coords $(x^1, \dots, x^n)$, write

$$g_{ij} = \langle \partial_i, \partial_j \rangle, \qquad g^{ij} = \langle dx^i, dx^j \rangle.$$

Let $X = \partial_i$, $Y = \partial_j$, $Z = \partial_k$, so

$$\langle \nabla_X Y, Z \rangle = g_{ab} (\nabla_i \partial_j)^a \delta^b_k = g_{ab} \Gamma^a_{ij} \delta^b_k = g_{ak} \Gamma^a_{ij}.$$

$$= \frac{1}{2} \left( \partial_X \langle Y, Z \rangle + \partial_Y \langle Z, X \rangle - \partial_Z \langle X, Y \rangle + \text{Lie bracket terms} \right)$$

$$= \frac{1}{2} \left( \partial_i g_{jk} + \partial_j g_{ki} - \partial_k g_{ij} \right)$$

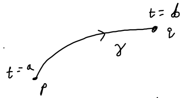
$$\Rightarrow \boxed{\Gamma^l_{ij} = \frac{1}{2} g^{kl} \left( \partial_i g_{jk} + \partial_j g_{ki} - \partial_k g_{ij} \right)}$$

ex: $(\mathbb{R}^n, g_E)$, $g_E$ = Euclidean inner product

$\Rightarrow (g_E)_{ij} = \delta_{ij}$ const. $\Rightarrow \Gamma^i_{jk} = 0$

$\Rightarrow$ geod. eqn. $\ddot{\gamma} = 0 \Rightarrow$ sols. are straight lines.

Assume $(M, g)$ a Riem. mfd.




Fix $a < b$ & $p, q \in M$,

$$C^\infty(p,q) := \{ \gamma : [a,b] \xrightarrow{C^\infty} M \mid \gamma(a) = p, \ \gamma(b) = q \}.$$

For $X \in TM$, $|X| := \sqrt{\langle X, X \rangle}$, a path $\gamma \in C^\infty(p,q)$ has <u>speed</u> $|\dot\gamma(t)|$ at time $t$.

The <u>length</u> of $\gamma$ is $\ell(\gamma) := \int_a^b |\dot\gamma(t)| \, dt$.

Its <u>energy</u> is $E(\gamma) := \int_a^b \langle \dot\gamma(t), \dot\gamma(t) \rangle \, dt$

<u>defn</u>: For a functional $F : C^\infty(p,q) \to \mathbb{R}$, a path $\gamma \in C^\infty(p,q)$ is <u>stationary</u> if $\forall$ smooth 1-param. families $\{\gamma_s \in C^\infty(p,q)\}_{s \in (-\varepsilon, \varepsilon)}$ (i.e. $(s,t) \mapsto \gamma_s(t)$ is $C^\infty$), (s.t. $\gamma_0 = \gamma$)

$$\frac{d}{ds} F(\gamma_s)\Big|_{s=0} = 0.$$

<u>prop</u>: $\gamma$ is a geodesic $\iff$ it is stationary for $E$.
$\iff$ it has constant speed & is stationary for $\ell$.

<u>pf</u>: Assume $\gamma_0 = \gamma$, $\partial_s \gamma_s|_{s=0} =: \eta \in \Gamma(\gamma^* TM)$.

$$\frac{d}{ds} E(\gamma_s)\Big|_{s=0} = \frac{d}{ds} \int_a^b \langle \dot\gamma_s(t), \dot\gamma_s(t) \rangle \, dt \, \Big|_{s=0}$$

$$= \int_a^b \partial_s \langle \dot\gamma_s(t), \dot\gamma_s(t) \rangle \Big|_{s=0} dt = 2 \int_a^b \langle \underbrace{\nabla_s \dot\gamma_s(t)}_{\nabla_s \partial_t = \nabla_t \partial_s}\Big|_{s=0}, \dot\gamma(t) \rangle \, dt$$

$$= 2 \int_a^b \langle \nabla_t \partial_s \gamma_s |_{s=0}, \dot\gamma(t) \rangle \, dt = 2 \int_a^b \langle \nabla_t \eta(t), \dot\gamma(t) \rangle \, dt.$$

Since $\eta(a) = \eta(b) = 0$, $\int_a^b \partial_t \langle \eta(t), \dot\gamma(t) \rangle \, dt = 0$

$$= \int_a^b \langle \nabla_t \eta, \dot\gamma \rangle \, dt + \int_a^b \langle \eta, \nabla_t \dot\gamma \rangle \, dt$$

$$\Rightarrow \frac{d}{ds} E(\gamma_s)\Big|_{s=0} = -2 \int_a^b \langle \eta(t), \nabla_t \dot\gamma(t) \rangle \, dt = 0$$

$\forall \eta$ satisfying $\eta(a) = \eta(b) = 0 \iff \nabla_t \dot\gamma \equiv 0$.