Recall: $\gamma \in C^\infty(p,q) := \{\gamma : [a,b] \to M \mid \gamma(a) = p,\ \gamma(b) = q\}$.

$$\ell(\gamma) := \int_a^b |\dot\gamma(t)|\, dt, \qquad E(\gamma) := \int_a^b \langle \dot\gamma(t), \dot\gamma(t) \rangle\, dt$$

proved: γ is stationary for $E \iff \nabla_t \dot\gamma = 0$ (geodesic eqn.)

observe (1) $\ell(\gamma) = \ell(\gamma \circ \varphi)$ for any reparam. $\varphi : [a,b] \xrightarrow{\cong} [a,b]$.

(2) geodesics have constant speed: $\partial_t \langle \dot\gamma(t), \dot\gamma(t) \rangle = 2 \langle \underbrace{\nabla_t \dot\gamma(t)}_{=0}, \dot\gamma(t) \rangle$

~~Spse~~ γ a geod., $\{\gamma_s \in C^\infty(p,q)\}_{s \in (-\varepsilon, \varepsilon)}$ s.t.

$\gamma_0 = \gamma$, $\partial_s \gamma_s \big|_{s=0} =: \eta \in \Gamma(\gamma^* TM)$.

WLOG, reparametrize each γ_s s.t. $v_s := |\dot\gamma_s(t)|$ indep. of t for each s.

Now
$$\frac{d}{ds} \ell(\gamma_s)\Big|_{s=0} = \int_a^b \frac{\partial}{\partial s} \sqrt{\langle \dot\gamma_s(t), \dot\gamma_s(t) \rangle}\,\Big|_{s=0} dt$$

$$= \int_a^b \frac{1}{\sqrt{\langle \dot\gamma_0(t), \dot\gamma_0(t) \rangle}} \frac{\partial}{\partial s} \langle \dot\gamma_s(t), \dot\gamma_s(t) \rangle \Big|_{s=0} dt = \frac{1}{v_0} \frac{d}{ds} E(\gamma_s)\Big|_{s=0} = 0.$$

$\Rightarrow$ cor: γ is geodesic $\iff$ it has constant speed & is stationary for ℓ. $\square$

normal coordinates

defn: Call a metric g on $\mathbb{R}^n$ *flat* if its cpnts $g_{ij} = \langle \partial_i, \partial_j \rangle$ are *constant*.

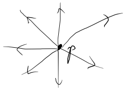
$$\xrightarrow{\text{coord. change}} (g_{ij}) = (\eta_{ij}) := \begin{pmatrix} 1 & & & & \\ & \ddots & & & \\ & & 1 & & \\ & & & -1 & \\ & & & & \ddots \\ & & & & & -1 \end{pmatrix}$$

$\underbrace{\quad}_{k}\ \underbrace{\quad}_{\ell}$

"signature (k,ℓ)"

Q: Given any ps.-Riem. mfd (M,g) of signature (k,ℓ) & $p \in M$, can we choose coords. near p s.t. $g_{ij} \approx \eta_{ij}$?

idea:



Choose an O-N basis $X_1, \dots, X_n$ of T_pM, defn. a chart near p as inverse of

$$\mathbb{R}^n \overset{\text{open}}{\supseteq} \underset{\underset{0}{\Psi}}{\mathcal{O}} \xrightarrow[\cong]{\varphi} \underset{\underset{p}{\Psi}}{\mathcal{U}} \overset{\text{open}}{\subseteq} M : (t^1, \dots, t^n) \longmapsto \exp_p(t^i X_i).$$

rays from 0 in $\mathbb{R}^n$ $\longleftrightarrow$ geods. from p in M

In coords., each ray from 0 in $\mathbb{R}^n$ = flow line of a const. vector field

$\Rightarrow$ any vec. fld $Y \in \mathfrak{X}(M)$ w/ const. cpnts. near p, $\nabla_{Y(p)} Y = 0$ since $\exists$ a geod. γ w/ $\dot\gamma(0) = Y(p)$ that is a flow line of Y. In particular,

$$0 = \nabla_{\partial_i + \partial_j}(\partial_i + \partial_j)\Big|_p = \underbrace{\nabla_i \partial_i}_{=0} + \underbrace{\nabla_j \partial_j}_{=0} + \underbrace{\nabla_i \partial_j}_{\Gamma_{ij}^k \partial_k} + \underbrace{\nabla_j \partial_i}_{\Gamma_{ji}^k \partial_k}\Big|_p$$

$$\Rightarrow \Gamma_{ij}^k(p) + \Gamma_{ji}^k(p) = 0 \xRightarrow{\text{symmetry of } \nabla} \Gamma_{ij}^k(p) = 0.$$

Since $X_1, \dots, X_n \in T_pM$ are O-N,

$$g_{ij}(p) = \langle \partial_i, \partial_j \rangle\big|_p = \eta_{ij}$$

$$\partial_k g_{ij}(p) = \partial_k \langle \partial_i, \partial_j \rangle\big|_p = \langle \underbrace{\nabla_k \partial_i}_{=0}, \partial_j \rangle + \langle \partial_i, \underbrace{\nabla_k \partial_j}_{=0} \rangle\big|_p = 0.$$

thm: In these coords. ("Riemann normal coords"), cpnts g_{ij} of metric satisfy $g_{ij}(p) = \eta_{ij}$ & $\partial_k g_{ij}(p) = 0$ $\forall$ i, j, k.

$\boxtimes$

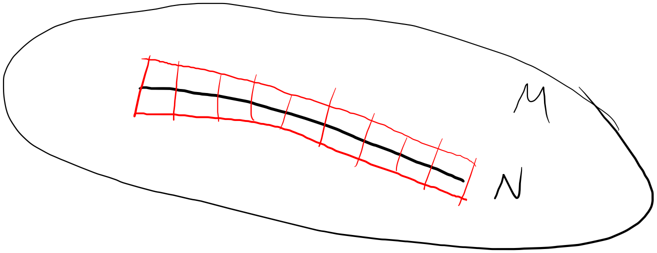
rk: Curvature $\Rightarrow$ can't do this for 2nd derivatives!

EX ("tubular nbhd thm"): Given a cpct submfd $N \subseteq M$ $(\partial N = \partial M = \emptyset)$, choose Riem. metric on M.

Prove: for $\varepsilon > 0$ suff. small, the map

$$\{X \in (TN)^\perp \subseteq TM|_N \mid |X| < \varepsilon\} \xrightarrow{\exp} M$$

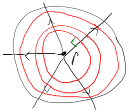
is a diffeo. onto a nbhd of N, sending 0-section of $(TN)^\perp$ to N.



shortest paths btwn pts.

thm: (M,g) a Riem. mfd, $p \in M$, $\exists$ a nbhd $\mathcal{U} \subseteq M$ of p s.t. $\forall$ $q \in \mathcal{U}$, $\exists !$ geodesic segment $\gamma : [0,1] \longrightarrow M$ from $\gamma(0)=p$, $\gamma(1)=q$ that is shorter than all other paths from p to q (except reparams of γ).

idea: Fix $\varepsilon > 0$ small s.t. $\mathcal{O} := \{ X \in T_pM \mid |X| < \varepsilon \} \xrightarrow{\exp_p} M$ is an embedding. For $r > 0$ small, consider sphere



$$\Sigma_r := \{ \exp_p(X) \mid X \in \mathcal{O},\ |X|^2 = r \}$$

Gauss lemma: The geodesics emerging from p hit the spheres Σ_r orthogonally.

pf: Suppose $q := \exp_p(X) \in \Sigma_r$, $Y \in T_q\Sigma_r$.

To show: $\langle \frac{d}{ds} \exp_p(sX)\big|_{s=1}, Y \rangle = 0$.

$Y = \partial_t \exp_p(X(t))\big|_{t=0}$ for some path $X(t) \in T_pM$ w/ $|X(t)|^2 = r$.

Let $f(s,t) := \exp_p(sX(t)) \in M$, so

$\partial_s f(1,0) = \frac{d}{ds} \exp(sX)\big|_{s=0}$, $\quad \partial_t f(1,0) = Y$.

main claim: $\langle \partial_s f, \partial_t f \rangle \equiv 0$.

$f(0,t) = p \ \forall\ t \Rightarrow \partial_t f(0,t) = 0 \Rightarrow$ claim is true at $s=0$.

$$\partial_s \langle \partial_s f, \partial_t f \rangle = \langle \underbrace{\nabla_s \partial_s f}_{\overset{\shortparallel}{0}\ \text{(geod. eqn.)}}, \partial_t f \rangle + \langle \partial_s f, \nabla_s \partial_t f \rangle$$

$$= \langle \partial_s f, \nabla_t \partial_s f \rangle = \boxed{\frac{1}{2} \partial_t \langle \partial_s f, \partial_s f \rangle}.$$

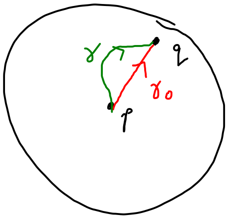
Each geod. $s \longmapsto f(s,t)$ has const. speed^2

$$\langle \partial_s f(s,t), \partial_s f(s,t) \rangle = \langle \partial_s f(0,t), \partial_s f(0,t) \rangle = \langle X(t), X(t) \rangle$$

$= r$ (indep. of t!)

$\Rightarrow \partial_s \langle \partial_s f, \partial_t f \rangle \equiv 0 \Rightarrow \langle \partial_s f, \partial_t f \rangle \equiv 0$. ∎

pf of thm on shortest paths:



Assuming $q = \exp_p(X)$ for $|X|^2 = r$

let $\gamma_0(t) = \exp_p(tX)$, $t \in [0,1]$,

$$\Rightarrow \quad \ell(\gamma_0) = \sqrt{r} .$$

Assume $\gamma(t) = \exp_p(\rho(t)\, X(t))$

s.t. $\rho(t) > 0$ (for $t > 0$), $|X(t)|^2 = r$,

$\rho(1) = 1$, $X(1) = X$, $\lim\limits_{t \to 0} \rho(t) = 0$.

$\gamma(t) = f(\rho(t), t)$ for $f(s,t) = \exp_p(s X(t))$,

$$\Rightarrow |\dot\gamma(t)|^2 = \left| \partial_s f(\rho(t), t) \frac{d\rho}{dt} + \partial_t f(\rho(t), t) \right|^2$$

$$= \underbrace{|\partial_s f(\rho(t),t)|^2}_{= r} \cdot |\dot\rho(t)|^2 + |\partial_t f(\rho(t),t)|^2$$

$$\geq r\, |\dot\rho(t)|^2 ,$$ strict ineq. unless γ is a reparam. of γ_0.

$$\Rightarrow \ell(\gamma) > \int_0^1 \sqrt{r}\, |\dot\rho(t)|\, dt \geq \sqrt{r} \int_0^1 \dot\rho(t)\, dt = \sqrt{r}$$

$$= \ell(\gamma_0).$$

∎