Übungen today:

13:30 – takehome questions

17:15 – geods. as Ham. system

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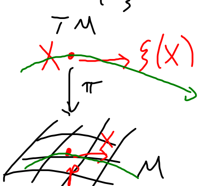
geodesic completeness:

(M, g) is geod. complete if all geodesics are def'd for all $t \in \mathbb{R}$ $\Longleftrightarrow$ the domain of exp is TM.

thm: all cpt Riem. mfds are geod. complete.

idea: $\exp = \pi \circ \varphi^1_\xi$ where $\pi : TM \to M$ & φ^t_ξ is flow of a vec. fld $\xi \in \mathfrak{X}(TM)$.



$X \longmapsto \xi(X) := \mathrm{Hor}_X(X) \in T_X(TM)$

$\Longrightarrow$ for any flow line $\alpha(t) \in TM$ of ξ, $\pi \circ \alpha(t) =: \gamma(t) \in M$ is a geodesic.

$\Longrightarrow$ exp has same domain as φ^1_ξ.

Each flow line of ξ is confined to a cpt subset of form $\{X \in TM \mid |X| = r\}$ for some $r > 0$

$\Longrightarrow$ flow of ξ exists globally! $\boxtimes$

notation for ps-Riem. metrics in coords:

$$g = g_{ij}\, dx^i \otimes dx^j \qquad (g_{ij} = g_{ji})$$

$$=: \sum_{i \le j} g_{ij}\, dx^i dx^j$$

where $dx^i dx^j := \frac{1}{2}\left(dx^i \otimes dx^j + dx^j \otimes dx^i\right)$.

ex: On $\mathbb{R}^2 \ni (x,y)$, Euclidean metric $g_E = dx^2 + dy^2$

Poincaré half-plane $h = \frac{1}{y^2}\left(dx^2 + dy^2\right)$.

isometries:

Defn: For ps.-Riem. mflds (M,g), (N,h), a diffeo $\varphi: M \longrightarrow N$ is an isometry $(M,g) \longrightarrow (N,h)$ if $\varphi^* h = g$.

i.e. $\forall\ p \in M$, $X, Y \in T_pM$, $h_{\varphi(p)}(\varphi_* X, \varphi_* Y) = g_p(X,Y)$

i.e. φ preserves lengths & angles.

Defn: φ is a conformal transformation $(M,g) \longrightarrow (N,h)$ if $\varphi^* h = f g$ for some fn. $f: M \longrightarrow (0,\infty)$.

i.e. $\forall\ p \in M$, $X, Y \in T_pM$, $h_{\varphi(p)}(\varphi_* X, \varphi_* Y) = f(p)\, g_p(X,Y)$.

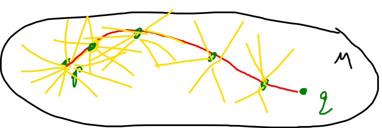
i.e. φ preserves angles (but not nec. lengths)

thm: Suppose (M,g) a connected ps.-Riem mfd, $p, q \in M$, $X_1, \dots, X_n \in T_pM$, $Y_1, \dots, Y_n \in T_qM$ O-N bases.

Then $\mathrm{Isom}(M,g) := \{\text{isometries } (M,g) \to (M,g)\}$ contains at most one φ s.t. $\varphi(p) = q$ & $\varphi_* X_i = Y_i$ $\forall\ i = 1, \dots, n$.

pf: Need to show: if $\varphi \in \mathrm{Isom}(M,g)$ s.t. $\varphi(p) = p$ & $T_p\varphi = \mathrm{Id}: T_pM \to T_pM$, then $\varphi \equiv \mathrm{Id}$ on M.

φ maps geods. to geods. $\Rightarrow$ $\varphi = \mathrm{Id}$ on the open subset of pts. reachable from p by geodesics.



$\square$

$\underline{Q}$: (M, g) a ps.-Riem. mfd, $\Sigma \subseteq M$ a submfd.

- Is $(\Sigma, g|_{T\Sigma})$ a ps.-Riem. mfd (i.e. is $g|_{T\Sigma}$ nondeg.?)

  (<u>defn</u>: If so, we call Σ a <u>ps.-Riem. submfd</u> of (M, g).)

- What are its geodesics?

<u>EX</u>: V a fin-dim. real vec. sp. w/ nondeg. symmetric bilinear form $\langle\, , \rangle$, $W \subseteq V$ a subspace; let

$$W^{\perp} := \{ v \in V \mid \langle v, w \rangle = 0 \ \forall\, w \in W \}.$$

Show: (1) $\dim W + \dim W^{\perp} = \dim V$

(2) $(W^{\perp})^{\perp} = W$

(3) $\langle\, , \rangle|_W$ is nondeg. $\iff$ $W \cap W^{\perp} = \{0\}$ $\iff$ $V = W \oplus W^{\perp}$.

<u>idea</u>: $W^{\perp} = \ker\left(V \longrightarrow W^{*} : v \longmapsto \langle v, \cdot \rangle|_W \right)$.

<u>cor</u>: $\Sigma \subseteq (M, g)$ is a ps.-Riem. submfd $\iff$

$\forall\, p \in \Sigma$, $T_pM = T_p\Sigma \oplus (T_p\Sigma)^{\perp}$. ⊡

$\Rightarrow$ $\exists$ ortho. proj. $\pi_\Sigma : TM|_\Sigma \longrightarrow T\Sigma$ (proj. along $(T\Sigma)^{\perp}$)

prop: For $\Sigma \subseteq (M, g)$ a ps.-Riem. submfd, ∇ the L-C conn. on (M, g). Then the L-C. conn. $\hat{\nabla}$ on Σ is given by

$$\hat{\nabla}_X Y = \pi_\Sigma(\nabla_X Y)$$ for any extension of $Y \in \mathcal{X}(\Sigma)$ to a vec. fld. on M.

idea of pf: Verify that $\pi_\Sigma \circ \nabla$ satisfies Leibniz rule ($\Rightarrow$ is a conn. on Σ), & is compat. w/ $g|_{T\Sigma}$ & is symmetric. ☒

cor: $\gamma : (a, b) \longrightarrow \Sigma$ is a geodesic iff for the L-C conn. ∇ on M, $\nabla_t \dot{\gamma}(t) \in (T_{\gamma(t)} \Sigma)^\perp \ \forall t$.

Canonical exs. of Riem. mfds

(1) $(\mathbb{R}^n, g_E)$ Euclidean space, $g_E = (dx^1)^2 + \dots + (dx^n)^2$

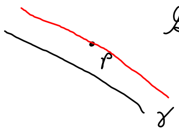
$\Rightarrow$ geod. eqn. is $\ddot{\gamma} \equiv 0$ $\Rightarrow$ straight lines.

properties: (E1) $\forall\ p \neq q,\ \exists !$ geod. segment

$\gamma : [0,1] \to M$ w/ $\gamma(0) = p,\ \gamma(1) = q$.

(E2) Geodesically complete: all geods. are def'd on $\mathbb{R}$.

– all the allowed isometries exist.

(E5) 

Given a geod. γ & $p \notin \gamma(\mathbb{R})$, $\exists$ at most one geod. thru p not intersecting γ.

(2) $S^n \subseteq \mathbb{R}^{n+1}$ Riem. submfd of $(\mathbb{R}^{n+1}, g_E)$.

$S^n = \{p \in \mathbb{R}^{n+1} \mid \langle p, p\rangle = 1\}$

Given $p \in S^n$, $v \in T_p S^n = p^\perp$ s.t. $|v| = 1$, then

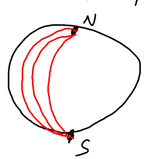
$\gamma(t) = (\cos t)p + (\sin t)v \in S^n$ is a geod. since

$\ddot{\gamma}(t) = -\gamma(t) \in (T_{\gamma(t)} S^n)^\perp$. $\gamma(\mathbb{R}) = S^n \cap \mathrm{Span}\{p, v\}$

"great circles".

$\mathrm{Isom}(S^n) \overset{=}{\supseteq} O(n+1)$ (preserves $S^n \subseteq \mathbb{R}^{n+1}$)

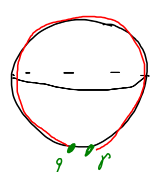
$\Rightarrow$ all allowed isometries exist.



(E2) ✓ S^n is geod. complete

(E1) ✗

Also: long geods. need not minimize length:

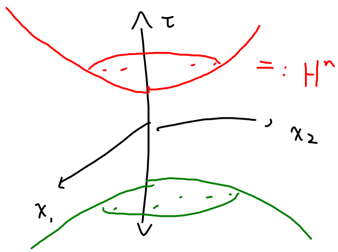


(3) <u>hyperbolic</u> <u>space</u>: Replace g_E on $\mathbb{R}^{n+1}$ by

<u>Minkowski</u> metric $g_M = -d\tau^2 + (dx^1)^2 + \dots + (dx^n)^2$

in coords. $(\tau, x^1, \dots, x^n) =: X$

$$H^n := \{ X \in \mathbb{R}^{n+1} \mid \langle X, X \rangle = -1, \text{ i.e. } \tau^2 - |x|^2 = 1, \underline{\text{and}}\ \tau > 0 \}$$



$T_p H^n = p^\perp$, so $(T_p H^n)^\perp = \mathbb{R} p$,

g_M is negative ($\Rightarrow$ nondeg.) on $\mathbb{R} p$

$\Rightarrow T_p \mathbb{R}^{n+1} = \mathbb{R} p \oplus T_p H^n$

$\Rightarrow g|_{TH^n}$ is <u>positive</u>, i.e. H^n is a Riem. submfd.

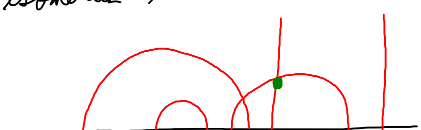
Given $p \in H^n$, $v \in T_p H^n$ w/ $|v| = \sqrt{\langle v, v \rangle} = 1$, let

$\gamma(t) = (\cosh t) p + (\sinh t) v$. This is a geodesic

$\gamma(\mathbb{R}) = H^n \cap \operatorname{Span}\{p, v\}$.

H^2 = "hyperbolic plane": satisfies all Euclid's axiom <u>except</u> (E5).

<u>EX</u>: H^2 is isometric to the Poincaré half-plane



∄ parallel postulate!