curvature of a connection ∇ on a VB $E \to M$

notation (bndl-valued forms):

$\Omega^k(M,E) :=$ set of smooth maps

$\underbrace{TM \oplus \dots \oplus TM}_{k} \longrightarrow E$ s.t. restriction to

each fiber is a multilinear antisym. map

$\omega_p : T_pM \times \dots \times T_pM \longrightarrow E_p$

Recall: We def'd a 2-form $\hat{\Omega}_K \in \Omega^2(E, VE)$ s.t.

$\hat{\Omega}_K(\eta, \xi) = -V([H(\eta), H(\xi)])$ where

we use projections $TE \overset{V}{\longrightarrow} VE$, $TE \overset{H}{\longrightarrow} HE$.

$\hat{\Omega}_K \equiv 0 \iff HE \subseteq TE$ is an integrable distr. on E

($\iff \nabla$ is flat).

If ∇ is a connection, $HE \subseteq TE$ satisfies $(m_\lambda)_* HE = HE$

for the map $m_\lambda : E \to E : v \mapsto \lambda v$ for each $\lambda \in \mathbb{F}$.

$\iff$ the map $\begin{tikzcd} T_vE \arrow[r, "V"] \arrow[rr, bend right, "K"'] & V_vE \arrow[r, "\mathrm{Vert}_v^{-1}", "\cong"'] & E_{\pi(v)} \end{tikzcd}$ satisfies

$K \circ Tm_\lambda = m_\lambda \circ K \quad \forall \lambda \in \mathbb{F}$.

claim: $\Omega_K(X,Y)v := \mathrm{Vert}_v^{-1}\left(\hat{\Omega}_K(\mathrm{Hor}_v(X), \mathrm{Hor}_v(Y))\right) \in E_{\pi(v)}$

defines a 2-form $\Omega_K \in \Omega^2(M, \underbrace{\mathrm{End}(E)}_{\cong \mathrm{Hom}(E,E)})$.

to prove: $\forall$ fixed X, Y, $v \mapsto \Omega_K(X,Y)v : E_p \to E_p$ is linear.

Suffi to show respects scalar mult. (since is smooth).

$$\Omega_K(X,Y)v = \mathrm{Vert}_v^{-1}\left(\hat{\Omega}_K\left(\mathrm{Hor}_v(X), \mathrm{Hor}_v(Y)\right)\right).$$

$$\hat{\Omega}_K(\eta,\xi) = -V\left([H(\eta), H(\xi)]\right).$$

Recall: $X \in \mathfrak{X}(M) \rightsquigarrow X^h \in \mathfrak{X}(E)$ s.t. $X^h(v) = \mathrm{Hor}_v(X(p))$
where $v \in E_p$.

$$\Rightarrow \Omega_K(X,Y)v = -K\left([X^h, Y^h](v)\right).$$

Since $(m_\lambda)_* HE = HE$, $X^h(\lambda v) = (m_\lambda)_*\left(X^h(v)\right) = Tm_\lambda \circ X^h(v)$

$\Rightarrow$ if $\lambda \neq 0$ so m_λ is a diffeo., $\underbrace{Tm_\lambda^{-1} \circ X^h \circ m_\lambda}_{= (m_\lambda)_* X^h} = X^h$

Sim. $Y^h = (m_\lambda)_* Y^h$.

Recall: for a diffeo ψ, $[\psi_* X, \psi_* Y] = \psi_*[X,Y]$.

$$\Rightarrow [(m_\lambda)_* X^h, (m_\lambda)_* Y^h] = (m_\lambda)_*[X^h, Y^h] = [X^h, Y^h]$$

$$\Rightarrow [X^h, Y^h](\lambda v) = Tm_\lambda\left([X^h, Y^h](v)\right).$$

Now for any $\lambda \neq 0$, $\Omega_K(X,Y)\lambda v = -K\left([X^h, Y^h](\lambda v)\right)$

$$= -K\left(Tm_\lambda\left([X^h,Y^h](v)\right)\right) = -m_\lambda \circ K\left([X^h,Y^h](v)\right)$$

$$= \lambda\, \Omega_K(X,Y)\, v.$$ $\blacksquare$

<u>EX</u>: $\Omega_K \in \Omega^2(M, \mathrm{End}(E))$ vanishes $\Leftrightarrow$
$\hat{\Omega}_K \in \Omega^2(E, VE)$ $\quad$ ($\Leftrightarrow \nabla$ is flat.)

<u>defn</u>: Ω_K is the <u>curvature 2-form</u> of ∇.

Riemann tensor

idea: Coord.-invt. version of the statement $\partial_i \partial_j = \partial_j \partial_i$

is $\mathcal{L}_X \mathcal{L}_Y - \mathcal{L}_Y \mathcal{L}_X = \mathcal{L}_{[X,Y]}$ for $X, Y \in \mathfrak{X}(M)$.

Does this also work for ∇_X, ∇_Y as operators on $\Gamma(E)$.

defn: The Riemann curvature tensor is the unique multilinear bndl map $R : TM \oplus TM \oplus E \longrightarrow E$

$$(X, Y, v) \longmapsto R(X,Y)v$$

s.t. $\forall$ $X, Y \in \mathfrak{X}(M)$ & $v \in \Gamma(E)$,

$$\boxed{R(X,Y)v = \nabla_X \nabla_Y v - \nabla_Y \nabla_X v - \nabla_{[X,Y]} v}$$

check: $R(X,Y)v$ is C^∞-linear in all 3 variables.

$\Rightarrow$ for $E = TM$, $R \in \Gamma(T^1_3 M)$.

In local coords., (chart $(x^1, \dots, x^n)$ & frame $e_1, \dots, e_m$ on E over $U \subseteq M$). one writes $R^a{}_{jkb}\, e_a = R(\partial_j, \partial_k) e_b$ $\left(\begin{matrix} a, b \in \{1, \dots, m\} \\ j, k \in \{1, \dots, n\} \end{matrix} \right)$

so $(R(X,Y)v)^a = R^a{}_{jkb} X^j Y^k v^b$.

EX: $R^a{}_{jkb} = \partial_j \Gamma^a_{kb} - \partial_k \Gamma^a_{jb} + \Gamma^a_{jm} \Gamma^m_{kb} - \Gamma^a_{km} \Gamma^m_{jb}$.

rk: For ∇ = L.C. conn. on a ps.-Riem. mfd (M, g), cpts $R^i{}_{jkl}$ are det'd by 2nd derivs. of g_{ij}.

EX: $\mathbb{R}^2 \overset{\text{open}}{\supseteq} V \xrightarrow{f} M$ & $v \in \Gamma(f^* E)$, then
$\quad\;\, \underset{(s,t)}{\cup}$

$$\boxed{\nabla_s \nabla_t v - \nabla_t \nabla_s v = R(\partial_s f, \partial_t f) v}$$

Hint: If f is ~~an~~ embedding, can deduce from defn.

big thm: $R(X,Y)v = \Omega_{12}(X,Y)v$.

cor: In coords., $\nabla_j \nabla_k = \nabla_k \nabla_j \iff$ conn. is flat.

EX: If ∇ is compat. w/ a bundl metric $\langle\, ,\, \rangle$ on E, then $\langle R(X,Y)v, w\rangle + \langle v, R(X,Y)w\rangle = 0$.

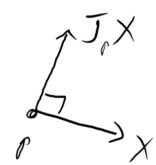
i.e. $R(X,Y): E_p \longrightarrow E_p$ for each $X, Y \in T_pM$ is antisymmetric w.r.t. $\langle\, ,\, \rangle$.

Example: (Σ, g) an oriented ps.-Riem. 2-mfd, $E = T\Sigma$, ∇ = L.C. conn. on $T\Sigma$

Here $\Lambda^2 T_p^*\Sigma$ and $\{A : T_p\Sigma \longrightarrow T_p\Sigma \mid \text{antisym.}\}$ are 1-dim.

multiple of $d\mathrm{vol} \in \Omega^2(\Sigma)$

multiple of $J : T\Sigma \longrightarrow T\Sigma$ def'd s.t.



$J_p : T_p\Sigma \longrightarrow T_p\Sigma$ is a $90°$ counterclockwise rotation.

$\Longrightarrow$ the Riemann tensor is determined a fn. $K : \Sigma \longrightarrow \mathbb{R}$

s.t. $\boxed{R(X,Y)Z = -K(p)\, d\mathrm{vol}(X,Y)\, JZ}$

K = "Gaussian curvature" of (Σ, g).

covariant exterior derivatives.

$\Omega^0(M,E) = \Gamma(E)$, so $\nabla : \Omega^0(M,E) \to \Omega^1(M,E)$.

prop: $\exists!$ linear map $d_\nabla : \Omega^k(M,E) \to \Omega^{k+1}(M,E)$ that matches ∇ for $k=0$ and satisfies $\left(\begin{array}{l}\alpha \in \Omega^k(M,E), \\ \beta \in \Omega^\ell(M)\end{array}\right)$

$$d_\nabla(\alpha \wedge \beta) = d_\nabla \alpha \wedge \beta + (-1)^k \alpha \wedge d\beta.$$

"pf": Locally, any $\omega \in \Omega^k(M,E)$ can be written as

$$\omega = \sum_{1 \le i_1 < \dots < i_k \le n} \omega_{i_1 \dots i_k} dx^{i_1} \wedge \dots \wedge dx^{i_k} \quad \text{for } \omega_{i_1\dots i_k} \in \Gamma(E|_U)$$

$$\Rightarrow d_\nabla \omega = \sum_{i_1 \dots i_k} \nabla \omega \wedge dx^{i_1} \wedge \dots \wedge dx^{i_k} = \sum_{i_1, \dots, i_k, j} \nabla_j \omega \, dx^j \wedge dx^{i_1} \wedge \dots \wedge dx^{i_k}.$$

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EX: In local coords, $\lambda \in \Omega^1(M,E)$,

$$(d_\nabla \lambda)_{ij} = \nabla_i \lambda_j - \nabla_j \lambda_i.$$

Coord-indep version: for $X, Y \in \mathfrak{X}(M)$,

$$\boxed{(d_\nabla \lambda)(X,Y) = \nabla_X(\lambda(Y)) - \nabla_Y(\lambda(X)) - \lambda([X,Y]).}$$

pf of big thm: Regard $K : TE \to E$ as $K \in \Omega^1(E, \pi^* E)$.

$$\begin{array}{ccc} \cup & & \\ T_v E & \longrightarrow & E_{\pi(v)} = (\pi^* E)_v \end{array}$$

Assign to $\pi^* E \to E$ the pullback connection.

main claim: $R(X,Y)v$ & $\Omega_K(X,Y)v$ are both

$$d_\nabla K(\mathrm{Hor}_v(X), \mathrm{Hor}_v(Y)).$$

pf for Ω_K: Recall $\Omega_K(X,Y) = -K([X^h, Y^h])$.

$$d_\nabla K(X^h(v), Y^h(v)) = \nabla_{X^h(v)}(\underbrace{K(Y^h)}_{=0}) - \nabla_{Y^h(v)}(\underbrace{K(X^h)}_{=0}) \underbrace{- K([X^h, Y^h](v))}_{= \Omega_K(X,Y) v}.$$

pf for R: Choose smooth $f(s,t) \in M$ s.t. $f(0,0) = p$ & $v(s,t) \in E_{f(s,t)}$ s.t. $\partial_s f(0,0) = X$, $\partial_t f(0,0) = Y$, $v(0,0) = v$ & $\nabla_s v(0,0) = \nabla_t v(0,0) = 0$. Now

$$R(X,Y)v = R(\partial_s f(0,0), \partial_t f(0,0))\, v(0,0)$$

$$= \nabla_s \nabla_t v - \nabla_t \nabla_s v \Big|_{s=t=0} = \nabla_s(K(\partial_t v)) - \nabla_t(K(\partial_s v))\Big|_{s=t=0}$$

$$= d_\nabla K(\partial_s v(0,0), \partial_t v(0,0)) = d_\nabla K(\mathrm{Hor}_v(X), \mathrm{Hor}_v(Y)).$$

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