(M,g) a ps.-Riem. mfd, ∇ = L.-C. conn. on TM

Riemann tensor: $R \in \Gamma(T^1_3 M)$,

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z.$$

<u>symmetries</u>: $R(Y,X) = -R(X,Y)$

$$\langle V, R(X,Y)Z \rangle = -\langle R(X,Y)V, Z \rangle$$

<u>covariant</u> <u>version</u>: $\mathrm{Riem} \in \Gamma(T^0_4)$, def'd by

$$\mathrm{Riem}(V,X,Y,Z) := \langle V, R(X,Y)Z \rangle.$$

Now can write $\mathrm{Riem}(V,Y,X,Z) = -\mathrm{Riem}(V,X,Y,Z)$

$$\mathrm{Riem}(Z,X,Y,V) = -\mathrm{Riem}(V,X,Y,Z)$$

In coords., cpnts of Riem:

$$R_{ijk\ell} := \mathrm{Riem}(\partial_i, \partial_j, \partial_k, \partial_\ell) = \langle \partial_i, R(\partial_j, \partial_k)\partial_\ell \rangle$$

$$= \langle \partial_i, R^m{}_{jk\ell}\, \partial_m \rangle = g_{im} R^m{}_{jk\ell}$$

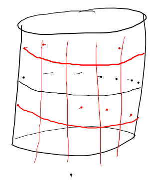
$$\Rightarrow R_{ikj\ell} = -R_{ijk\ell} = R_{\ell jki}.$$

<u>EX</u>: If $\dim M = 2$, all cpnts are det'd by R_{1122}.

defn: (M,g) is *locally flat* if $\forall\, p \in M$, $\exists$ a chart (U,x), $p \in U$ s.t. $g_{ij} = \langle \partial_i, \partial_j \rangle$ are constant fns.

$\iff$ (M,g) is *locally isometric* to the flat $\mathbb{R}^n$.

ex: $Z := \{x^2 + y^2 = 1\} \subseteq \mathbb{R}^3$



thm: (M,g) is locally flat $\iff R \equiv 0$.

pf: $\Rightarrow$: $g_{ij} = \text{const} \Rightarrow \Gamma^i_{jk} \equiv 0 \Rightarrow R^i{}_{jk\ell} \equiv 0$.

$\Leftarrow$: $R \equiv 0$, then $\forall\, p \in M$, $\exists$ nbhd $U \subseteq M$ of p & parallel vec. flds $X_1, \dots, X_n$ on U forming a frame for TM. Require O-N at p, then derivative of $\langle X_i, X_j \rangle = 0$ in all direction since $\nabla X_i \equiv \nabla X_j \equiv 0 \Rightarrow$ frame is O-N everywhere on U. $\Rightarrow g_{ij} := \langle X_i, X_j \rangle$ are const. fns.

∇ is symmetric $\Rightarrow$ $\underbrace{\nabla_{X_i} X_j - \nabla_{X_j} X_i}_{=0} = [X_i, X_j]$

$\Rightarrow \exists$ a chart near p s.t. $\partial_i = X_i$. $\square$

Case (Σ, g) = Riem. 2-mfd, embedded as Riem. submfd of $(\mathbb{R}^3, g := \pm dx^2 + dy^2 + dz^2)$ $\begin{cases} + = \text{"Euclidean"} \\ - = \text{"Minkowski"} \end{cases}$

Assume $\exists\ \nu \in \Gamma(T\Sigma^\perp) \subseteq \Gamma(T\mathbb{R}^3|_\Sigma)$ s.t. $\langle \nu, \nu \rangle = \pm 1$.

$\forall\ p \in \Sigma$, $\nu(p) \in S^2_\pm := \begin{cases} S^2 \text{ in Euclidean} \\ \{X \in \mathbb{R}^3 \mid \langle X, X \rangle = -1\} \\ = \{x^2 - y^2 - z^2 = 1\} \end{cases}$

$\rightsquigarrow$ Gauss map $\nu : \Sigma \longrightarrow S^2_\pm$

observation: $T_{\nu(p)} S^2_\pm = \nu(p)^\perp = T_p\Sigma$

$\Rightarrow$ $T_p\nu$ maps $T_p\Sigma$ to itself.

Lemma: $T_p\nu : T_p\Sigma \longrightarrow T_p\Sigma$ is self-adjoint w.r.t. $\langle\ ,\ \rangle$.

EX: Prove this by writing Σ near p as $f^{-1}(0)$ for a fn. $f : (\text{nbhd}(p) \subseteq \mathbb{R}^3) \longrightarrow \mathbb{R}$ s.t. $\nu = \nabla f$ along Σ.

$\Rightarrow$ $\exists$ O-N basis $X_1, X_2 \in T_p\Sigma$ s.t. $T_p\nu(X_i) = \kappa_i X_i$ for some $\kappa_1, \kappa_2 \in \mathbb{R}$.

κ_1, κ_2: "principal curvatures" at p

$\mathbb{R}X_i$: "principal directions" at p.

interpretation: For $p \in \Sigma$, $X \in T_p\Sigma$ w/ $|X| = 1$, defn. normal curvature of Σ in direction X as

$$\kappa_N(X) := \langle X, T_p\nu(X) \rangle = \left(\text{let } \gamma(t) := \exp_p(tX) \in \Sigma \right)$$

$$\partial_t \underbrace{\langle \dot\gamma(t), \nu(\gamma(t)) \rangle}_{=0}\Big|_{t=0} - \langle \nabla_t \dot\gamma(0), \nu(p) \rangle$$

$$= -\langle \nabla_t \dot\gamma(0), \nu(p) \rangle$$

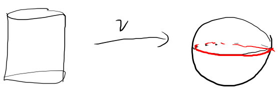
Here $\nabla :=$ conn. on $(\mathbb{R}^3, g)$ $\Rightarrow$ $\nabla_t \dot\gamma(0) \neq 0$ in general, but $\nabla_t \dot\gamma(0) \in \mathbb{R}\nu(p)$. $\Rightarrow$ $\kappa_N(X)$ measures how far γ deviates from being a straight line (i.e. geod. in $\mathbb{R}^3$).

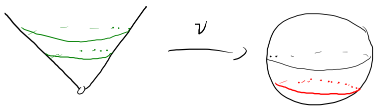
EX: $\{\kappa_N(X) \in \mathbb{R} \mid X \in T_p\Sigma \text{ w/ } |X| = 1\}$ = the interval bdd by κ_1, κ_2.

defn: The Gaussian curvature of $\Sigma \subseteq (\mathbb{R}^3, \pm dx^2 + dy^2 + dz^2)$ at $p \in \Sigma$ is $K_G(p) := \pm \kappa_1 \kappa_2 = \pm \det\left(T_p \nu : T_p\Sigma \to T_p\Sigma\right)$.

rk: If replace ν with $-\nu$, both κ_i change sign
$\Rightarrow K_G$ is unchanged. ($\Rightarrow$ can defn. K_G even if ν only exists locally.)

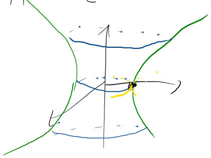
ex: $S^2 \subseteq \mathbb{R}^3$, $\nu : S^2 \to S^2$ is Id $\Rightarrow K_G \equiv 1$.

ex: $Z :=$  $\Rightarrow \det(T_p \nu) = 0 \Rightarrow K_G \equiv 0$.

$C := \{x^2 + y^2 = z^2,\ z > 0\}$ 

$\Rightarrow K_G \equiv 0$.

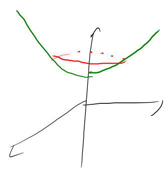
ex: $H := \{x^2 + y^2 - z^2 = 1\} \subseteq \mathbb{R}^3$



$K_G < 0$.

EX: $K_G(p) = -\dfrac{1}{|p|^4}$.

ex: Hyperbolic plane $H^2 \subseteq (\mathbb{R}^3, -dx^2 + dy^2 + dz^2)$
" upper sheet of S^2_-



$\nu = $ Id on upper sheet of S^2_-
$\Rightarrow K_G \equiv -1$.

thm of Hilbert: $\nexists$ isometric immersion of any geod. complete surface with const. negative K_G into Euclidean $\mathbb{R}^3$.

Recall: $\exists !$ fn. $K : \Sigma \to \mathbb{R}$ s.t. (defn: $J : T\Sigma \to T\Sigma$ rotation $90°$ counterclockwise)

$$R(X,Y)Z = -K \cdot dvol(X,Y)\, JZ$$

main thm: $K = K_G$.

cor (Gauss' "Theorema Egregium"): K_G is an isometry invt:
Given $\Sigma_1, \Sigma_2 \subseteq \mathbb{R}^3$ & an isometry $\varphi : \Sigma_1 \to \Sigma_2$,
then K_G on Σ_1 is K_G on Σ_2 composed w/ φ.

cor: If (Σ, g) is homogeneous (i.e. $\forall\, p, q \in \Sigma$, $\exists$ $\varphi \in Isom(\Sigma, g)$ s.t. $\varphi(p) = q$), $\Rightarrow K_G \equiv$ const.

K_G in coords

claim: $dvol(X,Y) = \langle JX, Y \rangle$.

pf: $J^2 = -Id$ & J is orthogonal $\Rightarrow$

$$\langle JY, X \rangle = \langle J^2 Y, JX \rangle = \langle -Y, JX \rangle = -\langle JX, Y \rangle$$

$\Rightarrow \exists$ fn. $f : \Sigma \to \mathbb{R}$ s.t. $\langle JX, Y \rangle = f \cdot dvol(X,Y)$.

Choose any X w/ $|X| = 1$, $dvol(X, JX) = 1 = |X|^2 = |JX|^2$
$= \langle JX, JX \rangle \Rightarrow f \equiv 1$. ✓

$$\langle R(X,Y)Y, X \rangle = \langle -K_G\, dvol(X,Y)\, JY, X \rangle =$$

$$= -K_G\, dvol(X,Y) \langle JY, X \rangle = K_G \cdot |dvol(X,Y)|^2$$

$\Rightarrow$ if X, Y are lin.-indep.,

$$K_G = \frac{\langle X, R(X,Y)Y \rangle}{|dvol(X,Y)|^2}.$$

In local coords., $dvol = \sqrt{\det g}\; dx^1 \wedge dx^2$ for $g = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix}$

Take $X = \partial_1$, $Y = \partial_2$, $K_G = \dfrac{\langle \partial_1, R(\partial_1, \partial_2)\partial_2 \rangle}{\det g} = \boxed{\dfrac{R_{1122}}{\det g}}$

EX: The Poincaré half-plane has $K \equiv -1$.