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Differentialgeometrie I
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Problem Set 1

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Problem 1

The standard *spherical coordinates* (r, θ, ϕ) on $\mathbb{R}^3$ are related to Cartesian coordinates by

$$x = r \cos\theta \cos\phi, \qquad y = r \sin\theta \cos\phi, \qquad z = r \sin\phi, \tag{1}$$

so θ plays the role of an angle in the xy-plane, and $\phi \in [-\pi/2, \pi/2]$ is the angle between the vector $(x, y, z) \in \mathbb{R}^3$ and the xy-plane.[1] Restricting to $r = 1$, the other two coordinates (θ, ϕ) can be used to describe points on the unit sphere $S^2 \subset \mathbb{R}^3$, though there are choices to be made since θ is only defined up to multiples of 2π (and it is not defined at all at the north and south poles $p_\pm := (0, 0, \pm 1) \in S^2$, where $\phi = \pm\pi/2$.)

(a) Find two subsets $\mathcal{U}_1, \mathcal{U}_2 \subset S^2$ with $\mathcal{U}_1 \cup \mathcal{U}_2 = S^2 \backslash \{p_+, p_-\}$ such that for $i = 1, 2$, there are 2-dimensional charts of the form $(\mathcal{U}_i, \alpha_i)$ with $\alpha_i = (\theta_i, \phi_i)$, where the coordinate functions $\theta_i, \phi_i : \mathcal{U}_i \to \mathbb{R}$ are continuous and satisfy the spherical coordinate relations (1), and have images $\alpha_1(\mathcal{U}_1) = (0, 2\pi) \times (-\pi/2, \pi/2) \subset \mathbb{R}^2$ and $\alpha_2(\mathcal{U}_2) = (-\pi, \pi) \times (-\pi/2, \pi/2) \subset \mathbb{R}^2$.

(b) One cannot use spherical coordinates to construct a chart on S^2 that contains either of the poles $p_\pm = (0, 0, \pm 1)$. Can you think of another way to construct charts on open subsets of S^2 that contain these two points?
Hint: On any sufficiently small neighborhood of p_+ or p_- in S^2, every point has its z-coordinate determined by the x and y-coordinates.

(c) Show that the charts you've constructed define a smooth atlas on S^2. (If they don't, you might want to modify the charts you constructed in part (b).)

Problem 2

The *real projective n-space* $\mathbb{RP}^n$ can be defined as the set of equivalence classes

$$\mathbb{RP}^n := (\mathbb{R}^{n+1} \backslash \{0\}) / \sim,$$

where two nonzero vectors $v, w \in \mathbb{R}^{n+1}$ are considered equivalent if and only if $v = \lambda w$ for some $\lambda \in \mathbb{R}$. This means that each element of $\mathbb{RP}^n$ is literally a line through the origin (excluding the origin itself) in $\mathbb{R}^{n+1}$.[2] It is convenient to denote points in $\mathbb{RP}^n$ via so-called *homogeneous coordinates*, in which the symbol

$$[x_0 : \ldots : x_n] \in \mathbb{RP}^n$$

means the equivalence class containing the vector $(x_0, \ldots, x_n) \in \mathbb{R}^{n+1} \backslash \{0\}$. Note that this symbol has no meaning unless at least one of the numbers $x_0, \ldots, x_n$ is nonzero. For

[1] Achtung: there are various conventions for spherical coordinates in use. I'm told that this is the standard convention learned by mathematics students in Germany. I learned a different convention as a physics student in the USA.

[2] Since each line through the origin in $\mathbb{R}^{n+1}$ contains exactly two unit vectors, which are antipodal points on S^n, one could alternatively define $\mathbb{RP}^n$ as $S^n / \sim$ for an equivalence relation on S^n such that every $x \in S^n$ is equivalent to its antipodal point $-x \in S^n$. This is the most popular way to define the *projective plane* $\mathbb{RP}^2$, in particular.

$j = 0, \dots, n$, define $\mathcal{U}_j := \{[x_0 : \dots : x_n] \in \mathbb{RP}^n \mid x_j \neq 0\}$ and a map $\varphi_j : \mathbb{R}^n \to \mathbb{RP}^n$ by $\varphi_j(t_1, \dots, t_n) := [t_1 : \dots : t_j : 1 : t_{j+1} : \dots : t_n]$. Show that φ_j is an injective map onto $\mathcal{U}_j$, so $(\mathcal{U}_j, \varphi_j^{-1})$ is a chart, and compute the transition maps relating any two of the charts constructed in this way for different values of $j = 0, \dots, n$. Show that these $n+1$ charts together form a smooth atlas, making $\mathbb{RP}^n$ into a smooth n-dimensional manifold.

Problem 3

The *Klein bottle* can be defined as the set of equivalence classes $K^2 := \mathbb{R}^2/\sim$, where $\sim$ is the smallest equivalence relation on $\mathbb{R}^2$ such that $(s,t) \sim (s, t+1)$ and $(s,t) \sim (s+1, -t)$ for all $(s,t) \in \mathbb{R}^2$.

(a) Do a google image search for "Klein bottle", or simply look at Figure 6 in the lecture notes, and try to understand how the definition of K^2 stated above gives rise to such pictures.
Hint: If one makes a slight modification to the equivalence relation so that (s,t) is equivalent to $(s+1,t)$ instead of $(s+1,-t)$, one obtains an alternative definition of the 2-torus $\mathbb{T}^2 = S^1 \times S^1$. Think of the Klein bottle as a variation on this.

(b) Show that K^2 is a smooth 2-dimensional manifold with a smooth atlas consisting of charts of the form $(\mathcal{U}, x)$ where $\mathcal{U} := \left\{[(s,t)] \in K^2 \mid (s,t) \in \widetilde{\mathcal{U}}\right\}$ for suitably chosen open subsets $\widetilde{\mathcal{U}} \subset \mathbb{R}^2$ and $x([(s,t)]) = (s,t)$.

Problem 4

Consider the two 1-dimensional charts $(\mathcal{U}, x)$ and $(\mathcal{V}, y)$ on $\mathbb{R}$ defined by $\mathcal{U} = \mathcal{V} := \mathbb{R}$, with $x(t) := t$ and $y(t) := t^3$.

(a) Show that $(\mathcal{U}, x)$ and $(\mathcal{V}, y)$ are C^0-compatible, but not C^1-compatible.

Since both charts are globally defined, they each define atlases $\mathcal{A} := \{(\mathcal{U}, x)\}$ and $\mathcal{A}' := \{(\mathcal{V}, y)\}$ on $\mathbb{R}$. Taking the unique maximal smooth atlas containing $\mathcal{A}$ gives the standard smooth structure on $\mathbb{R}$. If we instead take the unique maximal smooth atlas containing $\mathcal{A}'$, we obtain a different smooth structure on $\mathbb{R}$, and we will denote the smooth manifold obtained in this way by $\mathbb{R}'$: it is the same set as $\mathbb{R}$, but it carries a different smooth structure and is thus a different smooth manifold.

(b) Show that the natural topologies induced on $\mathbb{R}$ and $\mathbb{R}'$ by their respective smooth structures are the same. Equivalently, the identity map $\mathbb{R} \to \mathbb{R}'$ is a homeomorphism.

(c) Let $\mathcal{O} := (-1, 1) \subset \mathbb{R}'$, and show that the map

$$\varphi : \mathcal{O} \to \mathbb{R} : t \mapsto \tan\left(\frac{\pi}{2} t^3\right)$$

is a smooth chart on $\mathbb{R}'$.

(d) A continuous map $f : M \to N$ between two smooth manifolds is called *smooth* if for every pair of smooth charts $(\mathcal{U}, x)$ on M and $(\mathcal{V}, y)$ on N, the map $y \circ f \circ x^{-1}$ is smooth wherever it is defined. Show that this holds for the identity map $\mathbb{R} \to \mathbb{R}'$, but *not* for its inverse. (In technical terms, this implies that the identity map $\mathbb{R} \to \mathbb{R}'$ is a homeomorphism, but not a diffeomorphism.)

(e) Show that the map $\mathbb{R}' \to \mathbb{R} : t \mapsto t^2$ is not smooth.

(f) Find an actual smooth bijection $\mathbb{R} \to \mathbb{R}'$ whose inverse is also smooth. (That is, $\mathbb{R}$ and $\mathbb{R}'$ are distinct smooth manifolds, but they are nonetheless *diffeomorphic.*)