DRAFT: An overview of the set up of symplectic and algebraic Gromov-Witten invariants.

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Abstract

The goal of this talk is to motivate Gromov-Witten invariants from both the algebraic and symplectic perspective, explain what they are meant to be and why they cannot be what we would dream of. In particular, we give an idea why it solves the enumeration problem of rational curves passing through a certain number of points in the projective plane and why virtual fundamental classes are the crux of their algebraic and symplectic definition.

The first section provides some motivation for what the Gromov-Witten invariants can accomplish. Subection 2.1 presents what we would like those invariants to be under utopic assumptions. In 2.2 we explain what real life looks like from the algebraic perspective, which is then briefly compared to the symplectic pespective in the adjacent subsection. In 2.4 we hint at the structure the Gromov-Witten invariants carry. In section 3 we hope to provide some ideas on who the agents are in the differential-geometric side of things as well as some concrete and visual intuition for the moduli spaces we deal with. Finally, section 4 lists the seminar talks to provide some references. Overall, I would like to also frame most of the talks of the seminar as solutions to problems or techniques discussed here.

Disclaimer: This document means to convey ideas but it is definitely not a source for rigor, specially for the more algebraic aspects, where I present what I have the feeling is done from the algebraic perspective. Also, this begun as a personal account of things I wanted to say in the talk, so it is meant to convey a feeling for the subject rather than concrete knowledge. Proceed at your own peril.

1 Motivation

Let $\mathbb{P}^2$ be the complex projective plane. Let us frame some classical questions in a slightly less classical perspective.

Question 1. How many lines go through two distinct points in the plane $\mathbb{P}^2$?

Here is one way to see that there is only one line: parametrize $\mathbb{P}^2$ with homogeneous coordinates $[X; Y; Z]$, a line will be given by the zero set of a homogeous degree 1 polynomial. Therefore lines are parametrized by $a, b, c \in \mathbb{C}$ not all zero such that $aX + bY + cZ = 0$ and, since we only care about this up to non-zero constant factor, the parameter space for lines in the plane is $\mathbb{P}^2 = (\mathbb{C}^3 \setminus 0)/\mathbb{C}^*$. Passing through a point is a codimension one constraint, it determines a hyperplane in $\mathbb{P}^2$. Choosing two generic points (in this case this just means non-equal) we get two hyperplanes in the parameter space that intersect at a point (codimension 1+1 constraint). That point represents the unique line through those two points.

Remark 2. The projective plane $\mathbb{P}^2$ is better suited than $\mathbb{C}^2$ for enumerative purposes. Many geometric phenomena that appear different in the affine context are the same in the projective setup (a parabola and a circle, for example). More generally, we do not want information (intersections) to scape to infinity (this compactness phenomena will manifest in the fancier setup below).

Question 3. How many conics go through five generic points in the plane $\mathbb{P}^2$?

We follow the same strategy as above. Quadratic homogeneous polynomials live in $\mathbb{P}^5$ (since again we have six parameters for monomials $X^2, Y^2, Z^2, XY, XZ, YZ$ and substract one from homogeneity) and choosing five generic points constraints this space to five generic hyperplanes that will intersect at a single point: the conic through those points (there is a slight bit more work to check smoothness). The general pattern is clear: there will be exactly one degree d curve through $d(d+3)/2$ generic points in the plane (which is the dimension of the parameter space, $\binom{d+2}{2}$ monomials -1). So there will be a unique cubic through 9 generic points. What about, however, if we impose that the curves are rational (birrationally equivalent to a line or, equivalently, of genus 0)?

Question 4. How many rational plane cubics go though eight generic points in the plane? Or more generally, what is the number n_d of degree d rational curves through $3d-1$ generic points in the plane?

Remark 5. The number of points $3d-1$ is not random: the degree-genus formula characterizes the (geometric) genus g of a curve with its degree and number of nodes[1] δ: $g = (d-1)(d-2)/2 - \delta$. Recall that $\mathbb{P}^{d(d+3)/2}$ will be the parameter space for degree d homogenous polynomials in the plane, hence the subvariety of degree d plane rational curves will have dimension $d(d+3)/2 - \delta = 3d-1$. In other words, $3d-1$ is the right amount of constraints to add to enumerate.

For the case of rational cubics one can reason as follows: rationallity imposes a node, which is a codimension one constraint and so rational plane cubics are a hypersurface in $\mathbb{P}^9$, each point constraint corresponds to intersecting it with a hyperplane. One can see that there will be 12 points in the intersection. That is, $n_3 = 12$. This was known to Steiner already in 1848 and Zeuthen proved that $n_4 = 620$ in 1873. All this is classical. These methods cannot be pushed much further but notice that the philosophy is to parametrize a family of objects by an algebraic variety (i.e. a moduli space) and then understand each geometric incidence condition as a subvariety, the problem is then solved with intersection theory. Apparently, the great sophistication of this tool in algebraic geometry at the end of the last century enabled the computation of n_5, which is 87304. Aside from this, the general case was out of the reach of those techniques. The breakthrough came with Gromov-Witten theory.

In 1985 Gromov showed in [7] how the space of pseudo-holomorphic curves into a symplectic manifold with a tame almost-complex structure encodes deep invariants of its symplectic structure (three years prior Donaldson published the first instance of deep results on moduli spaces of solutions of elliptic PDE to resolve geometric questions using inspiration from physics). In 1987 Floer [5], inspired on Witten's *Suppersymmetry and Morse theory* [19] paper published five years before, developed an infinite dimensional version of Morse homology using Gromov's ideas on pseudo-holomorphic curves. In 1988 Witten's paper *Topological sigma models* [20] established the rudiments of what was to become Gromov-Witten theory yet another application of physical ideas in math. In 1994 Kontsevich, in a paper with Manin [12], deduced from a series of axioms certain relationships between the Gromov-Witten invariants which implies the following striking formula for the numbers n_d:

$$n_d = \sum_{d=k+l} n_k n_l \left(k^2 l^2 \binom{3d-4}{3k-2} - k^3 l \binom{3d-4}{3k-1} \right),$$

which allows one to compute all numbers n_d only from the information that $n_1 = 1$, which is the answer to Question 1. While most of the seminar will focus on the mathematics behind these ideas, we will have one or two lectures on the physics of it all.

Here are a list of provocative thoughts: this count is, in fact, somewhat independent of the complex structure of $\mathbb{P}^2$, it only depends on the symplectic deformation class of the Fubini-Study

[1] A node is the simplest kind of singularity, locally the algebraic curve will look like the transverse intersection of two planes, i.e. if C is said algebraic curve, $x \in C$ is a node if $\mathcal{O}^{\wedge}_{C,x} = \frac{\mathbb{C}[[z,w]]}{(zw)}$. If you understand algebraic curves over $\mathbb{C}$ as Riemann surface then you can visualize the node as a pinched closed curve.

form on the plane. What is more, this enumeration question is actually a manifestation of the associativity of the product in quantum cohomology, which is a deformation of the usual cohomology tracking "fuzzy intersections", c.f. 2.4.

The best sources to learn the basics of Gromov-Witten theory in textbook form are [9] from the algebraic perspective and [13] from the differential perspective (and also related, perhaps friendlier, earlier versions of that book). They both deal with genus 0 though the former only for $\mathbb{P}^n$. For more general constructions one can read the original papers which are cited in the last section.

2 Utopic algebraic Gromov-Witten invariants

2.1 The convinient lies we tell ourselves and friendly definitions

We now define the objects that are the basis of our enumeration.

Definition 6. A smooth irreducible algebraic curve $(\Sigma, p_1, \ldots, p_m)$ over $\mathbb{C}$ that is m-pointed (or marked) is such a curve Σ with m distinguished points $p_i \in \Sigma$. We say that two such objects are isomorphic when there is an isomorphism of the underlying curves sending the i-th marked point of one to the i-th marked point of the other. From the differential geometry perspective, one should replace curve by compact Riemann surface.

The **moduli space $\mathcal{M}_{g,m}$ of m-pointed genus g smooth curves** is the space of equivalence classes of such curves. In this utopia of ours we will assume that it is a smooth compact manifold whose topology portrays nicely the geometry of those curves (this means it is a fine moduli space, we will touch upon this later). This is definitely not true, but let us dream.

Definition 7. Let X be a smooth n-dimensional projective variety and Σ an m-pointed curve as above (slight abuse of notation). An m-pointed map is just a morphism of the underlying varieties $\Sigma \to X$ but the notion of equivalence of two m-pointed maps $f : \Sigma \to X$ and $f : \Sigma' \to X'$ is an isomorphisms of the corresponding m-pointed curves $\varphi : \Sigma \to \Sigma'$ commuting with the morphisms to X, that is $f = f' \circ \varphi$.

The **moduli space $\mathcal{M}_{g,m}(X)$ of m-pointed maps** whose domains are a smooth genus g curves is the space of equivalence classes of such pointed maps. Indeed, in this utopia of ours we will assume that it is a smooth compact manifold whose topology portrays nicely the geometry of those maps. Well, of course this is definitely not true. Alas, let us dream.

For $A \in H_2(X, \mathbb{Z})$, we will furthermore define $\mathcal{M}_{g,m}(X, A)$ as above but require that $f_* [\Sigma] = [A]$ for an m-pointed map $f : \Sigma \to X$. The way you should envision to equivalence classes of maps being close together is when their images are close together and maps that are close together in this sense will represent the same homology class nonetheless (see secton 3.1 for more intuition).

Example 8. It can be seen that a degree d rational curve in $\mathbb{P}^2$ is equivalent to an equivalence class of regular maps $f : \mathbb{P}^1 \to \mathbb{P}^2$ so $f_* \left[\mathbb{P}^1\right] = dP$ where P is a generator of $H_2(\mathbb{P}^2) = \mathbb{Z}$ represented by a line. The notion of equivalence is reparametrization: f and f' as above are equivalent if there is an isomorphism φ of $\mathbb{P}^1$ such that $f = f' \circ \varphi$. Then, our enumerative problem consists of counting objects in $\mathcal{M}_{0,0}(\mathbb{P}^2, dP)$.

We will later explain that the dimensions that these spaces ought to be are $3g-3+m$ for $\mathcal{M}_{g,m}$ and $(1-g)(n-3) + \int_A \mathrm{c}_1 T_X + m$ for $\mathcal{M}_{g,m}(X, A)$. These are called the **virtual dimensions** of those spaces. Computation of this Chern number yields that the virtual dimension of $\mathcal{M}_{0,0}(\mathbb{P}^2, dP)$ is exactly $3d-1$, which we deduced before.

Here are two obvious maps that are "nice" (they are nice also out of our utopia).

Definition 9. The evaluation map $\mathrm{ev} : \mathcal{M}_{g,m}(X, A) \longrightarrow X^{\times m}$ is the map that sends an equivalence class of m-pointed map $f : (\Sigma, p_1, \ldots, p_m) \to X$ to the m-tuples of points $(f(p_1), \ldots, f(p_m))$. The forgetful map $\Phi : \mathcal{M}_{g,m}(X, A) \longrightarrow \mathcal{M}_{g,m}$ that sends an equiivalence class of $f : (\Sigma, p_1, \ldots, p_m) \to X$ to one of $(\Sigma, p_1, \ldots, p_m)$.

Notice that these two maps embed geometric constraints (passing through certain m points and prescribing the type of curve that does that) within the language of our present set-up.

Example 10. Our original enumerative problem in the last section now consists of studying $\mathcal{M}_{0,3d-1}(\mathbb{P}^2, dP)$ where P is a generator of $H_2(\mathbb{P}^2)$. For example, say $d = 2$, take five ordered points in $\mathbb{P}^2$, that is a point in in $\mathbb{P}^{\times 5}$, then the preimage of this point by ev should consist of a unique equivalence class of maps that represent the unique conic through those five points (the word should is important, we are still in our dream world, see example 13 for the hard truth).

We now generalize this example and define the Gromov-Witten invariants in this utopia of ours.

Definition 11. Fix X, g, m and A as above, the Gromov-Witten invariants are a linear map

$$\mathrm{GW}^{\mathrm{A}}_{\mathrm{g,m}} : H^*(X)^{\otimes m} \otimes H_*(\mathcal{M}_{g,m}) \longrightarrow \mathbb{Z}$$

defined as

$$\mathrm{GW}^{\mathrm{A}}_{\mathrm{g,m}}(\alpha_1, \ldots, \alpha_m, \beta) = \int_{\mathcal{M}_{g,m}(X,A)} \mathrm{ev}_1^* \alpha_1 \wedge \ldots \mathrm{ev}_m^* \alpha_m \wedge \Phi^* \mathrm{PD}(\beta)$$

where $\mathrm{ev} = (\mathrm{ev}_1, \ldots, \mathrm{ev}_m)$ and PD denotes the Poincaré duality map. This definition really only makes sense when dimensions add up correctly, which is when

$$\dim(\mathcal{M}_{g,m}(X, A)) = \dim(\mathcal{M}_{g,m}) \sum_i |\alpha_i| - |\beta|$$

(here the $|\cdot|$ represents the degree).[2] Otherwise, we set $\mathrm{GW}^{\mathrm{A}}_{\mathrm{g,m}}(\alpha_1, \ldots, \alpha_m, \beta) = 0$.

This number is meant to be an algebraic count of maps in $\mathcal{M}_{g,m}(X, A)$ whose domain remain within a cycle representing β and that intersect the cycles representing $PD(\alpha_i)$ at their i-th marked point. So for example, the enumeratiev number we wanted to compute is now

$$n_d = \mathrm{GW}^{dP}_{0,3d-1}(p, \ldots, p, [\mathcal{M}_{0,3d-1}])$$

where P is as in the previous example and p a cohomology class Poincaré dual to a point.

Up to here we have the first third of the talk: motivating a fairy-tale definition of GW. It remains to see how this absurd machinery is actually useful, but first... we should do a reality check.

2.2 An algebraic virtual utopia

As we have shown, at the end of the day we want to do intersection theory on a space $\mathcal{M}_{g,m}(X, A)$. This requires being able to take unique limits (properness and separateness or compactness and Hausdorfness), a nice enough local structure that allows for some analogue of homology (for example, the Chow group on a stack or some notion of cycles on an orbifold) and, quite preferably, a fundamental class of our moduli space. We will deal with the algebraic and differential geometric fixes to this issues in the seminar. Let me slightly motivate some of them from the algebraic perspective with the very important disclaimer that I am very much not an algebraist, yet.

Taking unique limits. The spaces we are dealing with can be enlarged in a way all limits can be taken and they are unique, so this is not a major inconvenience. I will explain this more geometrically in section 3.3. We now give some intuitions. On the one hand, when we enumerated rational cubics through eight points it became necessary to consider nodal domains, which we have not in our definitions. But we should allow reducible curves as well.

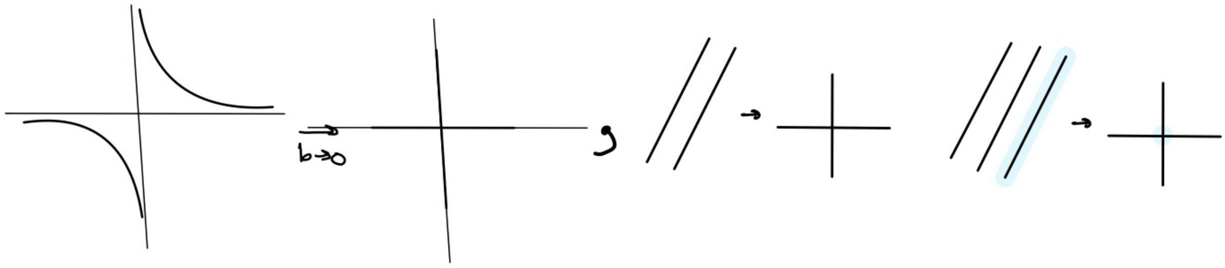

Figure 1: Left, failure of compactness. Right, failure of uniqueness of limit.

Example 12. Consider the one-dimensional family of conics $XY - aZ$ for $a \in \mathbb{C}^*$, it is clear that this should live in our moduli space (this is just a rough idea, we are actually cheating since we would have to first parametrize the curves). So if we expect to take reasonable limits then we should allow $a = 0$, that is the degenerate conic $XY = 0$. This brings up a certain issue, we can parametrize XY in several non-equivalent ways: parametrize $X = 0$ and $Y = 0$ by two projective lines and then take as many projective lines and send them to a single point. So our map would be given by two degree one maps and several degree 0. If we want our moduli space to be separated (or Hausdorff in the differential geometric language) we need limits to be unique and so the above issue needs to be dealt with. One obvious fix for this would just be to not allow those degree 0 superfluous maps. Unfortunately, this does not work: doing this would not yield enough maps in the boundary of the moduli space in order to compactify it. It turns out that demanding that those genus 0 components (resp. genus 1) mapped to a point have at least three non-smooth points or marked points (resp 1) solves the problem (marked points must be smooth and non-smooth ones can only be nodal). These considerations lead to Kontsevich's definition[3] of a m-pointed stable map (for which one now considers the arithmetic genus as opposed to the geometric one), see section 3.3 for a definition. This notion of stable map is equivalent to points in the moduli space having no infinitesimal automorphisms (more on this throughout the talk). The moduli space of such maps is denoted by $\overline{\mathcal{M}}_{g,m}(X, A)$ and it is proper and separated (or compact and Hausdorff in the differential setting, we will give a lot more information about this later).

Local structure of the moduli space. In both differential and algebraic settings we can present the moduli space as a quotient (acting by reparametrizing) of the stable maps, which have good structure. For now we briefly comment on the algeraic setting and elaborate further on the differential context in 3.1. Through this quotient one can see that the tangent space to an embedded curve $[f : \Sigma \to X]$ will be $H^0(f^*TX)$ (infinitessimal deformations) but that, unless the $H^1(f^*TX)$ vanishes (we then say it is unobstructed) the dimension of the tangent space will not match the virtual dimension. This makes our moduli space a bit weird, in the algebraic world stacks provide a good solution, though there will be no smoothness nor irreducibility in general. Even if the moduli spaces were smooth, strange things can happen. Here is a concrete example of our moduli space having different dimensions:

Example 13. Consider the space $\overline{\mathcal{M}}_{0,2}(\mathbb{P}^2, 2P)$. This should be the space of conics but it is not quite that: it contains all smooth conics; also double covers of lines (topologically a sphere double covering another sphere with two critical points); in the boundary $(\overline{\mathcal{M}}_{0,2}(\mathbb{P}^2, 2P) \setminus \mathcal{M}_{0,2}(\mathbb{P}^2, 2P))$ there are maps whose domain is nodal, each component mapping into two lines joint at a point; and even those maps whose domain is also nodal and both components map to the same line. Each

[2]This can easily be seen to be the simpler condition $\sum_i |\alpha_i| - |\beta| = n(1-g) + \int_A c_1 T_X$.

[3]Kontevich was inspired by Gromov's notion of a cusp curve and the algebraic notion of a stable curve. In fact, compactness of the corresponding moduli space in the symplectic setting was known since Gromov's paper in 85. A proof that the resulting space was Hausdorff was also known but it was only fully correct after adding Kontsevich's notion of stability. In this case, algebraic geometry was completely informed by symplectic topology, as Kontsevich himself explains.

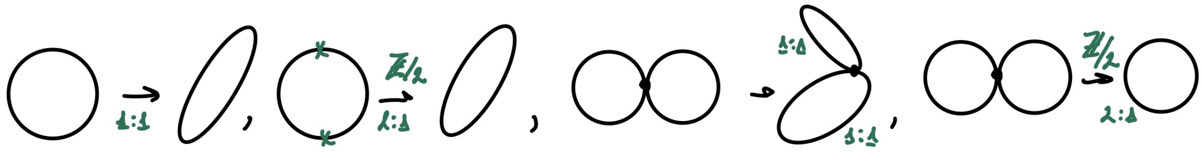

Figure 2: $\overline{\mathcal{M}}_{0,2}(\mathbb{P}^2, 2P)$.

of this strata has different dimensions: 5, 4, 4 and 3 respectively. That being said, it is not too bad because smooth conics are dense in that moduli space. But it can get bad: $\mathcal{M}_{1,0}(\mathbb{P}^2, P)$ is empty but it has a very non-empty boundary, in fact, $\overline{\mathcal{M}}_{1,0}(\mathbb{P}^2, P)$ is four-dimensional.

A wish for fine moduli spaces. Another thing that may trip is up is that stable maps have automorphisms: in example 13, we see that the maps in $\mathcal{M}_{0,2}(\mathbb{P}^2, 2P)$ that are branched double covers have $\mathbb{Z}/2$ symmetry because we can switch the sheets around, and maps in $\overline{\mathcal{M}}_{0,2}(\mathbb{P}^2, 2P) \setminus \mathcal{M}_{0,2}(\mathbb{P}^2, 2P)$ that are two lines joint at a node that map to a single line do as well because we can transpose the two domain lines. To explain why this phenomena is bad consider the following example:

Example 14. Grassmannians of k-planes in $\mathbb{R}^n$ are the main example of very nice moduli space. They are manifolds whose points are in bijective correspondence with the object they are parametrizing (as is true in our case) and close enough k-planes correspond to close enough points in the Grassmannian (this is a coarse moduli space). However, they have an even better property, which is that the tautological bundle is universal. This means that we can place the k-planes over the points representing them in the Grassmanian, assemble them into a vector bundle (the tautological bundle), and that for any other nice family of k-planes (a vector bundle over some space M) we can find a map of M to a Grassmannian so that that new family is just a pullback of the tautological bundle (roughly, this is a fine moduli space). Even more informally: any way to arrange our geometric object is somehow encoded in our moduli space and its universal family. Finally, notice that presenting the Grassmannian as a homogeneous space in the usual way presents it as a manifold modulo the action of a group acting by "reparametrizing". One can see that the Grassmannian is a manifold because the action is smooth, proper and free.

In our case, we deal with a very similar situation: we present $\overline{\mathcal{M}}_{g,m}(X, A)$ as the quotient of some maps by reparametrization. In our case, however, the action is not always free (the isotropies are precisely the automorphism groups). This phenomena usually does not allow for the construction of a universal family because there is somehow some ambiguity in which geometric object to place over the point representing it.

There is, however, a saving grace: the automorphism groups are finite (which is equivalent to the stability condition). This amounts to weighing the points in the moduli space differently: one will be worth $1/N$ if N is the order of automorphism group (as a consequence, we must define GW invariants over $\mathbb{Q}$ as opposed to $\mathbb{Z}$). This then requires the notion of a Delinge-Mumford stack (or an orbifold). Here is the first important resolution of this discussion, a theorem of Kontsevich [10].

Theorem 15. *The space $\overline{\mathcal{M}}_{g,m}(X, A)$ is a proper separated Deligne-Mumfort stack. If X is moreover convex (genus zero maps are unobstructed), then $\overline{\mathcal{M}}_{0,m}(X, A)$ is a smooth proper separated Deligne-Mumfort stack (it will in fact be the quotient of a smooth variety by a finite group).*

Happily enough, $\mathbb{P}^n$ is convex and so we can define Gromov-Witten invariants in a very reasonable way for genus 0. Moreover, in this unobstructed case it makes sense to consider a fundamental class and intersection theory will work as usual (accounting for those weighted points).

Upshot. There is a reasonable fix for our seemingly distopic reality for genus zero maps into a convex smooth projective variety. If we do not have convexity, we still have a space in which we can do intersection theory if we somehow manage to define a fundamental class (which is not clear since our moduli space is not equidimensional). In the algebraic context, however, this can be done [3]: there is a class in the Chow ring

$$\left[\overline{\mathcal{M}}_{g,m}(X,A)\right]^{\text{vir}} \in \mathcal{A}_{\text{vir-dim}(\overline{\mathcal{M}}_{g,m}(X,A))}(\overline{\mathcal{M}}_{g,m}(X,A),\mathbb{Q}),$$

called the virtual fundamental class, which behaves much like a fundamental class would and agrees with it in the unobstructed case.

This philosophy of moduli spaces will be covered in a much more precise way in the seminar as well as one or two talks on the virtual fundamental class.

2.3 What about a symplectic utopia?

We will explore the symplectic world in the next section but, for the sake of cohesion, let us now mirror the algebro-geometric statements in differential geometry. The projective variety X may be viewed as a symplectic manifold by restricting the natural symplectic form on $\mathbb{P}^N$ after the projective embedding. In fact, we can follow the same strategy as before to define Gromov-Witten invariants on a symplectic manifold by picking a compatible almost complex structure and using pseudo-holomorphic curves. It is truly remarkable that when we can define the invariants, they do not depend on the almost-complex structure chosen but only on the symplectic deformation class of the symplectic form. This means that GW-invariants are a symplectic invariant rather than a complex one.

Compactness and Hausdorffness can be achieved in an analogous way, see section 3.3. The local structure is both nicer and worse (see section 3.1): unlike in the algebraic setting, we can perturb the almost complex structure so embedded curves are generically unobstructed. However, even if this can be accomplished, their covers may not be well behaved in most cases (Calabi-Yau 3-folds, as we shall mention, are an example where things do work out). Around points in $\overline{\mathcal{M}}_{g,m}(X,A)$ that are unobstructed, we can show that the local structure is that of a smooth orbifold outside of the boundary and that of a log-smooth orbifold (probably).

In the symplectic setting defining a virtual fundamental class is harder and one of the main driving forces in the field for the last 20 years. One new(ish) approach of John Pardon consists on representing the moduli space in the category of (log-smooth, probably soon) derived orbifolds. That being said, Gromov-Witten invariants can still be defined for a very large class of manifolds, called semi-positive manifolds (e.g. all manifolds up to complex dimension 3 satisfy that condition). The first step is to achieve unobstructedness of all curves by destroying symmetries, this is done by considering approximately pseudo-holomorphic curves (technical words: we consider inhomogenous perturbations of the Cauchy-Riemann equation and they will generically yield unobstructedness). One can then show that the map (ev,Φ) on $\overline{\mathcal{M}}_{g,m}(X,A)$ is a pseudocyle (which is were semipositivity comes to effect, making naughty curves harmless by vanishing them to high enough codimension). Pseudocycles enable the use of intersection theory and, as in definition 11, we can write:

$$\mathrm{GW}^{\mathrm{A}}_{\mathrm{g,m}}(\alpha_1,\dots,\alpha_m,\beta) = (\text{ev},\Phi)_*\left[\overline{\mathcal{M}}_{g,m}(X,A)\right] \bullet \left[\overline{\alpha_1}\times\cdots\times\overline{\alpha_m}\times\overline{\beta}\right],$$

where the overbars indicate submanifolds representing of X and $\overline{\mathcal{M}}_{g,m}$ the corresponding $\mathbb{Q}$-homology classes and $\bullet$ the intersection product. There will be a talk in the seminar expanding on the definition of Gromov-Witten invariants for semi-positive symplectic manifolds via inhomogenous perturbations (which actually requires replacing $\overline{\mathcal{M}}_{g.m}$ by a fine moduli space that finitely covers it in order to parametrize the perturbations correctly and harmlessly).

2.4 Worlds out of nothing

We now explain why creating such an infrastructure can actually be helpful in important real life problems, such as enumeration of rational curves in the projective plane.

This abstract framework allows us to find non-trivial relationships between different Gromov-Witten invariants, so relationships between different enumerative problems. The hope is that one can reduce the computation of the GW-invariants to the simplest ones. In fact, Kontsevich and Manin listed some axioms GW-invariants ought to satisfy and deduced from it many good properties, such as the recursion formula for enumeration of plane curves (whatever definition you can rigorously establish for GW should verify the axioms). Since there will be a talk about this in the seminar, now we only give a flavor of what we could expect for the GW-invariants. For example, there are nice morphisms $\overline{\mathcal{M}}_{0,m+1} \times \overline{\mathcal{M}}_{0,m'+1} \to \overline{\mathcal{M}}_{0,m+m'+1}$ obtained from gluing together curves at specified marked points. There are also gluing maps to change genus by gluing one curve to itself at two different marked points. We then use these maps along with Ψ to, for example, find expressions of $\mathrm{GW}^{A}_{0,m+m'+1}$ via $\mathrm{GW}^{B}_{0,m+1}$ and $\mathrm{GW}^{B'}_{0,m'+1}$ for $A = B + B'$. This is one of the most important features of the GW-invariants and it usually goes under the name "Splitting axiom". The proof from the differential perspective requires delicate gluing theorems, see [13]. All of these information can be packaged efficiently though a generating function.

Generating functions are ubiquitous objects in discrete mathematics because they capture the behaviour of a discrete problem efficiently within a formal arithmetic system. A generating function for a discrete countable set of data is a formal power series whose coefficients are given by this data. Different coefficients of power series are easily related by differentiation and so differential equations of generating functions express relationships between these information. Multiplication of power series often capture how the problem the data is describing can be cut into two. In our case, one can set up a generating function encoding Gromov-Witten invariants (a discrete set of data), called the Gromov-Witten potential, and express those recursive relations explained above as differential equations.

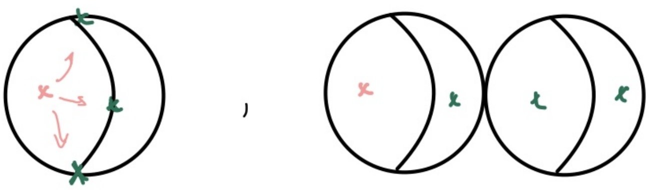

Figure 3: $\overline{\mathcal{M}}_{0,4}$.

Example 16. The space $\overline{\mathcal{M}}_{0,4}$ is isomorphic to $\mathbb{P}^1$. First, note that the space $\mathcal{M}_{0,4}$ is isomorphic to $\mathbb{P}^1$ with three points removed. The reason is that the isomorphism group of the projective line is 3-transitive and so we can fix three of the marked points and the last one is allowed to move around freely (it will be the cross-ratio). Compactifying one gets three nodal elements, each is two $\mathbb{P}^1$'s with two marked points that are joined at a node, see figure 3. Each of this points is a boundary divisor and they are all homologically equivalent since they live in $\mathbb{P}^1$. Pulling back this equivalence via the forgetful map Ψ (and a map that forgets map points and stabilizes) will yield a relationship of GW-invariants that is expressed by the WDVV equation on $\mathbb{P}^2$. This equation implies Kontsevich's formula by direct and easy computation. This is well explained in [9].

The WDVV equation, in general, is in fact a consequence of the aforementioned game of gluing and it captures the associativity of the product in quantum cohomology, which can be defined with the Gromov-Witten potential to capture the following behaviour: while the cup product in usual cohomology "counts" intersection of two cycles, the quantum product tracks the concatenations of stable maps connecting those two cycles.

3 Back to earth, one foot on symplectic ground

The reason that not both of our feet will be on symplectic ground is that we will continue to be sketchy even if we have come down from out utopia. A symplectic manifold is a smooth manifold M equipped with a non-degenerate closed 2-form. This structure is somehow topological: there are no local invariants and its isomorphism group is infinite-dimensional; these manifolds are somehow non-squeezable; even if the symplectic structure is defined using derivatives, its isomorphism group is $\mathcal{C}^0$-closed in the diffeomorphism group; the moduli space of symplectic forms is finite-dimensional. Notice this is quite the opposite to Riemannian geometry. Some of these curious behaviour will be explored in an upcoming talk in the seminar.

In any case, in this section, perhaps unexpectedly, we will only speak of symplectic structures at the end. For now, we study a generalization of complex manifolds and holomorphic maps.

Definition 17. A complex structure on a vector bundle $E \to M$ is a smooth section J of its endomorphism bundle that satisfies $J^2 = -\operatorname{id}$ (so it is an actual complex structure on each fiber of E that happens to vary smoothly). A manifold M with a complex structure on TM is called an almost complex manifold.

Complex manifolds locally look like $\mathbb{C}^n$ so we can pull-back the natural complex structure (e.g. component-wise multiplication by i) and patch it together with the holomorphic transition maps to a complex structure on the tangent bundle. Conversely, however, not all almost-complex manifolds are complex, i.e. the complex structure does not come from a holomorphic atlas (not integrable). In fact, a very deep theorem of Newlander–Nirenberg states that if the exterior derivative d decomposes exactly as $\partial + \overline{\partial}$ on forms, then almost complex structure is integrable. In particular, all (real) 2-dimensional almost complex manifolds are Riemann surfaces, i.e. (complex) 1-dimensional complex manifolds.

Definition 18. Let $(M, J), (N, J')$ be two almost complex manifolds, $f : M \to N$ is said to be pseudo-holomorphic if its differential $Tf : (TM, J) \to (TN, J')$ is complex linear, that is, $Tf \circ J = J' \circ Tf$ or, multiplying by J' on both sides, $Tf + J' \circ Tf \circ J = 0$. This equation is called the non-linear Cauchy-Riemann equation.

The theory of such maps makes the most sense when the domain is a Riemann surface.[4] Such a map is then called a pseudo-holomorphic curve and we will usually write $u : (\Sigma, j) \to (M, J)$. Notice that we can define $\mathcal{M}_{g,m}(M, J, A)$ in the same way we did before by taking pseudo-holomorphic curves with marked points up to mark-point-preserving reparametrizations. Notice as well that if we take M to be a single point, then $\mathcal{M}_{g,m}(M, J, A)$ is just $\mathcal{M}_{g,m}$ as defined before.

3.1 The elliptic deformation complex

In order to show manifold structure on our moduli space we want to use infinite-dimensional differential geometry. We will showcase the main ideas through a finite-dimensional toy model.

Toy model: parametrized moduli space. Say we have a rank k smooth vector bundle $E \to M$ over a smooth n-manifold M and $s : M \to E$ a smooth section of it. You should think about s as assigning an equation $s(p) = 0$ to each point p and we are interested in the points $p \in M$ that actually do verify the equation $s(p) = 0$, that is, $\mathcal{M} = s^{-1}(0)$. Say $p \in s^{-1}(0)$, we can linearize s at p to obtain a linear map $D_p s : T_p M \to E_p$ (choose a connection and show it is independent of the choice since $s(p) = 0$). The inverse function theorem says that if D_p is surjective, then $s^{-1}(0)$ is a submanifold of M around p of codimension k. Another way to go about this is to regard s as an embedding of M and $s^{-1}(0)$ as the intersection of s and the zero section, denoted $\mathbf{0}$. If s is transverse to $\mathbf{0}$, then the intersection will be a manifold of dimension $n - k$. Both approaches are equivalent.

[4] One typically expects not to be able to find general pseudo-holomorphic maps. For example, there can be no pseudo-holomorphic map from a complex manifold to an equidimensional non-integrable complex manifold.

So, in order to prove that our zero set is a manifold, we must check transversality of s and the zero section which is equivalent to the linearization of the section at a point in the intersection being surjective. The main tool was the IFT.

Toy model: unparametrized moduli space. Often times our moduli spaces are described by objects given by zero sets $s^{-1}(0)$ modulo reparametrization. So we need to extend the above to an equivariant setting. Let G be a finite-dimensional Lie group acting smoothly and properly on M in a way that it extends to E and our section s is G-equivariant, we are then interested in $\mathcal{M} = s^{-1}(0)/G$. The way you should think about this is that G captures the symmetries of the equation $s(p) = 0$. Here is a nice set up to show that $\mathcal{M} = s^{-1}(0)/G$ is a manifold. Consider the orbit map $G \to M$ given by $g \mapsto g \cdot p$, composing it with $s : M \to E$ and linearizing it at point $p \in M$ that is a zero of s we get:

$$T_{\mathrm{id}}G = \mathfrak{g} \xrightarrow{L^{-1}} T_pM \xrightarrow{L^0 = D_p s} E_p.$$

The above is a complex, $L^0 \circ L^{-1} = 0$, because s is constant along the orbit of p. Indeed, by equivariance $s(g \cdot p) = g \cdot s(p) = 0$. We call it the **deformation complex**, its cohomology $H^*(L^*)$ encodes the properties that we want:

Exercise 19.
- If $H^{-1}(L^*) = \ker L^{-1} = 0$, then L^{-1} is injective. This is the case when G acts on M with finite isotropy (i.e. stabilizers G_p are finite) or at least the isotropy at p is finite.
- If moreover $H^1(L^*) = \operatorname{coker} L^0 = 0$, then L^0 is surjective and $s^{-1}(0)/G$ is an orbifold around $[p]$ with isotropy G_p with tangent space given by $H^0(L^*) = \frac{\ker L^0}{\operatorname{im} L^{-1}}$.

We call $H^0(L^*)$ infinitesimal deformations of $[p] \in \mathcal{M} = s^{-1}(0)/G$ and $H^1(L^*)$ its infinitesimal obstructions.

The actual set up. Define $\overline{\partial}_J(j, u) = Tu + J \circ Tu \circ j$, the Cauchy-Riemann equation, which inputs pairs of (almost) complex structures on Σ and smooth maps $(j, u) \in \mathcal{J}(\Sigma) \times \mathcal{C}^\infty(\Sigma, M) =: \mathcal{B}$ and outputs an element of $\mathcal{E}_{j,u} = \overline{\mathrm{Hom}}_{\mathbb{C}}((T\Sigma, j), (u^*TM, J))$ (denoting complex antilinear maps). This assembles into an infinite dimensional vector bundle $\mathcal{E} \to \mathcal{B}$ and the zero set of $\overline{\partial}_J$ is made up of the maps $u : (\Sigma, j) \to (M, J)$ that are pseudo-holomorphic. We are interested in such maps up to reparametrization, the group on the background is the group of orientation-preserving diffeomorphisms $G = \mathrm{Diff}^+(M)$. PDE theory enters the stage here since $\overline{\partial}_J = 0$ is a non-linear partial differential equation (varying over $\mathcal{B}$) that we can linearize $D_p \overline{\partial}_J$ to use linear PDE theory. The right set-up to understand many of such PDE's is the theory of Sobolev spaces, which are Banach spaces generalizing L^p spaces by formally extending usual differentiation via integration (by parts).

In order for us to use the IFT in this infinite-dimensional setup we require our manifolds $\mathcal{E}$ and $\mathcal{B}$ to be locally modelled on Banach spaces (the IFT can be deduced from the Banach fixed point theorem), which either requires downgrading regularity from $\mathcal{C}^\infty$ to $\mathcal{C}^k$ or taking Sobolev completions of our spaces. The latter is preferred because it will exhibit better properties for $\overline{\partial}_J$, mainly that it will be a Fredholm map (linearization will be Fredholm and so have finite-dimensional kernel and cokernel) and the deformation complex will be elliptic. Ellipticity is an analytic condition that roughly says that derivatives of functions are somewhat controlled only by those derivatives dictated by the PDE. Usually one uses Sobolev completions of spaces to obtain results and is allowed to move it back to the smooth world via a property of elliptic operators, elliptic regularity. There will be a talk on the seminar explaining the functional analysis necessary for this and deducing things such as the Hodge decomposition theorem (where the Laplacian is an elliptic operator parametrezing real cohomology).

In conclusion, a set up analogous to the above allows us to obtain an elliptic deformation complex controlling the deformation theory of our moduli space. Ellipticity implies that the cohomology will be finite-dimensional (in fact, stability of maps implies the orbit map is injective) so infinitesimal deformations and obstructions are finite dimensional vector spaces:

Theorem 20. *The cohomology of the deformation complex of* $[(j,u)]$,

$$\Omega^0(\Sigma) \xrightarrow{L^{-1}} \Omega^{0,1}_j(T\Sigma) \oplus \Omega^0(u^*TM) \xrightarrow{L^0} \Omega^{0,1}_j(u^*TM),$$

is finite dimensional and encodes the infinitesimal deformations of the automorphisms of (Σ, j), *the infinitesimal deformations of the curve* $[(j,u)]$ *as well as the infinitesimal obstructions, that is:*

- $T_{\mathrm{id}}Aut(\Sigma, j) = H^{-1}(L^*)$ *which is zero in the stable case and, in that case,*
- *if* $H^1(L^*) = \operatorname{coker} L^0 = 0$, *then* $\mathcal{M}_{g,m}(M,J,A)$ *is a smooth orbifold around* $[(u,j)]$ *of isotropy* $Aut(j,u)$ *of the "right" real dimension*[5] $(2-2g)(n-3) + 2\int_A c_1 T_X + 2m$ *and* $T_{[(j,u)]}\mathcal{M}_{g,m}(M,J,A) = H^0(L^*) = \ker L^0 / \operatorname{im} L^{-1}$.

For a much more detailed and careful explanation of this perspective see the ongoing series of blog posts [16].

Visualizing deformations This section is brief and can be made rigorous with not too much effort. The main point is that we consider stable maps up to reparametrizations to capture the behaviour of the image of the map. It is then natural to interpret the deformations above yielding the local structure of the moduli space as deformations of this image. The way to think about this is to associate some normal bundle to the image and then check which sections of it are pseudo-holomorphic.[6]

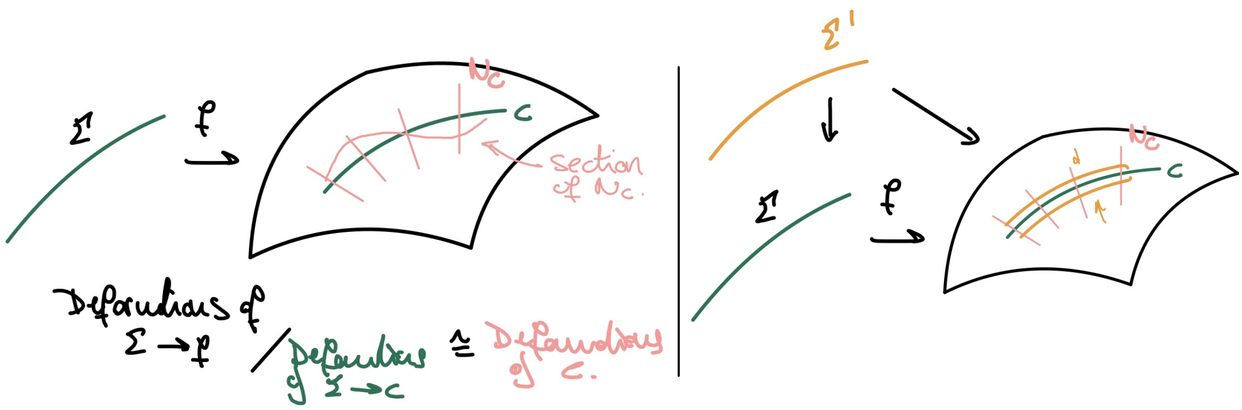

Figure 4: Left: visualization of deformations. Right: multiple cover collapsing.

Say $u : (\Sigma, j) \to M$ is an embedding and call $u(\Sigma) = C$, then the normal pseudo-holomorphic deformations of C (which should be conceived as deformations of C as a one-dimensional almost complex manifold) capture the deformations of the stable map $u : (\Sigma, j) \to M$ modulo the deformations of $\Sigma \to C$ (which basically mods out the variation on j) and the obstructions of (j,u) are exactly the same as the normal obstructions. In the following expressions GDC means global deformation complex:[7]

$$\frac{H^0(\mathrm{GDC}(u : (\Sigma, j) \to M))}{H^0(\mathrm{GDC}(u : (\Sigma, j) \to C)} \cong H^0(N_C) \text{ and } H^1(\mathrm{GDC}(u : (\Sigma, j) \to M)) \cong H^1(N_C).$$

[5]This corresponds to the index of the Fredholm operator in the background of the elliptic complex. It can be computed with a complex Riemann-Roch or Atiyah-Singer.

[6]For complex dimensions $n > 2$, pseudo-holomorphic curves are generically immersed. The space of immersions $\Sigma \to M$ modulo reparametrization (i.e. modulo diffeomorphism of Σ) is parametrized exactly by the normal bundle of $u(\Sigma)$. So, in this sense, normal deformations are the right context in which to look for deformations, we just must check which normal deformations are through pseudo-holomorphic curves.

[7]We also note that in the symplectic setting one can define the normal bundle for any pseudo-holomorphic curve, even if it has critical points, it will only have finitely many and the sheaf coker du can made torsionless to obtain this

Something worth explaining is that there are other "normal" deformations of C that are not captured by those of $u : (\Sigma, j) \to M$. There can be multiple covers of C collapsing onto C, which you can visualize as multivalued sections of N_C. Those can be captured by twisting N_C by a local system induced by a certain covering. This multiple cover contributions become very difficult to deal with in the context of symplectic Gromov-Witten theory. An example in which they are understood is for Calabi-Yau 3-folds, after Wendl's proof of the Superrigidity conjecture [18], we understand that multiple covers contribute in the simplest way possible.

3.2 When can transversality be achieved?

This, of course, is now the big question: if we can make the linearization of $\overline{\partial}_J$ surjective (i.e. $\operatorname{coker} L^0 = 0$) then our moduli space looks nice. The basic idea is to see when a generic enough perturbation of J (through an infinite-dimensional Sard's theorem) yields a generic enough perturbation of $\overline{\partial}_J$ so that $\overline{\partial}_J$ intersects $\mathbf{0}$ transversely.[8] This is possible for simple curves (those that do not factor through smaller Riemann surfaces) because, otherwise, symmetries will be present and will make L^0 not surjective because it will be valued in some equivariant subset. Here is the result:

Theorem 21 (see, for example, [17])**.** *For a generic choice of J, the moduli space of simple pseudo-holomorphic curves, denoted $\mathcal{M}^*_{g,m}(M, J, A)$, is a smooth orbifold of real dimension $(2 - 2g)(n - 3) + 2\int_A c_1 T_X + 2m$. Isotropy at a point is given by the automorphisms of the curve $[(j, u)]$ and $T_{[(j,u)]}\mathcal{M}^*_{g,m}(M, J, A) = H^0(L^*) = \ker L^0 / \operatorname{im} L^{-1}$. This continues to hold true for $\overline{\mathcal{M}}^*_{g,m}(M, J, A)$ if, instead of a smooth orbifold, we say a topological orbifold.*[9]

In fact, in [18], Wendl proved a stratification theorem for $\mathcal{M}_{g,m}(M, J, A)$ that helps us understand multiple covers, *roughly*: one can express the moduli space as a union of smooth finite-dimensional manifolds (generically) prescribing branching information so that each of those is generically stratified by walls (which are submanifolds of a given known codimension) given by maximally connected subsets in which the infinitesimal deformations and obstructions vary constantly. This, in retrospect, is the most natural thing that should happen but, of course, the exact formulation is difficult. This stratification can then be interpreted as a generalization of Brill and Noether theory to elliptic operators as in [4].

3.3 Some heuristics on Gromov compactness

I did not say anything about compactness in the symplectic setting in the talk other than it works out as nicely as in the algebraic case and, in fact, it the symplectic preceded the algebraic. Alas, I did tell someone I would write this section, so I will just go for brief introdution of the main points.

Definition 22. A tame almost complex structure J on a symplectic manifold (M, w) is an almost complex structure on M so that $w(v, Jv) > 0$ for all non-zero v.

By symmetrizing this expression one obtains a Riemannian metric and it is not too wrong to think about M then as a "Kähler manifold" where the complex structure is not necessarily integrable. Coming back to compactness, convergence of a sequence of maps in $\mathcal{M}_{g,m}(M, J, A)$

generalized normal bundle N_u. If D denotes the Weyl divisor of critical points of u in Σ and $\mathcal{O}_D$ its structure sheaf, then we can generalize the above to understand how there is a normal deformation theory related to any pseudo-holomorphic curve: the normal deformation theory will still control the obstructions perfectly and the deformations of the map modulo the deformations of the domain (including moving around critical points) yield the normal deformations (that is, $\frac{H^0(\mathrm{GDC}(u:(\Sigma,j)\to M))}{H^0(\mathrm{GDC}(u:(\Sigma,j)\xrightarrow{\mathrm{id}}\Sigma)\oplus\mathcal{O}_D} \cong H^0(N_C)$).

[8] Such arguments seem to always be carried out on symplectic manifolds to have a good hold on the space of almost-complex structures. I am not sure if this is necessary, but I can only say confidently that this statements of local structure hold for symplectic manifolds and the almost-complex structures we refer to are tamed by the symplectic form. This is defined in section 3.3

[9] This is a very technical point, it is likely that this is in fact a log-smooth orbifold.

can be thought as convergence of the domain complex structures together with convergence of the maps uniformly on compact sets for all derivatives (this can be made rigorous and it is the notion of Gromov convergence). One way to then think about non-convergent sequences is by separating what can go wrong in the domain Riemann surface $[(\Sigma, j)] \in \mathcal{M}_{g,m}$ and what can go wrong in the convergence of the map u given the that the domains are well behaved.

We will assume we have no marked points for brevity. Smooth irreducible marked algebraic curves over $\mathbb{C}$ are equivalent to marked compact Riemann surfaces, which are equivalent to hyperbolic surfaces for genus two or larger. By this we mean that $\mathcal{M}_{g,0}$ is equivalent to hyperbolic surfaces modulo isometry when $g > 1$. The only way compactness can essentially fail here is when we pinch one closed geodesic, as is exemplified in the picture. Then, adding nodal Riemann surface of arithmetic genus g (see figure) compactifies $\mathcal{M}_{g,0}$, see the figure as well.

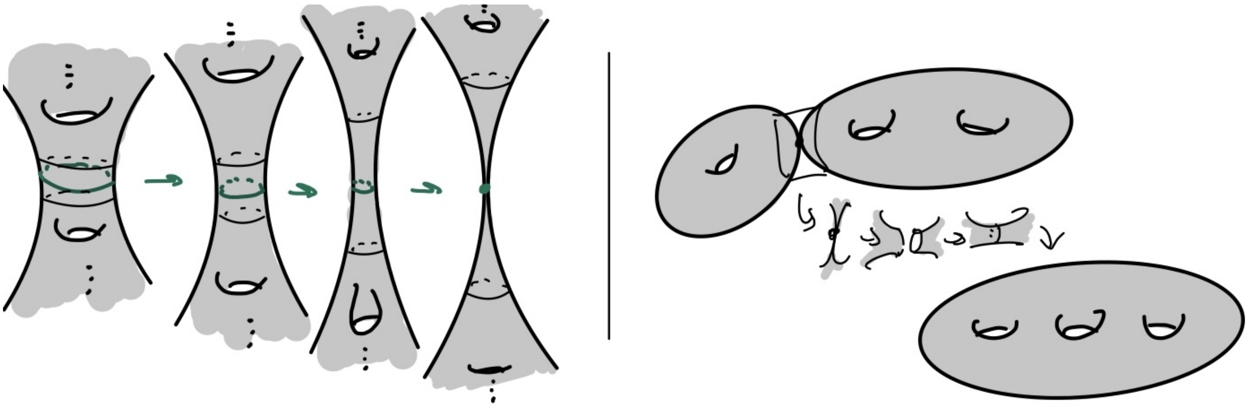

Figure 5: Left: closed geodesic (green) collapsing. Right: example of an arithmetic genus 3 surface and idea of reason why.

We now explain what can go wrong with the convergence of the map u if the domains are controlled. It is in here that the symplectic structure is crucial, it provides the concept of energy. The energy of a map $u : \Sigma \to M$ is defined as $E(u) = \int_\Sigma u^*w$, imposing uniform bounds on the energy of sequence of pseudo-holomorphic maps with controlled domain Riemann surfacse will yield a convergent subsequence modulo bubbling, which we explain in a second. Before, though, we remark that an important feature about the energy of a pseudo-holomorphic map on a compact Riemann surface, it is a homological invariant: $E(u) = \langle w, A\rangle$ where $A = u_*[\Sigma]$, the pairing is the cohomology-homology pairing and, consequently, energy of any map is always the same in $\mathcal{M}_{g,m}(M, J, A) = \{[(j, u)] \in \mathcal{M}_{g,m}(M, J) : A = u_*[\Sigma]\}$.

By convergence modulo bubbling we mean that the sequence will converge as nicely as possible outside a set of finitely many points, in which finitely many bubbles (pseudo-holomorphic spheres) appear maintaining the arithmetic genus.[10] So in the limit the domain Riemann surface will be the expected one (we assumed that aspect was controlled) but with certain nodal structure made of spheres tracking the bubbling.

[10]In fact, you can try to think about how to interpret these nodal Riemann surfaces of arithmetic genus g together with those stable maps as graphs, the ways the bubbles would dock means that their corresponding graph will be a tree and so one often hears the pretty term *bubble tree*

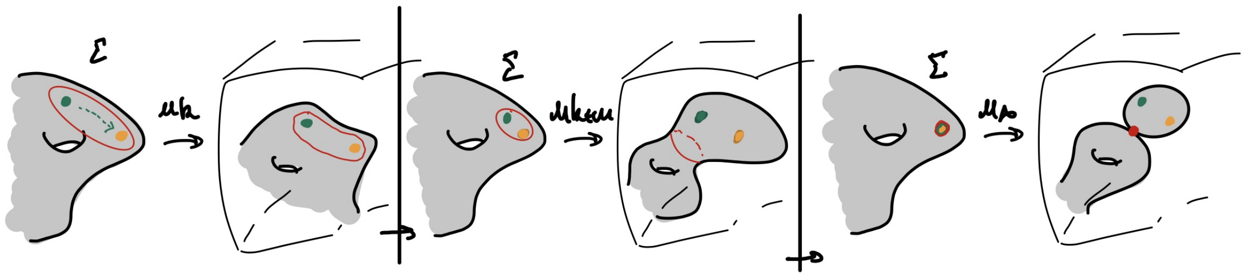

Figure 6: Formation of a bubble: we have a sequence of green points in Σ converging to an orange one as k goes to infinity but the images by u_k stay at a bounded distance. At this points the gradient of u_k blows up.

Finally, we define stability and show another example why the absence of stability does not provide a Hausdorff compactification.

Definition 23. An m-marked nodal Riemann surface is stable if each of its components has negative Euler characteristic (equivalently, it admits a hyperbolic metric) after removing the nodes and marked points. A pseudo-holomorphic curve $u : (\Sigma, j) \to (M, J)$ defined on m-marked nodal Riemann surface (Σ, j) is stable if whenever u is constant on a component of Σ, that component is stable as a marked nodal Riemann surface. This is equivalent to the automorphisms of (Σ, j) being finite.

The first formulation of stability seem like an ad-hoc condition. It is not for several reasons: requiring for finite automorphism group is natural and yields the first formulation; a similar way to say this is that stability can be though of as a condition of having enough special points, which can be understood as a gauge fixing condition to determine correctly the elements in the moduli space; stable Riemann surfaces are natural because hyperbolic geometry is the most common geometry for (real) surface theory; and, finally, it comes up naturally if one wants to make limits unique. We address this last point in figure below with an example of stability for algebraic curves. Here is the set up: given a nodal surface, one could define a neighbourhood of it in $\mathcal{M}_{g,m}$ by desingularizing. In this context, non-stable components may cause problems:

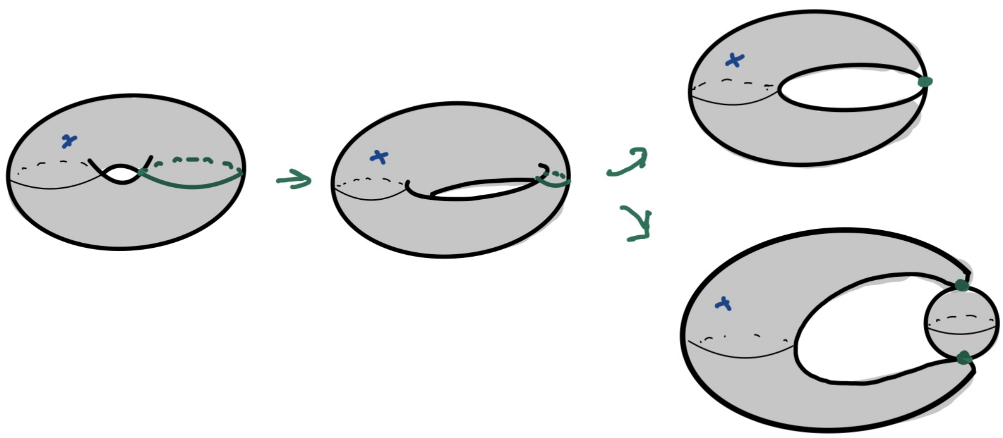

Figure 7: Two possible limits of eliptic curves $\mathcal{M}_{1,1}$. The (more or less) intuitive neighbourhood of those two limit curves by desingularizing are the same even if they are different elements, which is very much against Hausdorffness/separatedness. The problem is that that extra sphere in the bottom right corner is not stable.

Notice, however, that from the hyperbolic perspective this condition seems dumb: degenerations should happen within the hyperbolic world and so that middle sphere with two special points (a cylinder, a flat thing) would obviously not be there. Indeed, as I said before, stability is very natural from the hyperbolic point of view even if from the complex-algebraic or almost-complex point of view it seems very mysterious.

Even if we have not really defined Gromov-congergence (the topology of the moduli space) I hope this intuitive ideas make the following plausible:

Theorem 24 (Gromov compactness)**.** *The topological space of stable curves $\mathcal{M}_{m,g}$ and stable maps $\overline{\mathcal{M}}_{g,m}(M, J, A)$ are compact and Hausdorff.*

4 Talks on the seminar

Here we list the talks in the seminar (other than the one this document is for). These first four are already assigned to master's students and are more about general culture. References have already been provided for those in another document.

- Sheaves and sheaf cohomology. The DeRham and Dolbeaut theorems.
- Introduction to complex manifolds. Line bundles and divisors.
- Elliptic operators on closed manifolds. Hodge decomposition theorems.
- Introduction to symplectic manifolds. Stability and non-squeezing.

We now list the talks that tackle the foundations of Gromov-Witten theory. Differential geometric approach:

- **Gromov's compactness theorem.** This talk should introduce more precisely what the moduli space of stable pseudo-holomorphic curves is, along with its topology, and sketch proof of the compactness theorem. To provide an idea of neighbourhoods of boundary strata some rough ideas on gluing may be needed.

  Ref: [13] (genus 0 via bubbling analysis; to be combined with knowledge of Deligne-Mumford space), [8] (any genus via hyperbolic geometry).

- **The Sard-Smale theorem and transversality arguments.** This talk should introduce the infinite-dimensional set up to describe moduli spaces and exemplify transversality arguments (for example for simple curves) to understand the flavor of it all.

  Ref: [13] or [17].

- **Inhomogeneous perturbations and the Gromov-Witten pseudocycle.** The goal of this talk is to provide a rigorous definition of Gromov-Witten invariants on semipositive symplectic manifolds. For this one explains how inhomogeneous perturbations achieve transversality and how semi-positivity helps us show that a certain map is a pseudo-cycle. We cite the original papers.

  Ref: [15] (main), [14].

Algebraic geometric approach, an important disclaimer is that I do not understand these topics so what I say and cite is subject to my ignorance:

- **Moduli spaces in algebraic geometry.** The goal of this talk is to provide an overview of the philosophy of moduli spaces in algebraic geometry. Here is what *I* imagine could be said: coarse/fine moduli spaces, moduli functors and representability, moduli stacks, deformations and obstructions... The references for this are hard to give since it is quite broad, but here are some ideas:

Ref: Huybrechts chapter 6 for deformation of complex structures, Pardon's notes "Pseudo-holomorphic curves and virtual fundamental cycles" are great to apply whatever is said here to our situations, found this course of Vakil that seems complete with nice slides, https://math.stanford.edu/ vakil/22-245moduli/. I do not know what else or how okay this is...

- **Deformation theory of rational curves.** One should introduce rational curves (or more generally algebraic curves, over $\mathbb{C}$, of course), discuss their basic features and explain how one deals with their deformation theory.

  Ref: [1] (short), [2] (almost encyclopedic).

- **Virtual fundamental class.** The goal of this talk(s) is to introduce the construction of the virtual fundamental class. There seem to be two main equivalent constructions for our purposses but one seems to be preferred. It seems to be this requires maturity in algebraic geometry.

  Ref: [3] (prefered), [11] (secondary).

- **Localization of virtual classes.** It seems that the most successful way to compute arbitrary genus Gromov-Witten invariants consists of variations of the Atiyah-Bott localization formula. We give the main reference for this here and, again, while it is very well written, it requires maturity and having absorbed the above.

  Ref: [6].

The algebraic talks and items listed above need not be in one-to-one correspondance. Perhaps the deformation theory of rational curves can be absorbed into another talk and/or the virtual fundamental class can be two talks since it is so central (and interesting that it *can* be done from the algebraic perspective).

Last talks:

- **WDVV equations and the Kontsevich recursion formula.** Finally, in this talk we shall see some of the nice applications of our efforts. This is mostly in the land of genus 0 invariants.

  Ref: [12] (original), [9] (algebraic, well explained), [13] (analytical, a bit technical but very rigorous).

- **Gromov-Witten theory according to physics.** We have our friendly Neighborhood theoretical physicist willing to give this talk so I will not dare say anything about it.

Needless to say, but saying it just in case, that the goal is that differential and algebraic geometers understand each other. Therefore, the speakers should raise above the usual technical jargon in the field and do a good communication job.

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