PROBLEM SET 11
To be discussed: 17.01.2024

Instructions

This homework will not be collected or graded, but it is highly advisable to at least think through all of the problems before the next week's lectures, as the problems will often serve as mental preparation for the lecture material. Solutions will be discussed in the Übung.

1. A long time ago, I promised you a homework problem showing that for every simplicial complex $K = (V, S)$, the canonical chain map $\Phi : C^o_*(K) \to C^\Delta_*(K)$ sending ordered n-simplices $(v_0, \dots, v_n)$ to oriented n-simplices $[v_0, \dots, v_n]$ is a chain homotopy equivalence, thus proving that the two versions of simplicial homology are naturally isomorphic. This is that problem.

   Notice that for any subcomplex $L \subset K$, $\Phi(C^o_*(L)) \subset C^\Delta_*(L))$. This can be interpreted as a form of *naturality* if we view C^o_* and C^Δ_* as functors on the category of subcomplexes of K, with morphisms $L \to L'$ defined by inclusion: indeed, any nested pair of subcomplexes $L \subset L' \subset K$ gives rise to a commutative diagram

$$\begin{array}{ccc} C^o_*(L) & \xrightarrow{\Phi} & C^\Delta_*(L) \\ \downarrow & & \downarrow \\ C^o_*(L') & \xrightarrow{\Phi} & C^\Delta_*(L'), \end{array}$$

   where the two vertical maps are the chain maps induced by the inclusion $L \hookrightarrow L'$. We will now use the method of acyclic models to find a chain homotopy inverse $\Psi : C^\Delta_*(K) \to C^o_*(K)$ for Φ.

   As a preliminary step, we need to introduce a *reduced* version of simplicial homology. The definition should seem familiar: assume P is a simplicial complex with only one vertex, let $\epsilon : K \to P$ denote the unique simplicial map, and define

$$\widetilde{H}^o_*(K) := \ker\left(H^o_*(K) \xrightarrow{\epsilon_*} H^o_*(P)\right), \qquad \widetilde{H}^\Delta_*(K) := \ker\left(H^\Delta_*(K) \xrightarrow{\epsilon_*} H^\Delta_*(P)\right).$$

   (a) Prove that for any coefficient group G,

$$H^o_n(K;G) \cong \begin{cases} \widetilde{H}^o_n(K;G) \oplus G & \text{if } n = 0, \\ \widetilde{H}^o_n(K;G) & \text{if } n \ne 0, \end{cases}$$

   and that the analogous relation between $\widetilde{H}^\Delta_*(K;G)$ and $H^\Delta_*(K;G)$ also holds.

   (b) Show that $\widetilde{H}^o_*(K;G)$ is also the homology of an *augmented* chain complex $\widetilde{C}^o_*(K;G)$ of the form

$$\ldots \longrightarrow C^o_2(K;G) \xrightarrow{\partial} C^o_1(K;G) \xrightarrow{\partial} C^o_0(K;G) \xrightarrow{\epsilon} G \longrightarrow 0 \longrightarrow 0 \longrightarrow \ldots,$$

   i.e. $\widetilde{C}^o_n(K;G) = C^o_n(K;G)$ for all $n \ne -1$ but $\widetilde{C}^o_{-1}(K;G) = G$. Describe the *augmentation* map $\epsilon : C^o_0(K;G) \to G$ in this complex explicitly, and show that the analogous statement also holds for $\widetilde{H}^\Delta_*(K;G)$.

   We next define a simplicial analogue of the *cone* of a topological space. Let $CK = (CV, CS)$ denote the simplicial complex whose vertex set CV is the union of V with one extra element $v_\infty \notin V$, and whose simplices consist of all sets of the form $\sigma \cup \{v_\infty\}$ for $\sigma \in S$, plus all their subsets. It is not hard to show that the polyhedron $|CK|$ is then homeomorphic to the cone of $|K|$, thus it is contractible, and the isomorphism $H^\Delta_*(CK) \cong H_*(|CK|)$ implies $\widetilde{H}^\Delta_*(CK) = 0$. But this does not immediately imply $\widetilde{H}^o_*(CK) = 0$, since we haven't yet proved H^o_* and H^Δ_* are isomorphic.

(c) For integers $n \geqslant 0$, consider the homomorphism $h : C_n^o(CK) \to C_{n+1}^o(CK)$ determined by the formula $h(v_0, \dots, v_n) := (v_\infty, v_0, \dots, v_n)$. Find a definition of $h : \mathbb{Z} = \widetilde{C}_{-1}^o(CK;\mathbb{Z}) \to C_0^o(CK;\mathbb{Z})$ that makes $\widetilde{C}_*^o(CK;\mathbb{Z}) \xrightarrow{h} \widetilde{C}_{*+1}^o(CK;\mathbb{Z})$ into a chain homotopy between the chain maps $\mathbb{1} : \widetilde{C}_*^o(CK;\mathbb{Z}) \to \widetilde{C}_*^o(CK;\mathbb{Z})$ and $0 : \widetilde{C}_*^o(CK;\mathbb{Z}) \to \widetilde{C}_*^o(CK;\mathbb{Z})$, and deduce that $\widetilde{H}_*^o(CK;\mathbb{Z}) = 0$.

(d) For a given simplicial complex K, let us say that a chain map $\Psi : C_*^\Delta(K;\mathbb{Z}) \to C_*^o(K;\mathbb{Z})$ is *natural* if $C_0^\Delta(K;\mathbb{Z}) \xrightarrow{\Psi} C_0^o(K;\mathbb{Z})$ is determined by the formula $\Psi[v] := (v)$ and for every subcomplex $L \subset K$, Ψ sends $C_*^\Delta(L;\mathbb{Z})$ into $C_*^o(L;\mathbb{Z})$. Use the method of acyclic models to prove that a natural chain map $\Psi : C_*^\Delta(K;\mathbb{Z}) \to C_*^o(K;\mathbb{Z})$ exists and is unique up to chain homotopy.
Hint: Proceed by induction on the degree n, and construct Ψ first on "model" subcomplexes $L \subset K$ that consist of a single n-simplex and all its faces.

If you've gotten this far, then you can probably guess how the rest of the proof that $H_*^\Delta(K) \cong H_*^o(K)$ goes: one must similarly show the uniqueness up to chain homotopy of natural chain maps $C_*^o(K;\mathbb{Z}) \to C_*^\Delta(K;\mathbb{Z})$, $C_*^o(K;\mathbb{Z}) \to C_*^o(K;\mathbb{Z})$ and $C_*^\Delta(K;\mathbb{Z}) \to C_*^\Delta(K;\mathbb{Z})$, which would imply that $\Phi \circ \Psi$ and $\Psi \circ \Phi$ are each chain homotopic to the identity. Applying the functor $\otimes G$ then extends this result to an arbitrary coefficient group G. I suggest you work out the remaining details the next time you get bored on a long train ride.

2. For inverse systems over an arbitrary directed set $(I, \prec)$, prove each of the following:

(a) For any inverse system $\{X_\alpha, \varphi_{\alpha\beta}\}$ of topological spaces, abelian groups or chain complexes,

$$\varprojlim\{X_\alpha\} \cong \left\{ \{x_\alpha\}_{\alpha\in I} \in \prod_{\alpha\in I} X_\alpha \;\middle|\; \varphi_{\alpha\beta}(x_\beta) = x_\alpha \text{ for all } \beta \succ \alpha \right\},$$

with the associated maps $\varphi_\alpha : \varprojlim\{X_\beta\} \to X_\alpha$ defined for each $\alpha \in I$ as restrictions of the natural projections $\prod_\beta X_\beta \to X_\alpha$. Show moreover that in the case of topological spaces, the topology on $\varprojlim\{X_\alpha\}$ is the weakest one for which the maps φ_α are all continuous. In the case of chain complexes, how is the boundary operator on $\varprojlim\{X_\alpha\}$ defined?

(b) Consider the special case of part (a) in which the X_α are topological spaces that are all subsets (with the subspace topology) of some fixed space X, $\beta \succ \alpha$ if and only if $X_\beta \subset X_\alpha$ and the maps $\varphi_{\alpha\beta} : X_\beta \to X_\alpha$ are the natural inclusions. Show that $\bigcap_\alpha X_\alpha$ with the natural inclusions $\varphi_\alpha : \bigcap_\beta X_\beta \hookrightarrow X_\alpha$ defines an inverse limit of the system.

(c) Prove that for any inverse system $\{X_\alpha, \varphi_{\alpha\beta}\}$ of nonempty compact Hausdorff spaces, $\varprojlim\{X_\alpha\}$ is a nonempty space.
Hint: This depends on Tychonoff's theorem, which implies that $\prod_{\alpha\in I} X_\alpha$ in this case is a compact space. One possible approach is to construct a net $\{x^\beta \in \prod_\alpha X_\alpha\}_{\beta\in I}$ such that for each $\beta \in I$, the coordinates $x_\alpha^\beta \in X_\alpha$ of x^β satisfy $x_\alpha^\beta = \varphi_{\alpha\beta}(x_\beta^\beta)$ for every $\alpha \prec \beta$. One can then prove that every cluster point of this net is an element of $\varprojlim\{X_\alpha\}$.

(d) A subset $I_0 \subset I$ is called a **cofinal set** if for every $\alpha \in I$ there exists some $\beta \in I_0$ such that $\beta \succ \alpha$. Suppose $\{X_\alpha, \varphi_{\alpha\beta}\}$ is an inverse system over $(I, \prec)$ in any category, and $I_0 \subset I$ is a cofinal set with the property that for every $\alpha, \beta \in I_0$ with $\alpha \prec \beta$, $\varphi_{\alpha\beta} \in \mathrm{Mor}(X_\beta, X_\alpha)$ is an isomorphism. Prove that $\varprojlim\{X_\alpha\}$ is then isomorphic to X_γ for any $\gamma \in I_0$, and describe the associated morphisms $\varprojlim\{X_\beta\} \xrightarrow{\varphi_\alpha} X_\alpha$ for every $\alpha \in I$.
Advice: If universal properties in arbitrary categories make your head spin too much, feel free to work specifically in Top, Ab *or* Chain, *so that you can use the result of part (a).*

3. Can you think of an example of a path-connected space X for which $\check{H}_1(X;\mathbb{Z}_2) = 0 \neq H_1(X;\mathbb{Z}_2)$?
Hint: Take the suspension of something that is connected but not path-connected.

4. Find an example of a compact space X that is connected but not path-connected and is the inverse limit of a system $\{X_\alpha\}$ of path-connected spaces, and conclude: $H_*\big(\varprojlim\{X_\alpha\}\big) \not\cong \varprojlim\{H_*(X_\alpha)\}$. In other words, the singular homology functor is not *continuous* with respect to inverse limits.
Comment: In contrast, Čech homology is continuous on compact pairs. This is one of its selling points.