PROBLEM SET 4
To be discussed: 15.11.2023

Instructions

This homework will not be collected or graded, but it is highly advisable to at least think through all of the problems before the next week's lectures, as the problems will often serve as mental preparation for the lecture material. Solutions will be discussed in the Übung.

1. Assume $h_* : \mathsf{Top}_{\mathrm{rel}} \to \mathsf{Ab}_{\mathbb{Z}}$ is a functor satisfying all of the Eilenberg-Steenrod axioms for homology theories except possibly the additivity axiom. Given two spaces X and Y, use excision and the long exact sequences of the pairs $(X \amalg Y, X)$ and $(X \amalg Y, Y)$ to prove that for the natural inclusions $i^X : X \hookrightarrow X \amalg Y$ and $i^Y : Y \hookrightarrow X \amalg Y$, the map

$$i^X_* \oplus i^Y_* : h_*(X) \oplus h_*(Y) \to h_*(X \amalg Y) : (x, y) \mapsto i^X_* x + i^Y_* y$$

   is an isomorphism. Deduce that h_* does satisfy the additivity axiom for all *finite* disjoint unions.

2. Given a collection of pairs of spaces $\{(X_\alpha, A_\alpha)\}_{\alpha \in J}$, consider the pair

$$\coprod_{\alpha \in J} (X_\alpha, A_\alpha) := \left(\coprod_{\alpha \in J} X_\alpha, \coprod_{\alpha \in J} A_\alpha \right)$$

   with the natural inclusion maps $i^\alpha : (X_\alpha, A_\alpha) \hookrightarrow \coprod_{\beta \in J} (X_\beta, A_\beta)$. Prove that for any axiomatic homology theory h_*, the additivity axiom generalizes to pairs, producing an isomorphism

$$\bigoplus_{\alpha \in J} i^\alpha_* : \bigoplus_{\alpha \in J} h_*(X_\alpha, A_\alpha) \xrightarrow{\cong} h_*\Big(\coprod_{\alpha \in J} (X_\alpha, A_\alpha) \Big).$$

3. According to Hatcher, a *good* pair (X, A) is one for which the subset $A \subset X$ is closed and is a deformation retract of some neighborhood $V \subset X$ of itself. Show that the pair (X, A) with $X := [0, 1]$ and $A := \{1, 1/2, 1/3, 1/4, \ldots, 0\}$ is not good, and compare $H_1(X, A; \mathbb{Z})$ with $H_1(X/A; \mathbb{Z})$.
   Hint: X/A happens to be homeomorphic to a standard pathological example that you may have seen in Topology 1—it resembles an infinite wedge sum of circles, but has a much larger fundamental group.

4. Prove that the connecting homomorphism $\partial_* : H_n(X) \to H_{n-1}(A \cap B)$ in the Mayer-Vietoris sequence in singular homology of a space $X = \mathring{A} \cup \mathring{B}$ is given by the explicit formula

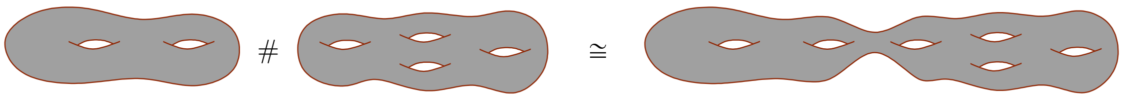
$$\partial_*[a + b] = [\partial a] \qquad \text{for } a \in C_n(A),\ b \in C_n(B).$$

   Use this to verify directly that the Mayer-Vietoris sequence is exact.

5. Use Mayer-Vietoris sequences to compute $H_*(X; \mathbb{Z})$ and $H_*(X; \mathbb{Z}_2)$, where X is
   (a) The projective plane $\mathbb{RP}^2$.
   (b) The Klein bottle.

   Hint: $\mathbb{RP}^2$ is the union of a disk with a Möbius band, and the latter admits a deformation retraction to S^1. The Klein bottle, in turn, is the union of two Möbius bands, also known as $\mathbb{RP}^2 \# \mathbb{RP}^2$.

6. Recall that given two connected topological n-manifolds X and Y, their **connected sum** $X \# Y$ is defined by deleting an open n-disk $\mathring{\mathbb{D}}^n$ from each of X and Y and then gluing $X \backslash \mathring{\mathbb{D}}^n$ and $Y \backslash \mathring{\mathbb{D}}^n$ together along an identification of their boundary spheres:



More precisely, we can choose topological embeddings $\iota_X : \mathbb{D}^n \hookrightarrow X$, $\iota_Y : \mathbb{D}^n \hookrightarrow Y$ of the closed unit n-disk $\mathbb{D}^n \subset \mathbb{R}^n$ and then define

$$X \# Y := \left(X \backslash \iota_X(\mathring{\mathbb{D}}^n) \right) \cup_{S^{n-1}} \left(Y \backslash \iota_Y(\mathring{\mathbb{D}}^n) \right),$$

where the gluing identifies the boundaries of both pieces in the obvious way with $S^{n-1} = \partial \mathbb{D}^n$. There are one or two subtle issues about the extent to which $X \# Y$ is (up to homeomorphism) independent of choices, e.g. in general this need not be true without an extra condition involving orientations, but don't worry about this for now. Last semester (see Problem Set 6 #3) we used the Seifert-van Kampen theorem to show that $\pi_1(X \# Y) \cong \pi_1(X) * \pi_1(Y)$ whenever $n \geqslant 3$. We can now use the Mayer-Vietoris sequence to derive a similar formula for the homology of a connected sum.

(a) Prove that for any $k = 1, \ldots, n-2$ and any coefficient group, $H_k(X \# Y) \cong H_k(X) \oplus H_k(Y)$. *Hint: There are two steps, as you first need to derive a relation between $H_k(X)$ and $H_k(X \backslash \mathring{\mathbb{D}}^n)$, and then see what happens when you glue $X \backslash \mathring{\mathbb{D}}^n$ and $Y \backslash \mathring{\mathbb{D}}^n$ together.*

(b) It turns out that the formula $H_{n-1}(X \# Y; \mathbb{Z}) \cong H_{n-1}(X; \mathbb{Z}) \oplus H_{n-1}(Y; \mathbb{Z})$ also holds if X and Y are both closed orientable n-manifolds with $n \geqslant 2$, and without orientability we still have $H_{n-1}(X \# Y; \mathbb{Z}_2) \cong H_{n-1}(X; \mathbb{Z}_2) \oplus H_{n-1}(Y; \mathbb{Z}_2)$. Prove this under the additional assumption that $X \backslash \mathring{\mathbb{D}}^n$ and $Y \backslash \mathring{\mathbb{D}}^n$ both admit (possibly oriented) triangulations.

(c) Find a counterexample to the formula $H_1(X \# Y; \mathbb{Z}) \cong H_1(X; \mathbb{Z}) \oplus H_1(Y; \mathbb{Z})$ where X and Y are both closed (but not necessarily orientable) 2-manifolds.