A BEGINNER'S OVERVIEW OF SYMPLECTIC HOMOLOGY

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Contents

My intention in these notes is to give an introductory overview of symplectic homology, including its historical origins, the main ideas behind it and a very brief sampling of applications and recent developments. This must necessarily begin with the disclaimer that the subject is almost as new to me as it is to my intended reader, perhaps even more so—I am not an expert, and you should not assume that everything I say is rooted in any deep understanding. If there's any advantage at all to this, perhaps it is that my perspective is still relatively unbiased (cf. [Sei]).

The second disclaimer involves signs: one notices quickly in surveying the literature that everyone has slightly different sign preferences on basic issues such as the definition of the standard symplectic form on $\mathbb{R}^{2n}$, or of a Hamiltonian vector field. In symplectic homology, these differences propagate to the point where one ends up often unsure whether one is talking about homology or cohomology, direct limits or inverse limits, arrows pointing to the right or to the left. For this exposition I've chosen to adopt the conventions used in [BO09] and *attempted* to maintain consistency throughout, but I make no guarantees.

With that out of the way, let us first make the point that there is not a *single* theory known as symplectic homology—there are several, which all have certain features in common. To attempt a unified definition, symplectic homology generally refers to the adaptation of Hamiltonian Floer homology into symplectic manifolds that are not closed; in fact symplectic homology is often referred to in the literature simply as "Floer (co)homology" in a specific setting. While Floer homology admits a more or less canonical definition in closed symplectic manifolds, this ceases to be true on open manifolds or manifolds with boundary, so that the exact details of the definition are highly dependent on the context and the intended applications. In practice, the version of symplectic homology that has received the most attention in recent years is a theory introduced by Viterbo in [Vit99], who simply called it "Floer homology on symplectic manifolds with contact type boundary".

Viterbo's theory is actually not an invariant of a compact manifold, but rather of the noncompact *completion* obtained by attaching a cylindrical end to a contact type boundary. In this sense it is considered *qualitative*: it doesn't detect any of the numerical parameters that can be defined e.g. for compact convex domains in $\mathbb{R}^{2n}$ but not for $\mathbb{R}^{2n}$ itself, though it can be applied to great effect for detecting symplectomorphism types or proving the existence of periodic orbits. The *original* symplectic homology, introduced some years beforehand in several papers by Floer, Hofer, Cieliebak and Wysocki [FH94, CFH95, CFH95, CFHW96] was developed with specifically *quantitative* applications in mind, such as classifying compact domains in $\mathbb{R}^{2n}$ up to symplectomorphism, and defining symplectic capacities.

In the following, we will begin by reviewing the main ideas of Hamiltonian Floer homology in the closed case, with an attempt to highlight details that will differ significantly in the generalization. We will then take a very brief look at the quantitative version of symplectic homology, its motivations and applications. In §3, we prepare the ground for more recent developments by reviewing the basic facts about symplectic manifolds with contact type boundary, so that we can describe the definition of Viterbo's theory and a smattering of its applications in §4.

A great deal of what follows is adapted from two excellent survey articles on symplectic homology, namely by Alexandru Oancea [Oan04] and Paul Seidel [Sei]. A nice introduction specifically to the quantitative theory can also be found in [HZ94, §6.6]. Naturally, they all use different sign conventions.

1. Review of Floer homology in the closed case

Both the following exposition and §4 are modeled largely on the summary of symplectic homology presented in [BO09, §2], though we will also borrow some details from [Sal99].

For this section, assume (W, ω) is a closed symplectic manifold of dimension $2n$, satisfying whatever assumption is required to avoid troubles with bubbling holomorphic spheres, e.g. at minimum (W, ω) should be semipositive (cf. [MS04]). In many cases of interest one may assume that (W, ω) is symplectically aspherical ($\omega|_{\pi_2(W)} = 0$), or even (in the non-closed case) that ω is exact. Depending how comfortable you are with virtual cycle techniques or abstract perturbations, you may or may not believe that these restrictions are removable.

For a given free homotopy class of loops

$$h \in [S^1, W],$$

we will define the Floer homology generated by 1-periodic Hamiltonian orbits in the homotopy class h. Floer homology is of course based on the symplectic action functional, which in the case of an exact symplectic form $\omega = d\lambda$, can be defined on the loop space $C^\infty(S^1, W)$ by

$$\mathcal{A}_H(\gamma) = -\int_{S^1} \Big(\gamma^*\lambda + H(t, \gamma(t))\, dt\Big).$$

Since we cannot actually assume ω to be exact and would rather avoid placing additional topological restrictions on (W, ω), the question of how to define $\mathcal{A}_H$ more generally is a bit delicate and requires Novikov rings. We shall abbreviate

$$H_2(W) := H_2(W; \mathbb{Z})/\text{torsion},$$

and for any given subgroup

$$\mathcal{R} \subset H_2(W),$$

define the *group ring* $\mathbb{Z}[H_2(W)/\mathcal{R}]$ to consist of all finite sums of the form

$$\sum_i c_i e^{A_i}$$

with $c_i \in \mathbb{Z}$ and $A_i \in H_2(W)/\mathcal{R}$, where multiplication is defined via the relation $e^A e^B := e^{A+B}$.

For any $A \in H_2(W)$, denote by

$$\omega(A),\ c_1(A) \in \mathbb{Z}$$

the evaluation of the cohomology classes $[\omega]$ and $c_1(TW)$ respectively on A, where $c_1(TW)$ is defined via any ω-compatible complex structure on TW. In order to define a suitable generalization of $\mathcal{A}_H$ that yields the required energy bounds for Floer trajectories, it will be essential to assume

$$\mathcal{R} \subset \ker \omega, \tag{1.1}$$

so ω descends to a homomorphism on the quotient $H_2(W)/\mathcal{R} \to \mathbb{R}$. One can then define the *Novikov ring*

$$\Lambda_\omega := \Lambda_\omega \left(\mathbb{Z}[H_2(W)/\mathcal{R}]\right)$$

as the completion of $\mathbb{Z}[H_2(W)/\mathcal{R}]$ obtained by including all infinite formal sums of the form

$$\sum_{i=1}^{\infty} c_i e^{A_i}$$

such that $\omega(A_i) \to +\infty$. Equivalently, Λ_ω is the ring of all formal sums $\sum_{i=1}^{\infty} c_i e^{A_i}$ such that for every $C \in \mathbb{R}$, the set $\{i \in \mathbb{N} \mid c_i \neq 0,\ \omega(A_i) \leq C\}$ is finite.

Note that in general, there may be some freedom in choosing $\mathcal{R}$: in the best case scenario (which will only be relevant when W is not closed), ω is exact, so we can choose $\mathcal{R} = \ker \omega = H_2(W)$, thus making $H_2(W)/\mathcal{R}$ trivial and forgetting the Novikov ring altogether. Alternatively, one can always take $\mathcal{R}$ to be the trivial subgroup and thus define Λ_ω as a completion of the full group ring $\mathbb{Z}[H_2(W)]$.

A less essential but convenient extra restriction to place on $\mathcal{R}$ is

$$\mathcal{R} \subset \ker c_1(TW), \tag{1.2}$$

so that $c_1(TW)$ also defines a homomorphism $H_2(W)/\mathcal{R} \to \mathbb{R}$. This condition will allow us to define an integer grading on Floer homology; without it we would have to settle for a $\mathbb{Z}_{2N}$-grading for some $N \in \mathbb{N}$. That is not the end of the world, but for convenience we will always assume that both (1.1) and (1.2) are satisfied.

Remark 1.1. Several variations on the above setup are possible. Many authors impose the condition

$$\int_{T^2} f^*\omega = 0 \qquad \text{for all } f : T^2 \to W,$$

which implies symplectic asphericity since there exists a map $T^2 \to S^2$ of positive degree. Whenever this condition holds, one can slightly modify the definitions we will give (see Remark 1.2) so that the action functional is well defined on the loop space (instead of a cover of the loop space), and the energy of a Floer trajectory depends only on its end points. For the case of contractible orbits $h = 0$, all this follows already from the weaker condition $\omega|_{\pi_2(W)} = 0$. Then Floer homology can be defined without using the Novikov ring, though it will not generally have an integer grading unless the group ring is included. The latter is roughly the approach taken in [BO09].

Note also that for the case $h = 0$, one can modify the definitions so as to replace $H_2(W)$ with the subgroup of *spherical* homology classes, i.e. the image of $\pi_2(W)$ under the Hurewicz homomorphism. This is done in [Sal99].

Given a smooth function $H : S^1 \times W \to \mathbb{R}$, we'll denote $H_t := H(t, \cdot) : W \to \mathbb{R}$ for $t \in S^1$, and define the corresponding time-dependent Hamiltonian vector field X_{H_t} on W by

$$\omega(X_{H_t}, \cdot) = dH_t. \tag{1.3}$$

Fix a "reference loop"

$$\ell_h : S^1 \to W$$

with $[\ell_h] = h$, and denote by $\mathcal{P}^h(H)$ the set of all 1-periodic orbits of X_{H_t} in the homotopy class h. For the important special case $h = 0$, we'll use $\mathcal{P}(H)$ to denote the set of all *contractible* orbits, and assume $\ell_0 : S^1 \to W$ is a constant map. Let $J = \{J_t\}_{t\in S^1}$ denote a smooth family of ω-compatible almost complex structures on W, so $g_t := \omega(\cdot, J_t\cdot)$ is a smooth family of Riemannian metrics, and defining the gradient vector field ∇H_t at time t with respect to this metric gives

$$X_{H_t} = -J_t \nabla H_t.$$

In the following we will always assume H is chosen so that all orbits in $\mathcal{P}^h(H)$ are nondegenerate, and the family $\{J_t\}_{t\in S^1}$ is chosen generically. We will refer to pairs (H, J) with these properties as *generic pairs.*

Denote by $\widetilde{C}^\infty(S^1, W)$ the set of all pairs

$$\tilde{\gamma} = (\gamma, [\sigma]),$$

where $\gamma \in C^\infty(S^1, W)$ and $[\sigma]$ is an equivalence class of smooth maps

$$\sigma : \Sigma \to W,$$

with Σ a compact oriented surface with two oriented boundary components $\partial\Sigma = \partial_1\Sigma \cup (-\partial_0\Sigma)$, $\sigma|_{\partial_1\Sigma} = \gamma$ and $\sigma|_{\partial_0\Sigma} = \ell_{[\gamma]}$, and we define

$$\sigma \sim \sigma' \quad \Leftrightarrow \quad [\sigma] - [\sigma'] = 0 \in H_2(W)/\mathcal{R}.$$

One can think of $\widetilde{C}^\infty(S^1, W)$ informally as an infinite dimensional manifold that is a covering of the loop space $C^\infty(S^1, W)$ and thus has tangent space $\Gamma(\gamma^* TW)$ at $\tilde{\gamma} = (\gamma, [\sigma])$. We now define the symplectic action functional

$$\mathcal{A}_H : \widetilde{C}^\infty(S^1, W) \to \mathbb{R} : (\gamma, [\sigma]) \mapsto -\int_\Sigma \sigma^*\omega - \int_{S^1} H(t, \gamma(t))\, dt,$$

whose linearization at $\tilde{\gamma} = (\gamma, [\sigma])$ is

$$d\mathcal{A}_H(\tilde{\gamma})\eta = \int_{S^1} \omega(\dot{\gamma} - X_{H_t}(\gamma), \eta)\, dt.$$

The critical points of $\mathcal{A}_H$ are thus the pairs $(\gamma, [\sigma])$ for which γ is a 1-periodic orbit; we shall denote these by

$$\widetilde{\mathcal{P}}^h(H) = \{(\gamma, [\sigma]) \in \mathrm{Crit}(\mathcal{A}_H) \mid [\gamma] = h\}.$$

Remark 1.2. It often makes sense to modify the definition of $\widetilde{C}^\infty(S^1, W)$ so that the maps σ are simply homotopies $[0,1] \times S^1 \to W$ between $\ell_{[\gamma]}$ and γ, or for the case of contractible loops, maps $\mathbb{D} \to W$ with $\sigma|_{\partial\mathbb{D}} = \gamma$. The latter in particular would allow us to replace $\mathbb{Z}[H_2(W)]$ with $\mathbb{Z}[\pi_2(W)]$, where $\pi_2(W)$ is identified with its image under the Hurewicz homomorphism.

Observe that there is a natural action of $H_2(W)/\mathcal{R}$ on $\widetilde{C}^\infty(S^1, W)$ which preserves $\mathrm{Crit}(\mathcal{A}_H)$: indeed, for $A \in H_2(W)/\mathcal{R}$ and $\tilde{\gamma} = (\gamma, [\sigma]) \in \widetilde{C}^\infty(S^1, W)$, we define

$$A \cdot \tilde{\gamma} = (\gamma, A + [\sigma]),$$

with $A + [\sigma]$ understood to mean any map in the correct relative homology class. We then have

$$\mathcal{A}_H(A \cdot \tilde{\gamma}) = \mathcal{A}_H(\tilde{\gamma}) - \omega(A). \tag{1.4}$$

The Floer chain complex can be defined using a similar finiteness condition as with the Novikov ring: let $FC_*^h(H)$ denote the additive abelian group consisting of all formal sums $\sum_{\tilde{\gamma} \in \widetilde{\mathcal{P}}^h(H)} c_{\tilde{\gamma}} \langle \tilde{\gamma} \rangle$ with $c_{\tilde{\gamma}} \in \mathbb{Z}$, such that for every $C \in \mathbb{R}$, the set

$$\{\tilde{\gamma} \in \widetilde{\mathcal{P}}^h(H) \mid c_{\tilde{\gamma}} \neq 0,\ \mathcal{A}_H(\tilde{\gamma}) \geq C\}$$

is finite. Such sums are necessarily countable, and it will be convenient to write them as

$$\sum_i c_i e^{[\sigma_i]} \langle \gamma_i \rangle,$$

where $c_i \in \mathbb{Z}$ and $e^{[\sigma]}\langle \gamma \rangle$ is alternative notation for $\langle (\gamma, [\sigma]) \rangle$. The obvious multiplication $e^A e^{[\sigma]} := e^{A + [\sigma]}$ now gives $FC_*^h(H)$ the structure of a Λ_ω-module, with the required finiteness condition satisfied due to (1.4).

To define a grading on $FC_*^h(H)$, we must choose a symplectic trivialization of TW along the reference loop ℓ_h. Note that this choice is arbitrary and the grading will generally depend on it, except in the special case $h = 0$ where it is natural to choose the constant trivialization. For any $\tilde{\gamma} = (\gamma, [\sigma]) \in \widetilde{\mathcal{P}}^h(H)$, the trivialization extends along σ to a unique (up to homotopy) trivialization along γ, which we can use to define the Conley-Zehnder index

$$\mu_{\mathrm{CZ}}(\tilde{\gamma}) \in \mathbb{Z}.$$

The action of $H_2(W)/\mathcal{R}$ affects $\mu_{\mathrm{CZ}}(\tilde{\gamma})$ by

$$\mu_{\mathrm{CZ}}(A \cdot \tilde{\gamma}) = \mu_{\mathrm{CZ}}(\tilde{\gamma}) + 2c_1(A). \tag{1.5}$$

Observe that here we're making use of the assumption $\mathcal{R} \subset \ker c_1(TW)$. Now for any $\tilde{\gamma} = (\gamma, [\sigma])$, we define the corresponding generator in $FC_*^h(H)$ to have degree

$$\left| e^{[\sigma]} \langle \gamma \rangle \right| = -\mu_{\mathrm{CZ}}(\tilde{\gamma}),$$

and we also assign degrees to the generators of Λ_ω by

$$|e^A| = -2c_1(A).$$

This notion of degree is then compatible with the action of Λ_ω on $FC_*^h(H)$ due to (1.5). We call elements of Λ_ω or $FC_*^h(H)$ *homogeneous* if they are (perhaps infinite) sums of generators of the same degree, and denote by $FC_k^h(H) \subset FC_*^h(H)$ the subgroup consisting of homogeneous elements of degree $k \in \mathbb{Z}$.

Remark 1.3. If we choose a fixed lift $\tilde{\gamma} = (\gamma, \sigma_\gamma) \in \widetilde{\mathcal{P}}^h(H)$ for every orbit $\gamma \in \mathcal{P}^h(H)$, then every generator $e^{[\sigma]}\langle\gamma\rangle$ can be written as $e^A e^{[\sigma_\gamma]}\langle\gamma\rangle$ for some unique $A \in H_2(W)/\mathcal{R}$, thus $FC_*^h(H)$ can be described as the "free Λ_ω-module generated by elements of $\mathcal{P}^h(H)$", as is done in [BO09]. One cannot however describe the grading quite so cleanly, as for instance it is not true that every element of $FC_*^h(H)$ can be written as a finite sum of homogenous elements, so $FC_*^h(H)$ is not technically the direct sum of the subgroups $FC_k^h(H)$ for all $k \in \mathbb{Z}$, but is rather a subgroup of its direct product.

Defining an L^2-product on the tangent spaces to $\widetilde{C}^\infty(S^1, W)$ via the metric $g_t = \omega(\cdot, J_t \cdot)$, the L^2-gradient of $\mathcal{A}_H$ is $\nabla \mathcal{A}_H(\gamma, [\sigma]) = J_t(\dot{\gamma} - X_{H_t}(\gamma))$, leading to the negative L^2-gradient flow equation for maps $u : \mathbb{R} \times S^1 \to W$,

$$\partial_s u + J_t(u)(\partial_t u - X_{H_t}(u)) = 0, \tag{1.6}$$

also known as the *Floer equation.* For any elements $\tilde{\gamma} = (\gamma, [\sigma]), \tilde{\gamma}' = (\gamma', [\sigma']) \in \widetilde{\mathcal{P}}^h(H)$ and $A \in H_2(W)/\mathcal{R}$, we define the moduli space of Floer trajectories

$$\begin{aligned}\mathcal{M}^A(\tilde{\gamma}, \tilde{\gamma}'; H, J) = \{u : \mathbb{R} \times S^1 \to W \mid{} & u \text{ satisfies (1.6)}, \\ & \lim_{s \to -\infty} u(s, \cdot) = \gamma,\ \lim_{s \to +\infty} u(s, \cdot) = \gamma', \\ & [u] + [\sigma] - [\sigma'] = A \in H_2(W)/\mathcal{R}\},\end{aligned}$$

where $[u]$, $[\sigma]$ and $[\sigma']$ should be understood as relative homology classes up to addition with elements of $\mathcal{R}$. Since $\{J_t\}$ is generic, this space is a smooth manifold of dimension

$$\begin{aligned}\dim \mathcal{M}^A(\tilde{\gamma}, \tilde{\gamma}'; H, J) &= \mu_{\mathrm{CZ}}(\tilde{\gamma}') - \mu_{\mathrm{CZ}}(\tilde{\gamma}) + 2c_1(A) \\ &= \left| e^{[\sigma]}\langle\gamma\rangle \right| - \left| e^{A + [\sigma']}\langle\gamma'\rangle \right|.\end{aligned}$$

Assigning coherent orientations to these spaces as described in [FH93], the 1-dimensional components divided by the natural $\mathbb{R}$-translation can be counted with signs, leading to a Λ_ω-module homomorphism $\partial : FC^h_*(H) \to FC^h_{*-1}(H)$ of degree -1, defined via

$$\partial\left(e^{[\sigma]}\langle\gamma\rangle\right) = \sum_{\substack{\tilde{\gamma}',A \\ \dim \mathcal{M}^A(\tilde{\gamma},\tilde{\gamma};H,J)=1}} \#\left(\frac{\mathcal{M}^A(\tilde{\gamma},\tilde{\gamma}';H,J)}{\mathbb{R}}\right) e^{A+[\sigma']}\langle\gamma'\rangle.$$

Note that this definition depends on a compactness theorem: in order to show that $\partial\left(e^{[\sigma]}\langle\gamma\rangle\right)$ belongs to $FC^h_*(H)$, we need to know that for any $C \in \mathbb{R}$, there are only finitely many 1-dimensional components $\mathcal{M}^A(\tilde{\gamma},\tilde{\gamma}';H,J)$ for which $\omega(A) \le C$. The corresponding compactness result for 2-dimensional components, together with a corresponding gluing theorem, then implies the relation $\partial^2 = 0$, and we define the Floer homology

$$FH^h_*(H,J) = H_*\left(FC^h_*(H),\partial\right).$$

In the non-closed case, the formal elements of this construction will be the same but the technical details will differ at a few crucial points, thus it's worth taking a moment to reflect on these technical issues. There are essentially three ingredients that are crucial for proving the required compactness theorem:

(1) Solutions $u \in \mathcal{M}^A(\tilde{\gamma},\tilde{\gamma}';H,J)$ must satisfy an a priori C^0-bound.
(2) Solutions $u \in \mathcal{M}^A(\tilde{\gamma},\tilde{\gamma}';H,J)$ must satisfy a uniform bound on the *energy*

$$E(u) := \frac{1}{2}\int_{\mathbb{R}\times S^1}\left(|\partial_s u|^2 + |\partial_t u - X_{H_t}(u)|^2\right) ds \wedge dt, \tag{1.7}$$

where the norm at time t is always defined via the metric $g_t = \omega(\cdot, J_t\cdot)$.
(3) All possible holomorphic spheres that could bubble off must live in spaces of dimension at most $\dim \mathcal{M}^A(\tilde{\gamma},\tilde{\gamma}';H,J) - 2$.

The third condition is the reason we required (W,ω) to be semipositive from the beginning, and it will be a complete non-issue when we later consider exact symplectic forms, for which no holomorphic spheres exist at all. To see why the energy is bounded, we can imagine $u \in \mathcal{M}^A(\tilde{\gamma},\tilde{\gamma}';H,J)$ as a smooth path

$$\tilde{\gamma}(s) := (u(s,\cdot),\sigma_s) \in \widetilde{C}^\infty(S^1,W)$$

from $\tilde{\gamma}(-\infty) = \tilde{\gamma} = (\gamma,[\sigma])$ to $\tilde{\gamma}(+\infty) = A\cdot\tilde{\gamma}' = (\gamma', A+[\sigma'])$, where σ_s is defined by concatenating σ with the half-cylinder $u|_{(-\infty,s]\times S^1}$. Then by interpreting the Floer equation as the negative gradient flow of $\mathcal{A}_H$, we find

$$\begin{aligned}
\mathcal{A}_H(\tilde{\gamma}) - \mathcal{A}_H(A\cdot\tilde{\gamma}') &= -\int_{-\infty}^{\infty}\frac{d}{ds}\mathcal{A}_H(\tilde{\gamma}(s))\,ds \\
&= -\int_{\mathbb{R}}\langle\nabla\mathcal{A}_H(\tilde{\gamma}(s)),\partial_s\tilde{\gamma}(s)\rangle_{L^2}\,ds \\
&= \int_{\mathbb{R}}|\partial_s\tilde{\gamma}(s)|^2_{L^2}\,ds = \int_{\mathbb{R}\times S^1}|\partial_s u|^2\,ds\wedge dt \\
&= E(u).
\end{aligned} \tag{1.8}$$

The first condition is automatic since W is closed, but here we will have to be much more careful when we allow W to be noncompact.

Next, we recall why $FH_*^h(H, J)$ is actually a symplectic invariant independent of (H, J), a detail in which the non-closed case will differ quite substantially from the closed case. Given two generic pairs (H^-, J^-) and (H^+, J^+), one can make use of the fact that the space of ω-compatible almost complex structures is contractible (see e.g. [MS98]) and thus find a generic homotopy $\{(H^s, J^s)\}_{s\in\mathbb{R}}$ which satisfies $(H^s, J^s) = (H^-, J^-)$ for $s \le -1$ and $(H^s, J^s) = (H^+, J^+)$ for $s \ge 1$. This gives rise to the *s-dependent Floer equation*

$$\partial_s u + J_t^s(u)\left(\partial_t u - X_{H_t^s}(u)\right) = 0, \tag{1.9}$$

and corresponding moduli spaces

$$\mathcal{M}^A(\tilde{\gamma}, \tilde{\gamma}'; \{H^s\}, \{J^s\})$$

with $\tilde{\gamma} \in \mathcal{P}^h(H^-)$ and $\tilde{\gamma}' \in \mathcal{P}^h(H^+)$. Counting the solutions in 0-dimensional components then yields a so-called *continuation map*, which is a Λ_ω-module homomorphism of degree 0,

$$\Phi_{(H^s,J^s)} : FC_*^h(H^-) \to FC_*^h(H^+). \tag{1.10}$$

The appropriate compactness and gluing theorems for $\mathcal{M}^A(\tilde{\gamma}, \tilde{\gamma}'; \{H^s\}, \{J^s\})$ imply that this is not only well defined but is also a chain map, and by a "homotopy of homotopies" construction, one can similarly show that the resulting map on homology

$$\Phi_{(H^s,J^s)} : FH_*^h(H^-, J^-) \to FH_*^h(H^+, J^+)$$

doesn't depend on the choice of homotopy (H^s, J^s). Moreover, composition of homotopies gives rise to composition of maps on the homology, so for any three generic pairs (H^i, J^i) for $i = 0, 1, 2$ and generic homotopies (H^{ik}, J^{ik}) from (H^i, J^i) to (H^k, J^k), we have

$$\Phi_{(H^{02},J^{02})} = \Phi_{(H^{12},J^{12})} \circ \Phi_{(H^{01},J^{01})}. \tag{1.11}$$

Choosing a "constant" homotopy for any given generic pair (H, J), it is easy to see that the only 0-dimensional moduli spaces to count are the constant solutions of (1.6) fixed at each orbit, thus the continuation map from $FH_*^h(H, J)$ to itself is manifestly the identity, implying that all the continuation maps are isomorphisms. This argument proves:

Theorem 1.4. *For any two generic pairs (H^0, J^0) and (H^1, J^1), there is a canonical Λ_ω-module isomorphism of degree 0,*

$$\Phi_{01} : FH^h(H^0, J^0) \to FH^h(H^1, J^1),$$

and given a third pair (H^2, J^2), the corresponding isomorphisms satisfy the relation $\Phi_{02} = \Phi_{12} \circ \Phi_{01}$.

For this reason one can sensibly write $FH_*^h(H, J)$ as $FH_*^h(W, \omega)$ without explicitly mentioning F and J.

Proving these results requires once again establishing suitable C^0-bounds and energy bounds for the solutions in $\mathcal{M}^A(\tilde{\gamma}, \tilde{\gamma}'; \{H^s\}, \{J^s\})$, where energy is now defined by the obvious analogue of (1.7) with both $X_{H_t^s}$ and the norm depending on s (the latter via J_t^s). The computation of the energy bound

(1.8) now acquires an extra term due to the s-dependence of H: writing $H(s,t,\cdot) = H_t^s$, we find

$$\mathcal{A}_{H^-}(\tilde{\gamma}) - \mathcal{A}_{H^+}(A \cdot \tilde{\gamma}') = E(u) + \int_{[-1,1]\times S^1} \partial_s H(s,t,\gamma(t))\, ds \wedge dt. \tag{1.12}$$

The extra term is uniformly bounded since $|\partial_s H|$ is bounded, but notice that in saying this, we're using the assumption that W is compact. We'll later find it important for various reasons to assume $\partial_s H \geq 0$, thus obtaining continuation maps that go in one direction but not the other!

The well known main theorem about the computation of Floer homology applies to the special case $h = 0$, i.e. we consider only contractible orbits, and abbreviate $FH_*(H,J) := FH_*^0(H,J)$. Recall that the grading on $FH_*(H,J)$ is indepedent of choices in this case. The computation then follows by using the continuation isomorphism $FH_*(H,J) \to FH_*(H^0,J^0)$ for a very special choice of pair (H^0,J^0): we assume namely that both are time-independent, and H^0 is a Morse function C^2-close to zero. The only 1-periodic orbits of X_{H^0} are thus the constant orbits $\gamma_x(t) = x$ located at critical points $x \in \mathrm{Crit}(H^0)$, and their Conley-Zehnder indices are related to the Morse indices $\mathrm{ind}(x;H^0) \in \mathbb{Z}$ by

$$\mu_{\mathrm{CZ}}^{\Phi_0}(\gamma_x) = \mathrm{ind}(x;H^0) - n,$$

where the superscript Φ_0 means we use the obvious constant trivialization of TW along γ_x. The constant orbits γ_x have distinguished lifts $\tilde{\gamma}_x = (\gamma_x, [\sigma_x]) \in \widetilde{\mathcal{P}}(H^0)$ for which $[\sigma_x] = 0 \in H_2(W)$, hence $\mu_{\mathrm{CZ}}(\tilde{\gamma}_x) = \mu_{\mathrm{CZ}}^{\Phi_0}(\gamma_x) = \mathrm{ind}(x;H^0) - n$. Then each solution $v : \mathbb{R} \to W$ of the gradient flow equation $\dot{v} - \nabla H^0(v) = 0$, or equivalently the *negative* gradient flow equation for the Morse function $-H^0 : W \to \mathbb{R}$, gives rise to a time-independent solution to the Floer equation in the form $u(s,t) = v(s)$, and for generic J^0 one can show that all 1-dimensional moduli spaces $\mathcal{M}^A(\tilde{\gamma}_x, \tilde{\gamma}_y; H^0, J^0)$ contain *only* these solutions, for which $A = 0$. Since

$$\mathrm{ind}(x;-H^0) = 2n - \mathrm{ind}(x;H^0) = n - \mu_{\mathrm{CZ}}(\tilde{\gamma}_x) = n + \left| e^{[\sigma_x]}\langle \gamma_x \rangle \right|,$$

it follows that $FH_*(H^0,J^0)$ is simply the Morse homology of $-H^0 : W \to \mathbb{R}$ with coefficients in Λ_ω and with its grading adjusted by $-n$, so for every generic (H,J) we have a natural isomorphism to singular homology,[1]

$$FH_*(H,J) = H_{*+n}(W;\Lambda_\omega).$$

Observe now what happens if we apply the same argument for a nontrivial free homotopy class $h \neq 0$: since H^0 has no non-contractible 1-periodic orbits at all, $FC_*^h(H^0) = 0$ and invariance implies that $FH_*^h(H,J)$ must always vanish. We summarize these results as follows.

Theorem 1.5. *For any generic pair (H,J), the Floer homology for contractible 1-periodic orbits admits a canonical isomorphism*

$$FH_*(H,J) \to H_{*+n}(W;\Lambda_\omega),$$

[1]There's a somewhat subtle issue in understanding what $H_*(W;\Lambda_\omega)$ is, considering that the coefficients Λ_ω are also graded. It's at least comparatively straightforward if $H_2(W)$ is replaced by $\pi_2(W)$ (see Remark 1.2) and $c_1(TW)|_{\pi_2(W)} = 0$, e.g. (W,ω) is Calabi-Yau.

and for any nontrivial free homotopy class of loops h,

$$FH_*^h(H,J) = 0.$$

Before leaving this discussion of the closed case, let us consider one more refinement of $FH_*^h(H,J)$: one can introduce a filtration on the Floer chain complex via the action. To simplify the following discussion, assume for now that ω vanishes on $\pi_2(W)$ and consider only contractible orbits: then by Remark 1.2, one can write down a slight modification of $\mathcal{A}_H$ that is well defined on the space of contractible loops in $C^\infty(S^1,W)$, so that $FC_*(H)$ can be defined simply as the free abelian group generated by elements of $\mathcal{P}(H)$, with $\mathbb{Z}_{2N}$-grading

$$|\langle\gamma\rangle| = -\mu_{\mathrm{CZ}}(\gamma) \mod 2N,$$

where N is the minimal Chern number. Now for any $a \in \mathbb{R}$, define

$$FC_*(H;a) \subset FC_*(H)$$

as the subgroup generated by all orbits $\gamma \in \mathcal{P}(H)$ with $\mathcal{A}_H(\gamma) < a$, and for $-\infty \le a < b \le \infty$, define

$$FC_*(H;[a,b)) = FC_*(H;b) \Big/ FC_*(H;a).$$

Then since all solutions to the Floer equation have nonnegative energy, we deduce from (1.8) that ∂ preserves $FC_*(H;a)$ and hence descends to $FC_*(H;[a,b))$, so that we can define the *filtered* Floer homology

$$FH_*(H,J;[a,b)) = H_*\left(FC_*(H;[a,b)),\partial\right).$$

We must now reexamine the question of whether the homology defined in this way is invariant: indeed, for any given (H^-,J^-) and (H^+,J^+) and a homotopy (H^s,J^s) between them, it is by no means clear that the map $FH_*(H^-,J^-) \to FH_*(H^+,J^+)$ defined as in (1.10) can be made compatible with the filtration, i.e. that it induces a map

$$\Phi_{(H^s,J^s)} : FC_*(H^-,J^-;[a,b)) \to FC_*(H^+,J^+;[a,b)).$$

The answer is provided by the energy bound (1.12): a map on the filtered chain complex can be defined if the left hand side of this equation is always nonnegative, which is not true in general, but is true whenever $\partial_s H \ge 0$. Keeping H constant is therefore fine, and we conclude that $FH_*(H,J;[a,b))$ is indeed independent of J, but in general one can define a map

$$FH_*(H^-,J^-;[a,b)) \to FH_*(H^+,J^+;[a,b))$$

if and only if $H^- \le H^+$, and this map will not generally be invertible. This foreshadows an issue that will arise repeatedly in our discussion of symplectic homology: generic choices of Hamiltonians H will not generally suffice to define a symplectic invariant, but an invariant can nonetheless be defined as a direct limit for increasing sequences of Hamiltonians.

2. Quantitative symplectic homology

The original motivation for defining Floer homology in non-closed settings came from the direction of quantitative symplectic invariants, i.e. parameters that measure the symplectic embedding properties of subdomains within larger symplectic manifolds. One of the simplest and deepest results in this area is the famous non-squeezing theorem: let us denote by B^{2n}_r the open ball of radius r in $\mathbb{R}^{2n}$ with standard symplectic form ω_0, and let $B^{2n} \subset \mathbb{R}^{2n}$ denote the open unit ball.

Theorem (Gromov [Gro85])**.** *There exists a symplectic embedding of* (B^{2n}_r, ω_0) *into* $(B^2_R \times \mathbb{R}^{2n-2}, \omega_0)$ *if and only if* $r \le R$.

Gromov's proof used J-holomorphic curves, but soon afterwards, alternative proofs appeared that seem at first glance to have nothing to do with holomorphic curves. One such proof comes from the existence of a *symplectic capacity* on $\mathbb{R}^{2n}$. This notion, defined originally by Ekeland and Hofer [EH89], associates to every open subset $\mathcal{U} \subset \mathbb{R}^{2n}$ a number $c(\mathcal{U}) \in [0, \infty]$ satisfying the following properties:

- (Monotonicity) If $(\mathcal{U}, \omega_0)$ admits a symplectic embedding into $(\mathcal{U}', \omega_0)$, then $c(\mathcal{U}) \le c(\mathcal{U}')$.
- (Conformality) For all $\alpha > 0$, $c(\alpha\mathcal{U}) = \alpha^2 c(\mathcal{U})$.
- (Normalization) $c(B^{2n}) = \pi = c(B^2 \times \mathbb{R}^{2n-2})$.

The non-squeezing theorem itself implies the existence of a symplectic capacity, namely the *Gromov width* (see [HZ94]), but Ekeland and Hofer constructed another capacity that was defined in terms of a variational principle for periodic orbits of Hamiltonian systems; of course the existence of such an object implies the non-squeezing theorem.

It may seem surprising at first that periodic orbits of Hamiltonian systems have anything to do with symplectic embedding obstructions, but Floer and Hofer [FH94] give the following heuristic explanation for this phenomenon. Imagine $\varphi : B^{2n}_r \hookrightarrow B^2_r \times \mathbb{R}^{2n-2}$ is an "optimal" symplectic embedding of the ball into the cylinder; indeed, the non-squeezing theorem tells us that we cannot squeeze the image $\varphi(B^{2n}_r)$ into any smaller cylinder $B^2_{r'} \times \mathbb{R}^{2n-2}$ for $r' < r$. But let's try to do this anyway and see what can go wrong. For simplicity, since B^{2n}_r is maximally squeezed into the cylinder, it seems not altogether unreasonable to assume that the set

$$\Sigma := \overline{\varphi(B^{2n}_r)} \cap (\partial \overline{B}^2_r \times \mathbb{R}^{2n-2})$$

is an open subset of $\partial \overline{B}^2_r \times \mathbb{R}^{2n-2}$, and hence a smooth hypersurface in $\mathbb{R}^{2n}$. Then one way to squeeze $\varphi(B^{2n}_r)$ symplectically into an even smaller cylinder would require finding a Hamiltonian vector field X_F which points transversely into the cylinder everywhere along Σ. This is not possible in general, and in fact there's one very simple obstruction one can imagine: suppose Σ itself has a closed characteristic, meaning $\Sigma \subset H^{-1}(0)$ for some Hamiltonian $H : \mathbb{R}^{2n} \to \mathbb{R}$ such that X_H has a periodic orbit $\gamma : [0, T] \to \mathbb{R}^{2n}$ contained in Σ. Then since X_F is transverse to Σ, we have $dH(X_F) \neq 0$ everywhere along Σ, so

$$dF(\dot\gamma(t)) = dF(X_H) = \{H, F\} = -dH(X_F) \neq 0,$$

implying F must always increase or decrease along the periodic orbit γ, which is clearly impossible.

The above discussion is rather simplistic, but regardless of whether you're now convinced that periodic orbits give obstructions to symplectic embeddings, you already know what periodic orbits have to do with holomorphic curves: an elegant relationship between them is provided by Floer homology. Since the Ekeland-Hofer capacity was defined by measuring the symplectic actions of 1-periodic orbits, Floer and Hofer [FH94] were motivated to define a more refined invariant using the action filtration on Floer homology. We shall now describe the basic idea of this construction. The following is actually a somewhat simplified version of the theory defined in [FH94], with a few details borrowed from [CFH95] and [HZ94, §6.6].

We identify $\mathbb{R}^{2n}$ with $\mathbb{C}^n$ and write the standard symplectic form as

$$\omega_0 = \sum_{j=1}^{n} dx_j \wedge dy_j,$$

where $z_j = x_j + iy_j$ are the standard coordinates on $\mathbb{C}^n$. The standard complex structure i is then compatible with ω_0. For any open subset $\mathcal{U} \subset \mathbb{C}^n$, define the set of *admissible* Hamiltonians $\mathcal{H}(\mathcal{U})$ to consist of all smooth $H : S^1 \times \mathbb{C}^n \to \mathbb{R}$ with the following properties:[2]

- $H \geq 0$,
- $\operatorname{supp}(H_t)$ is compact and contained in $\mathcal{U}$ for all $t \in S^1$,
- All contractible 1-periodic orbits $\gamma \in \mathcal{P}(H)$ with $\mathcal{A}_H(\gamma) < 0$ are nondegenerate.

We define also a special class of time-dependent almost complex structures $\mathcal{J}$ on $\mathbb{C}^n$ by saying $\{J_t\} \in \mathcal{J}$ if and only if the following properties are satisfied for all $t \in S^1$:

- J_t is compatible with ω_0,
- $J_t = i$ outside of a compact subset.

We claim that for any $H \in \mathcal{H}(\mathcal{U})$, $J \in \mathcal{J}$ and $-\infty \leq a < b \leq 0$, the filtered Floer chain complex $(FC_*(H, J; [a, b)), \partial)$ and corresponding homology $FH_*(H, J; [a, b))$ can be defined exactly as in §1. Note that since $H_2(\mathbb{C}^n) = 0$, the generators of $FC_*(H, J; [a, b))$ are simply orbits $\gamma \in \mathcal{P}(H)$, there is no Novikov ring, and ∂ is defined by counting the 1-dimensional moduli spaces $\mathcal{M}(\gamma, \gamma'; H, J)$ of Floer trajectories connecting two nondegenerate orbits γ, γ' with negative action. To show that $FH_*(H, J; [a, b))$ is well defined, we must check that solutions in $\mathcal{M}(\gamma, \gamma'; H, J)$ satisfy the required C^0 and energy bounds (there is no danger of bubbling since ω_0 is exact). Energy bounds follow by the same argument as before, but C^0-bounds now require an extra ingredient: we must take advantage of the *convexity* of $(\mathbb{C}^n, J)$ at infinity.

[2]The exposition in [HZ94] requires the opposite sign for Hamiltonians $H \in \mathcal{H}(\mathcal{U})$. This can probably be attritubed to the fact that they write down the definition of the Hamiltonian vector field (1.3) with an extra minus sign. That is, of course, the *right* way—perhaps I am slightly biased.

Proposition 2.1. *If* (Σ, j) *is a Riemann surface and* $u : (\Sigma, j) \to (\mathbb{C}^n, i)$ *is a holomorphic map, then the function*

$$\Sigma \to [0, \infty) : z \mapsto |u(z)|^2$$

has no local maximum.

One can prove this by showing that the Cauchy-Riemann equation for u implies that the function $\log\left(|u(z)|^2\right)$ is subharmonic and thus satisfies a maximum principle. It also follows from a more general result that we'll discuss in the next section; see Prop. 3.4.

Given Prop. 2.1 we conclude that Floer trajectories $u \in \mathcal{M}(\gamma, \gamma'; H, J)$ can never escape from some large ball $B_R^{2n} \subset \mathbb{C}^n$, outside of which $J = i$ and $H = 0$, as they would then become holomorphic and force the function $z \mapsto |u(z)|^2$ to attain a maximum. This implies the required C^0-bound, and the rest of the compactness argument for $\mathcal{M}(\gamma, \gamma'; H, J)$ is the same as usual, so that $FH_*(H, J; [a, b))$ is well defined.

As we already saw at the end of §1, we can expect $FH_*(H, J; [a, b))$ to be invariant under changes in J, but not H: the energy relation (1.12) implies that a continuation map compatible with the filtration

$$FH_*(H^-, J^-; [a, b)) \to FH_*(H^+, J^+; [a, b)) \tag{2.1}$$

can only be defined in general for *monotone* homotopies, i.e. we must assume the homotopy $\{H^s\}_{s \in [-1,1]}$ from H^- to H^+ satisfies $\partial_s H^s \geq 0$. Such maps therefore exist (and are independent of the chosen homotopy, by the usual chain homotopy argument) whenever $H^+ \geq H^-$, and they also satisfy the composition relation (1.11), but they cannot in general be inverted. Thus instead of defining a symplectic invariant simply as $FH_*(H, J; [a, b))$ for a suitable choice of H and J, we are led naturally to a *direct limit*: defining a partial order $\prec$ on $\mathcal{H}(\mathcal{U}) \times \mathcal{J}$ by

$$(H^-, J^-) \prec (H^+, J^+) \quad \Longleftrightarrow \quad H^- \leq H^+,$$

the existence of the maps (2.1) and their compatibility under compositions allows us to define the direct limit

$$SH_*^{[a,b)}(\mathcal{U}) = \varinjlim_{(H,J) \in \mathcal{H}(\mathcal{U}) \times \mathcal{J}} FH_*(H, J; [a, b)).$$

This is a simplified version of the *symplectic homology* first defined by Floer and Hofer.

For an intuitive notion of what $SH_*^{[a,b)}(\mathcal{U})$ measures, consider for example the irrational ellipsoid

$$E(r_1, r_2) = \{F(z_1, z_2) < 1\} \subset \mathbb{C}^2$$

where

$$F(z_1, z_2) := \frac{|z_1|^2}{r_1^2} + \frac{|z_2|^2}{r_2^2}$$

for $r_1, r_2 > 0$, $r_1/r_2 \notin \mathbb{Q}$. Let us attempt a rough guess at the computation of $SH_*^{[a,b)}(E(r_1, r_2))$. We claim in fact that the direct limit can be computed as the filtered Floer homology for a particular "infinitely large" Hamiltonian, which we can approximate using an increasing sequence of cutoff functions.

To define the divergent Hamiltonian, choose a smooth function $h_\infty : (0,1] \to [0,\infty)$ such that for some $\epsilon > 0$,

(1) $h_\infty(s) = 0$ for all $s \in [1-\epsilon, 1]$,
(2) $h'_\infty(s) < 0$ and $h''_\infty(s) > 0$ for all $s \in (0, 1-\epsilon)$,
(3) $\lim_{s\to 0} h_\infty(s) = \infty$.

Now for any $\tau > 0$, we can "smooth the divergence" of h_∞ to define a smooth function $h_\tau : [0,1] \to [0,\infty)$ such that

(1) $h_\tau = \tau$ on a closed neighborhood of 0,
(2) $h_\tau = h_\infty$ on $h_\infty^{-1}[0, \tau - 1]$,
(3) $h'_\tau < 0$ everywhere else.

This gives rise to a family of nonnegative Hamiltonians $H^\tau : \mathbb{C}^2 \to \mathbb{R}$ with support in $E(r_1, r_2)$, namely

$$H^\tau(z_1, z_2) = h_\tau(F(z_1, z_2)),$$

which match the singular Hamiltonian $H^\infty := h_\infty \circ F$ on increasingly large subsets that exhaust $\mathbb{C}^2 \setminus \{0\}$ as $\tau \to +\infty$. We can also assume without loss of generality that

$$H^\tau \geq H^{\tau'} \quad \Longleftrightarrow \quad \tau \geq \tau'.$$

We are now going to cheat a bit and pretend $H^\tau \in \mathcal{H}(E(r_1, r_2))$, which is not true because H^τ is time-independent; in particular every nonconstant 1-periodic orbit of X_{H^τ} comes in a degenerate S^1-parametrized family related by time translation. However, there is a standard way to perturb autonomous Hamiltonians so that degenerate S^1-families of this sort are replaced by pairs of nondegenerate orbits of almost the same period, whose Conley-Zehnder indices differ by 1, thus one can still deduce properties of $SH_*^{[a,b)}(E(r_1, r_2))$ by examining the orbits of H^τ. In fact, the *interesting* orbits will be the orbits of X_{H^∞}, since all of these are also orbits of X_{H^τ} when τ is sufficiently large. The Hamiltonian vector field determined by H^∞ on $\mathbb{C}^2 \setminus \{0\}$ is

$$X_{H^\infty}(z_1, z_2) = -2h'_\infty(F(z_1, z_2)) \left(\frac{iz_1}{r_1^2}, \frac{iz_2}{r_2^2} \right),$$

thus since $r_1/r_2 \notin \mathbb{Q}$, its 1-periodic orbits come in two types:

- For every $k \in \mathbb{N}$, there is a unique $\rho_k \in (0, r_1)$ such that $-2h'_\infty(\rho_k^2/r_1^2) = 2\pi r_1^2 k$, producing orbits $\gamma_k^1(t) = (e^{2\pi i k t}\rho_k, 0)$ and their S^1-translations, all of which have action
$$\mathcal{A}_{H^\infty}(\gamma_k^1) = -\pi\rho_k^2 - h_\infty(\rho_k^2/r_1^2).$$
- For every $k \in \mathbb{N}$, there is a unique $\sigma_k \in (0, r_2)$ such that $-2h'_\infty(\sigma_k^2/r_2^2) = 2\pi r_2^2 k$, producing orbits $\gamma_k^2(t) = (0, e^{2\pi i k t}\sigma_k)$ and their S^1-translations, with action
$$\mathcal{A}_{H^\infty}(\gamma_k^2) = -\pi\sigma_k^2 - h_\infty(\sigma_k^2/r_2^2).$$

Notice that in both lists of orbits, ρ_k and σ_k must approach zero as $k \to \infty$, so that the action diverges to $-\infty$. Thus for any given finite $a < b \leq 0$, only a finite subset of these orbits have action in $[a, b)$, and we can choose $\tau > 0$ large enough so that all of them are also 1-periodic orbits of X_{H^τ} with the same action. From this we infer that $SH_*^{[a,b)}(E(r_1, r_2))$ will always be generated by finite subsets of the orbits listed above, whose actions depend on r_1 and r_2, so that the rank of $SH_*^{[a,b)}(E(r_1, r_2))$ is a function of r_1 and r_2.

A complete computation of the symplectic homology of $E(r_1, r_2)$ was carried out by Floer-Hofer-Wysocki [FHW94], in fact in any dimension and without assuming irrationality. It leads to a complete symplectic classification of ellipsoids in $\mathbb{C}^n$, which can be stated for the $n = 2$ case as follows:

Theorem 2.2 ([FHW94])**.** *Suppose $r_1 \le r_2$ and $r_1' \le r_2'$. Then $(E(r_1, r_2), \omega_0)$ and $(E(r_1', r_2'), \omega_0)$ are symplectomorphic if and only if $r_1 = r_1'$ and $r_2 = r_2'$.*

The same paper [FHW94] explains how symplectic homology can be used to define a new symplectic capacity for open subsets of $\mathbb{C}^n$. In a related pair of papers [CFH95, CFHW96] together with Cieliebak, the same authors define a related symplectic homology theory for open sets of compact symplectic manifolds with contact type boundary, and use it to show that the symplectomorphism type of the interior "sees the boundary" in some sense: namely, one can define an invariant of the interior that detects the periods of orbits on the boundary.

3. Convexity and contact type boundaries

The fact that the theory in the previous section can be defined on the noncompact manifold $\mathbb{C}^n$ depends on a certain convexity property (Prop. 2.1) in order to prove C^0-bounds for the space of Floer trajectories. We will now consider a generalization of the crucial convexity property, which leads naturally into the setting of symplectic manifolds with *contact type* boundary.

For this and the next section, assume (W, ω) is a compact $2n$-dimensional symplectic manifold with boundary $\partial W = M$. We say that the boundary is (symplectically) *convex*, or of *contact type*, if near ∂W there exists a vector field η that points transversely outward at ∂W and is a so-called *Liouville* vector field, meaning

$$\mathcal{L}_\eta \omega = \omega.$$

There are several equivalent ways to state this condition: for instance, given η as above, define the 1-form $\lambda = \iota_\eta \omega$. Then $\mathcal{L}_\eta \omega = \omega$ implies $d\lambda = \omega$, and it is an easy exercise to show that η points transversely outward at the boundary $M = \partial W$ if and only if the restriction $\alpha := \lambda|_{TM}$ satisfies

$$\alpha \wedge (d\alpha)^{n-1} > 0, \tag{3.1}$$

with M understood to carry the natural boundary orientation. The condition (3.1) means that α is a positive *contact form* on M. Its *contact structure* is the co-oriented hyperplane field $\xi = \ker \alpha \subset TM$, and the pair (M, ξ) is then called a *contact manifold*. The relation (3.1) is equivalent to requiring that $d\alpha|_\xi$ be nondegenerate, hence giving the bundle $\xi \to M$ a symplectic structure whose induced orientation is compatible with the co-orientation determined by α. It can also be interpreted as a "maximal non-integrability" condition for ξ, e.g. it implies that ξ has no integral submanifolds of dimension greater than $n - 1$. In contact geometric language, the boundary of (W, ω) is convex if and only if ω admits a primitive λ near ∂W that restricts to a positive contact form on the boundary.

Notice that Liouville vector fields transverse to the boundary are far from unique: if any such vector field exists, then one can obtain more by adding any sufficiently small Hamiltonian vector field. Thus the contact form induced on $M = \partial W$ is not unique, but it turns out that up to a natural notion of equivalence, the contact *structure* is.

Proposition 3.1. *If (W, ω) has contact type boundary and ξ is the contact structure induced on $M = \partial W$ by a choice of Liouville vector field as described above, then ξ is uniquely determined up to isotopy.*

This follows easily from a basic result of contact geometry. Notice that the space of Liouville vector fields pointing transversely outward at the boundary is a convex set, thus by interpolating between any two choices, we obtain a smooth family of contact structures. This reduces Prop. 3.1 to the following result, which can be proved by a Moser deformation argument (see e.g. [Gei08]).

Theorem (Gray's stability theorem)**.** *If ξ_0 and ξ_1 are contact structures on M that are homotopic through a smooth family of contact structures, then they are also isotopic.*

In light of Prop. 3.1, we regard the contact structure ξ (but not the contact *form* α) as the natural structure induced on the boundary of a symplectic manifold (W, ω) satisfying the convexity condition. In contact geometry, one then says that (W, ω) is a *strong symplectic filling* of the contact manifold (M, ξ).

For the rest of this section we assume (W, ω) has convex boundary $M = \partial W$ and fix the notation η, $\lambda = \iota_\eta \omega$, $\alpha = \lambda|_{TM}$ and $\xi = \ker \alpha$ as described above. A neighborhood of ∂W then admits a convenient normal form: let φ_η^t denote the flow of η, and choose $\epsilon > 0$ small enough so that there is an embedding

$$\Phi : (-\epsilon, 0] \times M \hookrightarrow W : (a, m) \mapsto \varphi_\eta^a(m).$$

It is now easy to check that $\Phi^*\lambda = e^a\alpha$, hence $\Phi^*\omega = d(e^a\alpha)$. The open symplectic manifold

$$(\mathbb{R} \times M, d(e^a\alpha))$$

is called the *symplectization* of (M, ξ). We thus see that a neighborhood of ∂W in W can be identified symplectically with the subset $(-\epsilon, 0] \times M$ in the symplectization, so that one can smoothly attach a cylindrical end to define a larger, open symplectic manifold

$$(\widehat{W}, \hat{\omega}) = (W, \omega) \cup_{\partial W} ([0, \infty) \times M, d(e^a\alpha)).$$

This is called the *completion* of (W, ω).

Remark 3.2. Contrary to appearances in the above presentation, the symplectization $(\mathbb{R} \times M, d(e^a\alpha))$ does not actually depend (up to symplectomorphism) on the choice of contact form α with $\ker \alpha = \xi$, and the completion $(\widehat{W}, \hat{\omega})$ can be regarded as containing a cylindrical end of the form $([T, \infty) \times M, d(e^a\alpha))$ for *any* choice of α if $T \in \mathbb{R}$ is taken sufficiently large. Indeed, given α as above, suppose α' is a different contact form related to α by $\alpha = e^f\alpha'$ for some smooth function $f : M \to \mathbb{R}$. Then by a minor generalization of the above construction, one can identify a neighborhood of ∂W in (W, ω) with a slightly different subset of $(\mathbb{R} \times M, d(e^a \alpha'))$, namely with a neighborhood of the boundary of the domain

$$\{(a, m) \in \mathbb{R} \times M \mid f(m) \leq a\}.$$

Then the completion can instead be defined by attaching the complement of this domain in $(\mathbb{R} \times M, d(e^a \alpha'))$.

Definition 3.3. For any contact manifold (M, ξ) with contact form α, we define the *Reeb vector field* X_α to be the unique vector field satisfying

$$d\alpha(X_\alpha, \cdot) \equiv 0 \quad \text{and} \quad \alpha(X_\alpha) \equiv 1.$$

The condition $d\alpha(X_\alpha, \cdot)$ determines the direction of X_α uniquely: it must be transverse to ξ since $d\alpha|_\xi$ is symplectic, so we can then use α for normalization. Notice that the Reeb vector field X_α on the convex boundary of our symplectic manifold (W, ω) spans the *characteristic line bundle* of ∂W, i.e. the kernel of $\omega|_{TM}$. Thus for any Hamiltonian $H : W \to \mathbb{R}$ that has ∂W as a regular energy level, the closed orbits of X_H in ∂W are precisely the closed orbits of X_α on M.

We now define a special class of compatible almost complex structures on the symplectization $(\mathbb{R} \times M, d(e^a \alpha))$. Define $\mathcal{J}(M, \alpha)$ to be the (contractible) space of all almost complex structures J on $\mathbb{R} \times M$ with the following properties:

- J is invariant under the natural action by $\mathbb{R}$-translation,
- $J\partial_a = X_\alpha$ and $JX_\alpha = -\partial_a$, where ∂_a denotes the unit vector in the $\mathbb{R}$-direction,
- $J(\xi) = \xi$, and $J|_\xi$ restricts to a compatible complex structure on the symplectic vector bundle $(\xi, d\alpha) \to M$.

We can now prove a useful generalization of Prop. 2.1:

Proposition 3.4. *Suppose (Σ, j) is a Riemann surface, $J \in \mathcal{J}(M, \alpha)$ and $u = (f, v) : (\Sigma, j) \to (\mathbb{R} \times M, J)$ is J-holomorphic. Then the function $f : \Sigma \to \mathbb{R}$ has no local maximum.*

Proof. The point is to prove that $f : \Sigma \to \mathbb{R}$ is subharmonic, so that the result follows from the maximum principle. It suffices to prove this in local conformal coordinates (s, t) on any small open subset of Σ, so the nonlinear Cauchy-Riemann equation takes the local form $\partial_s u + J(u)\partial_t u = 0$. Writing $\pi_\alpha : TM \to \xi$ for the projection along X_α, this is equivalent to the three equations

$$\begin{aligned} \partial_s f - \alpha(\partial_t v) &= 0, \\ \partial_t f + \alpha(\partial_s v) &= 0, \\ \pi_\alpha \partial_s v + J\pi_\alpha \partial_t v &= 0. \end{aligned}$$

Now observe that since $d\alpha$ vanishes on X_α and $d\alpha(\cdot, J\cdot)$ defines a bundle metric on ξ,

$$d\alpha(\partial_s v, \partial_t v) = d\alpha(\pi_\alpha \partial_s v, \pi_\alpha \partial_t v) = d\alpha(\pi_\alpha \partial_s v, J\pi_\alpha \partial_s v) \geq 0,$$

with equality if and only if $\pi_\alpha \partial_s v = 0$. Thus we compute

$$0 \leq d\alpha(\partial_s v, \partial_t v) = \partial_s [\alpha(\partial_t v)] - \partial_t [\alpha(\partial_s v)] = (\partial_s^2 + \partial_t^2) f,$$

as claimed. □

Remark 3.5. This result implies Prop. 2.1 for the following reason: one must first observe that the standard symplectic $(\mathbb{C}^n, \omega_0)$ admits a global radial Liouville vector field η_0, so that the balls $\overline{B}_r^{2n}$ all have symplectically convex boundaries. The 1-form $\lambda_0 = \iota_{\eta_0}\omega_0$ then restricts to $S^{2n-1} = \partial\overline{B}^{2n}$ as the so-called *standard contact form* on the sphere, $\alpha_0 = \lambda_0|_{TS^{2n-1}}$. By flowing along the Liouville field from $\partial\overline{B}^{2n}$, one can then construct a diffeomorphism

$$\Phi : \mathbb{R} \times S^{2n-1} \to \mathbb{C}^n \setminus \{0\}$$

which takes each of the sets $(-\infty, T] \times S^{2n-1}$ to a punctured ball, and it is easy to show that $\Phi^* i \in \mathcal{J}(S^{2n-1}, \alpha_0)$.

We now list a few of the most important special cases of symplectic manifolds with contact type boundaries, progressing from more to less general.

Example 3.6. If in addition to the conditions stated above, the Liouville field η (or equivalently the primitive λ) exists *globally* on W, then we call (W, ω) a *Liouville domain*, or an *exact symplectic filling of* (M, ξ), and say that the boundary is of *restricted contact type*. Observe that the completion $(\widehat{W}, \hat{\omega})$ is also an exact symplectic manifold, as the primitive can be extended to the cylindrical end as $e^a\alpha$.

Example 3.7. A *Weinstein domain* $(W, \omega, \eta, \varphi)$ is a Liouville domain (W, ω) with a global Liouville field η and a smooth Morse function $\varphi : W \to \mathbb{R}$ for which ∂W is a regular level set and $d\varphi(\eta) > 0$ except at the critical points.

Example 3.8. A *Stein domain* is a compact complex manifold (W, J) with boundary which admits a smooth Morse function $\varphi : W \to \mathbb{R}$ such that ∂W is a regular level set and

$$\omega_\varphi := -d(d\varphi \circ J)$$

is a symplectic form compatible with J. One can use the resulting metric $\omega_\varphi(\cdot, J\cdot)$ to define a gradient vector field $\nabla\varphi$, which makes $(W, \omega_\varphi, \nabla\varphi, \varphi)$ into a Weinstein domain. The contact structure ξ induced on $M = \partial W$ can also be described as the maximal complex-linear subbundle of TM.

Observe that the topology of Weinstein domains is quite restricted, as one can show that $\varphi : W \to \mathbb{R}$ may only have critical points of index $k \le n$. Thus by Morse theory, a manifold diffeomorphic to W can be constructed from the ball $\overline{B}^{2n}$ by attaching finitely many $2n$-dimensional k-handles for $k = 0, \ldots, n$; the absence of k-handles for $k > n$ implies for instance that ∂W must be connected. (This is not true for every Liouville domain, as shown by McDuff [McD91] and Geiges [Gei95, Gei94].) Relatedly, Eliashberg proved [Eli90b] quite surprisingly that in complex dimensions greater than 2, any compact almost complex manifold with boundary that satisfies this topological condition can be deformed to a Stein domain; there are also results of this nature that hold in complex dimension 2 but are more complicated to state. Full details on these topics may be found in the monograph [CE12]. Since it often arises in discussions of symplectic homology, we now mention one more special case, whose topology is even more strongly restricted:

Definition 3.9. A *subcritical Weinstein (or Stein) domain* is a Weinstein domain $(W, \omega, \eta, \varphi)$ such that φ only has critical points of index strictly less than n.

4. Viterbo's theory and its applications

We now adapt the presentation from §1 to define Viterbo's version of symplectic homology for a compact symplectic manifold (W, ω) with contact type boundary

$$\partial(W, \omega) = (M, \xi).$$

We will again assume (W, ω) is semipositive, choose a subgroup

$$\mathcal{R} \subset \ker \omega \cap \ker c_1(TW) \subset H_2(W)$$

for which to define the Novikov completion Λ_ω of $\mathbb{Z}[H_2(W)/\mathcal{R}]$, and a free homotopy class of loops $h \in [S^1, W]$. An important class of examples is provided by Liouville domains, for which ω is exact.

A large part of the motivation for this theory comes from the following well known contact counterpart to the Arnold conjecture:

Conjecture (Weinstein). *For any contact manifold (M, ξ) with contact form α, the Reeb vector field X_α admits a periodic orbit.*

For contact structures that are strongly symplectically fillable, this amounts to the conjecture that every compact symplectic manifold with contact type boundary has a closed characteristic on its boundary. The goal is thus to define a Floer theory for (W, ω) generated by two types of periodic orbits:

(1) All 1-periodic orbits of a suitable Hamiltonian $H : W \to \mathbb{R}$,
(2) All closed characteristics of the characteristic line field on ∂W.

By analogy with the closed case, we intuitively expect the 1-periodic orbits of X_H to give us essentially topological information about W, i.e. something analogous to Morse homology. What turns out to be true in fact is that if there are no closed characteristics on ∂W, then the theory we define will indeed be isomorphic to some version of Morse homology and hence *singular* homology on W, as in the closed case. More generally, we will find that there is a map between symplectic homology and singular homology, but it need not be an isomorphism—and whenever it is not, this implies the Weinstein conjecture for ∂W.

To see why defining a theory generated by the two types of orbits mentioned above might be a reasonable thing to do, we consider a special class of Hamiltonians on the symplectic completion $(\widehat{W}, \hat{\omega})$. Recall from Remark 3.2 that for any contact form α with $\ker \alpha = \xi$, $(\widehat{W}, \hat{\omega})$ contains a cylindrical end of the form

$$([T_0, \infty) \times M, d(e^a \alpha))$$

for sufficiently large $T_0 > 0$. Thus we can change α if necessary and assume without loss of generality that all closed orbits of X_α are nondegenerate,[3]

[3]Since X_α is time-independent, we mean "nondegenerate" in the *transversal* sense, i.e. the linearized Reeb flow has no eigenvectors with eigenvalue 1 in directions transverse to the orbit.

a generic condition. Now consider a time-dependent Hamiltonian $H : S^1 \times \widehat{W} \to \mathbb{R}$ which takes the form

$$H(t, a, m) = \bar{H}(t, e^a)$$

for $(a, m) \in [T_0, \infty) \times M$, where $\bar{H}(t, a)$ is a smooth function on $S^1 \times [e^{T_0}, \infty)$. It is instructive to see what the periodic orbits X_{H_t} look like in $[T_0, \infty) \times M$: by an easy computation, we find

$$X_{H_t}(a, m) = -\partial_a \bar{H}(t, e^a) X_\alpha(m). \tag{4.1}$$

Thus a τ-periodic orbit of X_α gives rise to a 1-periodic orbit of X_{H_t} in $\{a_0\} \times M$ if and only if

$$\tau = \left| \int_0^1 \partial_a \bar{H}(t, e^{a_0})\, dt \right|,$$

and conversely every 1-periodic orbit in $\{a_0\} \times M$ for $a_0 \ge T_0$ is of this form. Note that the set of all periods of orbits of X_α, the so-called *action spectrum*

$$\mathrm{Spec}(M, \alpha) \subset (0, \infty),$$

is discrete if α is nondegenerate. Thus if we choose $\bar{H}$ with $|\partial_a \bar{H}(t, e^{T_0})|$ sufficiently small for all $t \in S^1$ and allow $|\partial_a \bar{H}(t, e^a)|$ to grow to infinity as $a \to \infty$, we find that all periodic orbits of X_α, of all periods, appear as 1-periodic orbits of X_{H_t}. This is therefore the type of Hamiltonian for which we'd like to define our Floer homology—though as we'll see below, the technical details are a bit more complicated.

We will of course have to be careful about our choices of H and J to ensure suitable C^0-bounds and energy bounds for Floer trajectories in the noncompact manifold $\widehat{W}$. In light of Prop. 3.4, it is now at least easy to guess what conditions should be placed on J: recalling the space $\mathcal{J}(M, \alpha)$ that was defined in the previous section, let $\mathcal{J}(W, \omega, \alpha)$ denote the space of all almost complex structures J on $\widehat{W}$ satisfying the following conditions:

- J is everywhere compatible with $\hat{\omega}$
- J matches an almost complex structure in $\mathcal{J}(M, \alpha)$ on $[T, \infty) \times M$ for some $T \ge T_0$.

Now let's see what kinds of Hamiltonians we can get away with using to define Floer homology on $(\widehat{W}, \hat{\omega})$ with $J \in \mathcal{J}(W, \omega, \alpha)$. We will need C^0-bounds for solutions $u : \mathbb{R} \times S^1 \to \widehat{W}$ to the s-dependent Floer equation

$$\partial_s u + J_t^s(u) \left(\partial_t u - X_{H_t^s}(u) \right) = 0. \tag{4.2}$$

Here $\{J_t^s\}_{(s,t) \in \mathbb{R} \times S^1}$ is a smooth family of almost complex structures in $\mathcal{J}(W, \omega, \alpha)$ that are s-independent for $|s| \ge 1$, so for $T_1 \ge T_0$ large enough we can assume all of them are in $\mathcal{J}(M, \alpha)$ on $[T_1, \infty) \times M$. Likewise, $H_t^s = H(s, t, \cdot) : \widehat{W} \to \mathbb{R}$ is a smooth family of Hamiltonians, s-independent for $|s| \ge 1$, which in light of the above discussion we shall assume to take the form

$$H(s, t, a, m) = \bar{H}(s, t, e^a)$$

for $(a, m) \in [T_1, \infty) \times M$, with $\bar{H}(s, t, a)$ a smooth function on $\mathbb{R} \times S^1 \times [T_1, \infty)$. For any $T \ge T_0$, denote by $\widehat{W}_T$ the complement of $(T, \infty) \times M$ in

$\widehat{W}$, so

$$\widehat{W} = \widehat{W}_T \cup_{\partial \widehat{W}_T} ([T,\infty) \times M).$$

Proposition 4.1. *Given the data described above, assume $\partial_s \partial_a \bar{H}$ is everywhere nonnegative. Then for any $T \ge T_1$, every bounded solution to (4.2) whose image intersects $\widehat{W}_T$ is contained in $\widehat{W}_T$.*

Proof. Suppose $u : \mathbb{R} \times S^1 \to \widehat{W}$ is a solution to (4.2) and $\mathcal{U} \subset \mathbb{R} \times S^1$ is an open set on which $u(\mathcal{U}) \subset [T_1, \infty) \times M$, thus for $(s,t) \in \mathcal{U}$ we can write

$$u(s,t) = (f(s,t), v(s,t)) \in \mathbb{R} \times M.$$

The result will again follow by showing that $f : \mathcal{U} \to \mathbb{R}$ satisfies a maximum principle. Proposition 3.4 showed this for the case where u is a J-holomorphic curve, i.e. $H_t^s = 0$ and J_t^s is independent of s and t. More generally, plugging in the formula (4.1) for $X_{H_t^s}$ in $[T_1,\infty) \times M$, we find that $u = (f, v) : \mathcal{U} \to \mathbb{R} \times M$ now satisfies the equations

$$\begin{aligned}
\partial_s f - \alpha(\partial_t v) - \partial_a \bar{H}(s,t,e^f) &= 0,\\
\partial_t f + \alpha(\partial_s v) &= 0,\\
\pi_\alpha \partial_s v + J_t^s \pi_\alpha \partial_t v &= 0.
\end{aligned}$$

We have $0 \le d\alpha(\pi_\alpha \partial_s v, J_t^s \pi_\alpha \partial_s v) = d\alpha(\partial_s v, \partial_t v)$ just as before, so repeating the same calculation gives

$$\begin{aligned}
0 \le d\alpha(\partial_s v, \partial_t v) &= \partial_s \left[\alpha(\partial_t v)\right] - \partial_t \left[\alpha(\partial_s v)\right]\\
&= (\partial_s^2 + \partial_t^2) f - \partial_s \left[\partial_a \bar{H}\left(s,t,e^{f(s,t)}\right)\right]\\
&= (\partial_s^2 + \partial_t^2) f - e^f \partial_a^2 \bar{H}(s,t,e^f) \partial_s f - \partial_s \partial_a \bar{H}(s,t,e^f).
\end{aligned}$$

Then since $\partial_s \partial_a \bar{H} \ge 0$, f satisfies the second order elliptic partial differential inequality

$$(\partial_s^2 + \partial_t^2) f - e^f \partial_a^2 \bar{H}(s,t,e^f) \partial_s f \ge 0,$$

which implies the maximum principle. □

This result reveals another monotonicity property that will be needed to define maps between Floer homologies for different Hamiltonians: if $H(s,t,a,m) = \bar{H}(s,t,e^a)$ near infinity, then we must assume

$$\partial_s \partial_a \bar{H} \ge 0, \tag{4.3}$$

i.e. the slopes will need to get steeper under homotopies. With this assumption in place, C^0-bounds are assured, and the required energy bounds are then provided again by (1.12).

We are now ready to proceed with the construction of the invariant. For any number $\tau > 0$ that is not a period of any orbit of X_α, define the class $\mathcal{H}_\tau$ of τ-*admissible* Hamiltonians $H : S^1 \times \widehat{W} \to \mathbb{R}$ to have the following properties:

1. $H_t < 0$ on W,
2. $H_t(a,m) = \tau e^a + c$ on $[T,\infty) \times M$ for some large $T \ge T_0$ and any constant $c \in \mathbb{R}$,
3. Every 1-periodic orbit of X_{H_t} is nondegenerate.

Observe that since $\tau \notin \mathrm{Spec}(M, \alpha)$, the linear behavior of H_t near infinity means that there are *no* 1-periodic orbits in $[T, \infty) \times M$, hence there are finitely many in total. Then for a generic pair $(H^\tau, J) \in \mathcal{H}_\tau \times \mathcal{J}(W, \omega, \alpha)$, the Floer homology $FH_*^h(H^\tau, J)$ for periodic orbits of $X_{H_t^\tau}$ in the free homotopy class h is well defined. It is not an invariant, nor should we expect it to be since it clearly doesn't see all the Reeb orbits we're interested in at the boundary, but only those up to a bounded period determined by τ. Fortunately, the condition (4.3) allows us to define continuation maps

$$FH_*^h(H^\tau, J) \to FH_*^h(H^{\tau'}, J')$$

for any pairs $(H^\tau, J) \in \mathcal{H}_\tau \times \mathcal{J}(W, \omega, \alpha)$ and $(H^{\tau'}, J') \in \mathcal{H}_{\tau'} \times \mathcal{J}(W, \omega, \alpha)$ such that $\tau' \geq \tau$. The usual arguments show that these maps are independent of the choice of (asymptotically monotone) homotopy and compatible under composition, so that we can define symplectic homology as the obvious direct limit

$$SH_*^h(W, \omega) = \varinjlim_{\tau \to \infty} FH_*^h(H^\tau, J).$$

Remark 4.2. As we've defined it, it's still not obvious whether $SH_*^h(W, \omega)$ depends on the choice of contact form α at the boundary, as our space of admissible almost complex structures $\mathcal{J}(W, \omega, \alpha)$ clearly depends on this choice. In light of Remark 3.2 however, such dependence would be surprising, as the construction of the completion $(\widehat{W}, \hat{\omega})$ does not really depend on α. Indeed, $SH_*^h(W, \omega)$ is in fact independent of α, and one can prove it via continuation maps for homotopies of J between $\mathcal{J}(W, \omega, \alpha)$ and $\mathcal{J}(W, \omega, \alpha')$ for any two contact forms α, α'. This just requires a slightly more careful convexity argument to obtain C^0-bounds, details of which may be found in [Sei]. Another proof that $SH_*^h(W, \omega)$ doesn't depend on α follows from the fact that it is invariant under *symplectic deformations*, see Theorem 4.6 below.

Remark 4.3. One might wonder whether $SH_*^h(W, \omega)$ could have been defined without a direct limit, just by a sufficiently intelligent choice of Hamiltonian. Whatever "sufficiently intelligent" means, it clearly must be one that includes all the orbits of X_α as 1-periodic orbits, which can be achieved by choosing H_t^∞ to satisfy

$$H^\infty(t, a, m) = \bar{H}^\infty(t, e^a)$$

for $(a, m) \in [T_0, \infty) \times M$, where $\partial_a \bar{H}^\infty(t, a)$ is always increasing and satisfies

$$0 < \partial_a \bar{H}^\infty\big(t, e^{T_0}\big) < \inf\{\tau > 0 \mid \tau \in \mathrm{Spec}(M, \alpha)\},$$

$$\lim_{a \to \infty} \partial_a \bar{H}^\infty(t, e^a) = \infty.$$

Then for any $J \in \mathcal{J}(M, \omega, \alpha)$, Prop. 4.1 allows us to define the Floer homology $FH_*^h(H^\infty, J)$, as well as a continuation map

$$FH_*^h(H^\tau, J) \to FH_*^h(H^\infty, J) \tag{4.4}$$

for every $H^\tau \in \mathcal{H}_\tau$. After taking the direct limit we thus obtain a map

$$SH_*^h(W, \omega) \to FH_*^h(H^\infty, J). \tag{4.5}$$

Is it an isomorphism? As explained in [Sei], this is not too hard to see in at least one important case: assume for simplicity

$$[\omega] = c_1(TW) = 0,$$

which is true for instance if (W, ω) is the unit disk bundle $D^*Q \subset T^*Q$ in a cotangent bundle with its standard symplectic structure. Then we can take $\mathcal{R} = H_2(W)$, drop the Novikov ring and define $FC_*^h(H^\infty)$ simply as the free group generated by all 1-periodic orbits; in particular, infinite formal sums are not allowed. Now by a clever choice of a sequence $H^\tau \in \mathcal{H}_\tau$ as $\tau \to \infty$, one can arrange for all generators of $FC_*^h(H^\tau)$ to be generators of $FC_*^h(H^\infty)$ as well, and for the maps (4.4) to be induced by the natural inclusions of the chain complexes. Then since $FC_*^h(H^\tau)$ eventually exhausts $FC_*^h(H^\infty)$, (4.5) is an isomorphism.

One trouble with defining $SH_*^h(W, \omega) := FH_*^h(H^\infty, J)$ right from the beginning is that in the absence of continuation maps between arbitrary choices of Hamiltonians, it's quite hard to see why $FH_*^h(H^\infty, J)$ should be independent of the choice of H^∞. In fact, the easiest proof of this fact is the one we just outlined, for which one needs the alternative definition of $SH_*^h(W, \omega)$ as a direct limit. It is nonetheless common in the literature to define $SH_*^h(W, \omega)$ for certain special cases as the Floer homology of a Hamiltonian with specified growth conditions at infinity, e.g. this is typically done for the case of cotangent bundles. Invariance is then proved by *computing* the invariant, which for $\widehat{W} = T^*Q$ means constructing an isomorphism to the homology of the loop space (see [AS06, SW06, Vit]).

Consider now the special case $h = 0$, so we take only contractible 1-periodic orbits and write

$$SH_*(W, \omega) := SH_*^0(W, \omega).$$

One of the fundamental properties of symplectic homology is an analog of the isomorphism between the Floer homology of a closed symplectic manifold and its singular homology. It is not hard to see from the above construction: choose $\tau > 0$ less than the smallest number in $\mathrm{Spec}(M, \alpha)$ and a Hamiltonian $H^\tau \in \mathcal{H}_\tau$ that is a time-independent Morse function that takes the form $H^\tau(a, m) = \tau e^a + c$ on $[T_0, \infty) \times M$ and is C^2-small everywhere else, with all critical points in the interior of W. Then just as in the closed case, the only 1-periodic orbits are the constant orbits at critical points, and for a generic time-independent $J \in \mathcal{J}(W, \omega, \alpha)$, the 1-dimensional spaces of Floer trajectories consist only of negative gradient flow lines for the Morse function $-H^\tau : \widehat{W} \to \mathbb{R}$. Since the gradient flow in question flows outward through the boundary, the corresponding Morse homology is isomorphic to $H_*(W, \partial W)$. Then using the natural map from $FH_*(H^\tau, J)$ to the direct limit, we obtain a natural map

$$H_{*+n}(W, \partial W; \Lambda_\omega) \to SH_*(W, \omega).$$

Moreover, if ∂W has no closed characteristics contractible in W, then $FC_*(H^\tau)$ already contains all the generators of $SH_*(W, \omega)$, and one can find a sequence $H^{\tau_k} \in \mathcal{H}_{\tau_k}$ with $\tau_k \to \infty$ such that all the natural maps $FH_*(H_k^\tau, J) \to FH_*(H^{\tau_\ell}, J)$ for $\ell > k$ are isomorphisms, implying that the above map is an isomorphism as well. This is the idea behind the following result proved in [Vit99]:

Theorem 4.4. *There is a natural Λ_ω-module homomorphism*

$$H_{*+n}(W, \partial W; \Lambda_\omega) \to SH_*(W, \omega), \tag{4.6}$$

and it is an isomorphism if ∂W has no closed characteristics that are contractible in W.

In light of this result, one says that (W, ω) satsifies the *algebraic Weinstein conjecture* if the map (4.6) is not an isomorphism.

Example 4.5. The algebraic Weinstein conjecture is not really a conjecture, as it is easy to show that it doesn't always hold: consider for instance the unit disk bundle $D^*\Sigma \subset T^*\Sigma$ over a closed oriented surface Σ with a Riemannian metric. For the natural choice of contact form on $\partial(D^*\Sigma)$, the Reeb orbits are lifts of geodesics on Σ, and thus cannot be contractible in $D^*\Sigma$ unless the corresponding geodesics are contractible. But clearly contractible geodesics do not always exist; take for instance the flat metric on T^2. It follows that (4.6) is an isomorphism when $W = D^*T^2$. Note that this algebraic fact never *disproves* the existence of contractible Reeb orbits: e.g. if (M, ξ) is a contact 3-manifold, then for any contractible knot $K \subset M$ transverse to ξ (these always exist), one can choose a contact form for ξ that makes K a Reeb orbit. For the case of $\partial(D^*T^2) = T^3$, that is a different contact form from the canonical one.

We conclude this introduction with a quick survey of the properties of $SH_*(W, \omega)$ and a few of its applications.

Deformation invariance. It turns out that $SH_*(W, \omega)$ is invariant under not only symplectomorphisms, but also *symplectic deformations.*

Theorem 4.6. *Suppose $\{\omega_\tau\}_{\tau \in [0,1]}$ is a smooth family of cohomologous*[4] *symplectic forms on W such that ∂W has convex boundary for all ω_τ. Then*

$$SH_*(W, \omega_0) \cong SH_*(W, \omega_1).$$

One consequence of this is the fact that $SH_*(W, \omega)$ doesn't depend on the choice of Liouville vector field at ∂W (and hence the choice of contact form on M), since one can always deform the boundary of the neighborhood

$$((-\epsilon, 0] \times M, d(e^a \alpha))$$

to a graph in $\mathbb{R} \times M$ to produce some multiple of any desired contact form (for the same contact structure) on the boundary. Put another way, this means that $SH_*(W, \omega)$ really only depends on the completion $(\widehat{W}, \hat{\omega})$, not on (W, ω) itself. In this way it is radically different from the quantitative invariant

[4]The condition that the forms ω_τ all be cohomologous can sometimes be weakened, but one must be careful since the definition of the Novikov ring we use for coefficients generally depends on $[\omega] \in H^2_{\mathrm{dR}}(W)$. Recall that on closed manifolds, cohomologous deformations of symplectic forms always arise from isotopies due to the Moser stability theorem, but this is not generally true on manifolds with boundary. For an example of what can be proved about non-cohomologous deformations, see [Rit10, §6].

discussed in §2.[5] As Seidel [Sei, §7] explains, this hints at the fact that symplectic homology can be extended over a larger category that includes certain types of noncompact symplectic manifolds: examples include the completions of Liouville domains as well as Stein manifolds with infinitely many critical points.

Viterbo functoriality. One of the main results proved by Viterbo in [Vit99] was that a codimension 0 embedding

$$(W_0, \omega_0) \hookrightarrow (W, \omega)$$

of one symplectic manifold with boundary into a larger one induces a so-called *transfer map* on the symplectic homologies between them. For technical reasons, this only works under some extra assumptions, the most natural of which is to suppose that both are Liouville domains, and one is a *Liouville subdomain* of the other.

Theorem 4.7. *Suppose $(W, d\lambda)$ is a Liouville domain with $\lambda|_{T(\partial W)}$ a positive contact form, and $W_0 \subset W$ is a compact codimension 0 submanifold such that $\lambda|_{T(\partial W_0)}$ is also a positive contact form. Then there exists a natural homomorphism*

$$SH_*(W, d\lambda) \to SH_*(W_0, d\lambda).$$

Moreover, this map fits together with the map from Theorem 4.4 and the natural map on relative singular homology $H_*(W, \partial W) \to H_*(W_0, \partial W_0)$ to form the following commutative diagram:

$$\begin{array}{ccc} H_{*+n}(W, \partial W) & \longrightarrow & H_{*+n}(W_0, \partial W_0) \\ \downarrow & & \downarrow \\ SH_*(W, d\lambda) & \longrightarrow & SH_*(W_0, d\lambda) \end{array}$$

Some computations and applications. The simplest computation of symplectic homology is for the standard symplectic ball: the answer is

$$SH_*(\overline{B}^{2n}, \omega_0) = 0. \qquad (4.7)$$

This result should not be mistaken for a lack of information: considering that $H_*(\overline{B}^{2n}, \partial\overline{B}^{2n})$ does not vanish, this computation together with Theorem 4.4 implies the Weinstein conjecture for the standard contact structure on S^{2n-1}. Various methods for carrying out the computation are described in [Oan04] and [Sei].

A substantial generalization of this computation follows from the work of Oancea [Oan03] and Cieliebak [Cie], strengthening a previous result of Viterbo [Vit99]:

Theorem 4.8. *If (W, ω) is any subcritical Stein domain, then $SH_*(W, \omega) = 0$.*

This of course implies the Weinstein conjecture for all contact manifolds that admit subcritical Stein fillings. Cieliebak [Cie02] proved in fact that

[5] This is not to say that one cannot also extract more quantitative information from $SH_*(W, \omega)$, which does admit an action filtration similar to the discussion in §2.

attaching subcritical handles to Stein domains does not change their symplectic homology, so Theorem 4.8 may be seen as a corollary of this result together with the computation (4.7) for the ball.

Lest one should get the impression that symplectic homology always vanishes, we should mention the computation [Vit, AS06, SW06] for the cotangent bundle T^*Q of a smooth manifold Q, or rather for the unit disk bundle $D^*Q \subset T^*Q$ with its canonical exact symplectic structure $d\lambda_{\text{can}}$. The result provides an isomorphism to the singular homology of the free loop space of Q:

Theorem 4.9. $SH_*(D^*Q, d\lambda_{can}) = H_*(\Lambda Q)$.

This yields an especially interesting result when combined with Theorems 4.7 and 4.8. Suppose $(W, d\lambda)$ is a subcritical Stein domain and

$$L \subset W$$

is an exact Lagrangian submanifold, i.e. $\lambda|_{TL}$ is exact. In this case, a neighborhood of L can be identified with D^*L, embedded into $(W, d\lambda)$ as a Liouville subdomain, so Theorem 4.7 provides a transfer map

$$0 = SH_*(W, d\lambda) \to SH_*(D^*L, d\lambda_{\text{can}}) = H_*(\Lambda L).$$

When combined with the maps of Theorem 4.4 and the resulting commutative diagram, one can derive a contradiction and thus the following generalization of a theorem of Gromov [Gro85]:

Theorem 4.10. *If (W, ω) is a subcritical Stein domain, then it does not admit any exact Lagrangian embeddings.*

Finally, we mention a pair of results due to Seidel and Smith (see [Sei]) that concern the existence of exotic symplectic and contact structures.

Theorem 4.11. *If (W, ω) is a $2n$-dimensional Liouville domain whose boundary is a standard contact sphere, then $SH_*(W, \omega) = 0$.*

This fits in with a pattern of classification results for symplectic fillings of the sphere: in fact it's obvious for $n = 2$ due to a result of Gromov [Gro85] and Eliashberg [Eli90a], that the standard contact S^3 has only one Stein filling up to symplectic deformation. No such precise result is known in higher dimensions, though a theorem usually attributed to Eliashberg-Floer-McDuff (see [McD91]) states that every Stein filling of the standard S^{2n-1} must at least be *diffeomorphic* to the ball, and this fact is used in the proof of the above result. Using results from [SS05], Seidel and Smith constructed a Liouville domain that is diffeomorphic to $\overline{B}^8$ but has nonvanishing symplectic homology, hence:

Corollary. *There exists a symplectic structure on $\overline{B}^8$ with restricted contact type boundary that is not contactomorphic to the standard S^7.*

More results along these lines have been produced recently by Mark McLean [McL09], using symplectic homology to distinguish the symplectic structures on diffeomorphic Stein manifolds constructed via Lefschetz fibrations.

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