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Greedy Algorithm And Matroid Intersection Algorithm

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Talk Summary

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Abstract

Many combinatorial optimization problems can be formulated in terms of independence systems. In the first part I want to introduce the class of greedy algorithms, which find an optimal solution for all weight functions if and only if the independence system is a matroid. For some problems like finding the matching in a bipartite graph or the travelling salesman problem the system is not a matroid. But I prove that every independence system is the finite intersection of matroids. The problem of finding the matching in a bipartite graph, can be described as intersection of two matroids. For this case I present the Edmond's Intersection Algorithm which solve the problem in polynomial time.

1 Greedy Algorithm

The maximization problem is defined as the task to find a set $X \in \mathcal{I}$ such that the weight $w(X) = \sum_{e \in X} w(e)$ for $w : E \to \mathbb{R}$ is maximized. For the maximization problem, we can define the best-in-greedy algorithm for a given independence oracle:

1.1 Best-In-Greedy Algorithm

`Sort` $E = \{e_1, ..., e_n\}$ `s.t.` $w(e_1) \geq \ ... \ \geq \ w(e_n)$.
`Set` $X := \emptyset \in \mathcal{I}$.
`For i:=1 to n do: If` $F \cup \{e_i\} \in \mathcal{I}$ `then set` $X := X \cup \{e_i\}$.
`Output:` $X \in \mathcal{I}$.

The complexity of sorting E is $\mathcal{O}(n \cdot log(n))$, but we have to ask the oracle in each step of the loop. Therefore the complexity of the algorithm mainly depends on the complexity of the independence oracle.

1.2 Feasibility In Matroids

In general the greedy algorithm find only a local optimum and the cardinality of the output has not to be maximum, the output is just maximal (i.e. there exists no independent superset). But for matroids we can prove the following theorem if we assume that the greedy algorithm work correctly in the independence system:

Theorem (Feasability)**.** *An independence system is a matroid if and only if the best-in-greedy algorithm for all weight functions finds an optimal solution for the maximization problem.*

Proof : First we prove that in a matroid the best-in-greedy algorithm finds an optimal solution for all weight functions. Let w be an abitrary weight function and $X = \{x_1, ..., x_r\}$ the output of the greedy algorithm. Without loss of generality we may assume $w(x_1) \geq ... \geq w(x_r)$ (if not we change the numbering). We show that X has a maximum cardinality by contradiction, we then use that to prove by contradiction that $w(X)$ is maximum[1]:

Assume there exists a set $Y \in \mathcal{I}$ with greater cardinality than X. By the augmentation property follows, that there exists a $y \in Y \setminus X$ such that $X \cup \{y\} \in \mathcal{I}$. Therefore a t has to exist such that $\{x_1, ..., x_t, y, x_{t+1}, ..., x_r\} \in \mathcal{I}$ with $w(x_t) \geq w(y) \geq w(x_{t+1})$. Hence $\{x_1, ..., x_t\} \subseteq \{x_1, ..., x_t, y\} \in \mathcal{I}$. But then y should be choosen in step $t + 1$ of the greedy algorithm, what contradicts the correctness.

Next we assume that there exists a $Y = \{y_1, ..., y_q\} \in \mathcal{I}$ such that $w(Y) > w(X)$ and $w(y_i) \geq w(y_{i+1})$. By the definiton of the weight function follows that $\sum_{y_i \in Y} w(y_i) > \sum_{x_i \in X} w(x_i)$ and because $|Y| \leq |X|$ there exists a k such that $w(y_k) > w(x_k)$. Define $X' := \{x_1, ..., x_{k-1}\}$ ($= \emptyset$ if $k = 1$) and $Y' := \{y_1, ..., y_k\}$. Because of the augmentation property there exists a $y_t \in Y' \setminus X'$ with $t \leq k$ such that $\{x_1, ..., x_{k-1}, y_t\} = X' \cup \{y_t\} \in \mathcal{I}$. But then y_t should be choosen before step k of the greedy algorithm, because $w(y_t) \geq w(y_k) > w(x_k)$. This contradicts the assumption that X is the correct output of the greedy algorithm.

Secondly we want to prove by contradiction that the feasibility of the greedy algorithm for all weight functions implies that the independence system is a matroid. We assume that the independence system is not a matroid, i.e. there exists sets $I, J \in \mathcal{I}$ with $|I| < |J|$ such that for all $e \in J \setminus I$ the set $I \cup \{e\}$ is dependent. Under this assumption the greedy algorithm would not work for the following weight function[2] with a variable $\varepsilon > 0$ which we specify later:

$$w(e) := \begin{cases} 1 + \varepsilon & , if \quad e \in \quad I \\ 1 & , if \quad e \in \quad J \setminus I \\ 0 & , if \quad e \in \quad E \setminus \{I \cup J\} \end{cases}$$

For this weight function the greedy algorithm would first choose all elements of the set I, because they got the heighest weight. After this it can not choose any element of J, because of the assumption. Only elements of $E \setminus \{I \cup J\}$ could be choosen afterwards. Therefore the weight of the output X of the greedy algorithm is

$$w(X) = |I| \cdot (1 + \varepsilon) + 0.$$

[1]Due to (Oxley, 2006, pp. 63-64).
[2]Due to (Lee, 2004, p. 60).

But for the weight of J holds the inequality

$$w(X) = |I|(1+\varepsilon) < w(J) = |J \setminus I| + |I \cap J|(1+\varepsilon)$$

for $\varepsilon < \frac{|J \setminus I|}{|I| - |I \cap J|} - 1$. This implies that the greedy algorithm would not have found the optimal solution for that ε, and that is a contradiction to the correctness of the algorithm for all weight functions. □

2 Matroid Intersection

Proposition. *Any independendence system is a finite intersection of matroids.*

Proof : For each circuit $C \in \mathcal{C}$ in $(E, \mathcal{I})$ we define $\mathcal{I}_C := \{F \subseteq E | C \setminus F \neq \emptyset\}$. We will prove that every $(E, \mathcal{I}_C)$ is an independence system, even a matroid[3] and that the intersection of all $\mathcal{I}_C$ is indeed $\mathcal{I}$.

Obviously is $\emptyset \in \mathcal{I}_C$ because $C \setminus \emptyset \neq \emptyset$.
Also for $I \subseteq J$ with $J \in \mathcal{I}_C$ is $I \in \mathcal{I}_C$ because $C \setminus J \neq \emptyset$ implies $C \setminus I \neq \emptyset$.

The independence system $(E, \mathcal{I}_C)$ is a matroid because for all subsets of E the rank r is equal to the lower rank ϱ:

If $X \in \mathcal{I}_C$ then X has an unique maximal independent subset in $\mathcal{I}_C$ namely X itself. Hence $r(X) = \varrho(X) := \min\{|Y| : Y \subseteq X, Y \in \mathcal{I} \text{ and } Y \cup \{x\} \notin \mathcal{I}\ \forall x \in X \setminus Y\} = |X|$

If $X \notin \mathcal{I}_C$ then every maximal independent subset of X is of the form $X \setminus \{c\}$ for any $c \in C$, because $C \setminus (X \setminus \{c\}) \neq \emptyset$. Hence $r(X) = \varrho(X) = |X| - 1$

As the rank is equal to the lower rank for all subsets of E the rank quotient of $(E, \mathcal{I}_C)$ is one and therefore the independence system is a matroid.[4]

Now we show that $(E, \mathcal{I}) = (E, \cap_{C \in \mathcal{C}} \mathcal{I}_C)$:
Every $F \subseteq \cap_{C \in \mathcal{C}} \mathcal{I}_C$ is independent. If not there would exist a circuit $C \in \mathcal{C}$ with $C \subseteq F$ such that $C \setminus F = \emptyset$, what would contradict the assumption that F is out of the intersection.

If $F \in \mathcal{I}$ then for all $C \in \mathcal{C}$ would be $C \setminus F \neq \emptyset$ and therefore F would be an element in the intersection. □

To find a maximum cardinality set X out of the intersection of two matroids $(E, \mathcal{I}_1)$ and $(E, \mathcal{I}_2)$ we can use Edmonds' matroid intersection algorithm. The idea of the algorithm is to start with $X = \emptyset \in \mathcal{I}_1 \cap \mathcal{I}_2$ and augment by one element in each step. When you find no more $e \in E$ such that $X = X \cup \{e\} \in \mathcal{I}_1 \cap \mathcal{I}_2$,

[3] Due to (Du, 2008, p. 24).
[4] See Propositon 13.7. (Korte and Vygen, 2007, p. 308).

then construct a special directed bipartite graph G_X over the disjoint vertice sets X and $E \setminus X$. Search in this graph for a certain shortest alternating path P. With $X = X \triangle V(P) = X \setminus V(P) \cup V(P) \setminus X$ augment X by one element and repeat until you can not augment X anymore.

For $i = 1, 2$ let $C_i(X, e)$ be the unique circuit in the matroid $(E, \mathcal{I}_i)$ which is a subset of $X \cup \{e\}$. For all $X \in \mathcal{I}_1 \cap \mathcal{I}_2$, we can define the directed auxiliary graph G_X by:

$$A_X^{(1)} := \{(x, y) | y \in E \setminus X, x \in C_1(X, y) \setminus \{y\}\}$$

$$A_X^{(2)} := \{(y, x) | y \in E \setminus X, x \in C_2(X, y) \setminus \{y\}\}$$

$$G_X := (X \cup E \setminus X, A_X^{(1)} \cup A_X^{(2)}).$$

Then we are searching for the shortest path from $S_X := \{y \in E \setminus X | X \cup \{y\} \in \mathcal{I}_1\}$ to $T_X := \{y \in E \setminus X | X \cup \{y\} \in \mathcal{I}_2\}$ to augment X.

Fig 13.2. from (Korte and Vygen, 2007, p. 324)

If $S_X \cap T_X \neq \emptyset$ and therefore $X \cup \{e\} \in \mathcal{I}_1 \cap \mathcal{I}_2$ for all $e \in S_X \cap T_X$, we can augment X by any element in $S_X \cap T_X$.

If $S_X \cap T_X = \emptyset$ and the length of the shortest $S_X T_X$-path is greater then zero, we augment the set by using the symmetric difference of X and the vertices of the path.

If there exists no $S_X T_X$-path at all we are done and $X \in \mathcal{I}_1 \cap \mathcal{I}_2$ got its maximum size.

This leads to the following algorithm with polynomial complexity on the maximum of the two independence oracles[5]:

2.1 Edmonds Matroid Intersection Algorithm

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Set X := ∅.
While we can augment X do:
    Add all elements e such that X = X ∪ {e} ∈ I1 ∩ I2.
    For all y ∈ E \ X compute Ci(X, y).
    Compute SX, TX, and GX.
    Find shortest SXTX-path P in GX.
    If no P exists stop, else set X = X△V(P) and repeat.
Output: X.
```

To prove the correctness of this algorithm we need three lemma from (Korte and Vygen, 2007, pp. 323-325), the first two show that indeed $X \triangle V(P) \in \mathcal{I}_1 \cap \mathcal{I}_2$

[5] A short proof of the complexity you can find in (Korte and Vygen, 2007, Theorem 13.32 p. 326), a more detailed proof you can find in (Papadimitriou and Steiglitz, 1982, pp. 297-298).

and the third shows that the algorithm finds the set of maximum cardinality in the intersection:

2.2 Correctness Of The Algorithm

Lemma (13.27)**.** *Let* $(E, \mathcal{I})$ *be a matroid and* $X \in \mathcal{I}$*. Let* $x_1, ..., x_s \in X$ *and* $y_1, ..., y_s \notin X$ *with*

(a) $x_k \in C(X, y_k)$ *for* $k = 1, ..., s$ *and*

(b) $x_j \notin C(X, y_k)$ *for* $1 \leq j < k \leq s$*.*

Then is $(X \setminus \{x_1, ..., x_s\}) \cup \{y_1, ..., y_s\} \in \mathcal{I}$*.*

Proof : Let be $X_r := (X \setminus \{x_1, ..., x_r\}) \cup \{y_1, ..., y_r\}$, then we want to prove that $X_r \in \mathcal{I}$ for all $r \in \{1, ..., s\}$ by induction:

For $r = 0$ we have nothing to prove because $X_0 = X \in \mathcal{I}$.
In the inductive step we are assuming that $X_{r-1} \in \mathcal{I}$ and want to prove that $X_r \in \mathcal{I}$ by case destinction:

If $X_{r-1} \cup \{y_k\} \in \mathcal{I}$ it follows by $I2$ that $(X_{r-1} \setminus \{x_r\}) \cup \{y_r\} = X_r \in \mathcal{I}$.

If $X_{r-1} \cup \{y_k\} \notin \mathcal{I}$, we prove that $X_r \in \mathcal{I}$ by contradiction. If we assume that $X_{r-1} \setminus \{x_r\} \cup \{y_r\} \notin \mathcal{I}$, there has to exists a circuit $C \subseteq X_{r-1} \setminus \{x_r\} \cup \{y_r\}$. Additionally there exists a circuit $C(X, y_r) \subseteq X \cup \{y_r\}$ for which even $C(X, y_r) \subseteq X \setminus \{x_1, ..., x_{r-1}\} \cup \{y_r\}$ because of b). This circuits are distinct, because $x_r \notin C$ by assumption and $x_r \in C(X, y_r)$ by a). Hence $y_r \in C \cup C(X, y_r)$, otherwise $C \subseteq X_{r-1}$ or $C(X, y_r) \subseteq X$ would be and therefore independent, what would contradict the fact that both are circuits. Now by $C3$[6] follows that there exists a circuit $C_3 \subseteq (C \cup C(X, y_r)) \setminus \{y_r\}$, for which holds:

$$\begin{aligned} C_3 &\subseteq ([X_{r-1} \setminus \{x_r\} \cup \{y_r\}] \cup [X \setminus \{x_1, ..., x_{r-1}\} \cup \{y_r\}]) \setminus \{y_r\} \\ &\subseteq [X \setminus \{x_1, ..., x_r\} \cup \{y_1, ..., y_{r-1}\}] \cup X \setminus \{x_1, .., x_{r-1}\} \\ &\subseteq X \setminus \{x_1, ..., x_r\} \cup \{y_1, ..., y_{r-1}\} \\ &\subseteq X_{r-1} \in \mathcal{I} \end{aligned}$$

This would imply that there exists an independent circuit, what is a contradiction and therefore is $X_r \in \mathcal{I}$ in contrary to the assumption. □

[6] See (Korte and Vygen, 2007, Theorem 13.12 (C3) pp. 312-313).

Lemma (13.28)**.** *Let $X \in \mathcal{I}_1 \cap \mathcal{I}_2$. Let $y_0, x_1, y_1, ..., x_s, y_s$ be in this order the vertices of a shortest y_0-y_s-path in G_X, with $y_0 \in S_X$ and $y_s \in T_X$. Then is*

$$X' := (X \cup \{y_0, .., y_s\}) \setminus \{x_1, ..., x_s\} \in \mathcal{I}_1 \cap \mathcal{I}_2.$$

Proof : Because $y_0, ..., y_s$ is a shortest y_0-y_s-path in G_X we know that

1. $(x_j, y_j) \in A_X^{(1)} := \{(x, y) | y \in E \setminus X, x \in C_1(X, y) \setminus \{y\}\}$ and
2. $(y_{j-1}, x_j) \in A_X^{(2)} := \{(y, x) | y \in E \setminus X, x \in C_2(X, y) \setminus \{y\}\}$

for all $j \in 1, ..., s$. From this we can show that the requirements of lemma 13.27 and the properties a) and b) are fulfilled in both matroids.

First we want to show that $X' \in \mathcal{I}_1$:
We define $\widetilde{X} := X \cup \{y_0\}$ which is independent because $y_0 \in S_X$. We know that $x_j \in \widetilde{X}$ because $x_j \in C_1(X, y_j) \setminus \{y_j\} \subseteq X \subseteq \widetilde{X}$ and we know $y_j \in E \setminus \widetilde{X}$ because $y_j \neq y_0$ for all $j = 1, ..., s$ and therefore the requirements of the lemma are fulfilled for $\widetilde{X}$.

Property a) of the lemma is satisfied, because from 1. for all $j \in \{1, ..., s\}$ follows that $x_j \in C_1(X, y_j) \subseteq C_1(\widetilde{X}, y_j)$. Property b) is satisfied, because if for any $k < s$ a $j < k$ exists such that $x_j \in C_1(\widetilde{X}, y_k) = C_1(X, y_k)$[7] there would be a shortcut $x_j y_k$, what contradicts the fact that $y_0 x_1 y_1 ... x_j y_j ... x_k y_k ... y_s$ is the shortest x_0-y_s-path.

Altogether lemma 13.27 implies that $\widetilde{X} \setminus \{x_1, ..., x_s\} \cup \{y_1, ..., y_s\} = X' \in \mathcal{I}_1$.

Analog we want to show that $X' \in \mathcal{I}_2$:
We define $\widetilde{X} := X \cup \{y_s\}$ which is independent because $y_s \in T_X$. We know that $x_j \in \widetilde{X}$ because $x_j \in C_2(X, y_{j-1}) \setminus \{y_{j-1}\} \subseteq X \subseteq \widetilde{X}$ and we know $y_{j-1} \in E \setminus \widetilde{X}$ because $y_{j-1} \neq y_0$ for all $j = 1, ..., s$ and therefore the requirements of the lemma are fulfilled for $\widetilde{X}$.

Property a) of the lemma is satisfied, because from 1. for all $j \in \{1, ..., s\}$ follows that $x_j \in C_2(X, y_{j-1}) \subseteq C_2(\widetilde{X}, y_{j-1})$. Property b) is satisfied, because if for any $j < s$ a $i < j$ exists such that $x_j \in C_2(\widetilde{X}, y_{i-1}) = C_2(X, y_{i-1})$[8] there would be a shortcut $y_{i-1} x_j$, what contradicts the fact that $y_0 x_1 ... y_{i-1} x_i ... y_{j-1} x_j ... y_s$ is the shortest x_0-y_s-path.

Now lemma 13.27 implies that $\widetilde{X} \setminus \{x_1, ..., x_s\} \cup \{y_0, ..., y_{s-1}\} = X' \in \mathcal{I}_2$. □

[7] See (Korte and Vygen, 2007, Lemma 13.12 b)): $X \cup \{y_k\} \subseteq X \cup \{y_0\} \cup \{y_k\}$ contains at most one circuit and therefore both cycles are equal.

[8] See (Korte and Vygen, 2007, Lemma 13.12 b)): $X \cup \{y_{i-1}\} \subseteq X \cup \{y_s\} \cup \{y_{i-1}\}$ contains at most one circuit and therefore both cycles are equal.

Lemma (13.30)**.** *$X \in \mathcal{I}_1 \cap \mathcal{I}_2$ is maximum if and only if there is no S_X-T_X-path in G_X.*

Proof : We prove by contradiction that if $X \in \mathcal{I}_1 \cap \mathcal{I}_2$ is maximum there exists no S_X-T_X-path in G_X. If we assume there exists a S_X-T_X path, there also exists a shortest one. We apply lemma 13.28 and obtain a set $X' \in \mathcal{I}_1 \cap \mathcal{I}_2$ with $|X| < |X'|$ what contradicts the requirement that X is maximum.

Then we prove that the set X is maximum, if there is no S_X-T_X-path. Let R be the set of vertices reachable from S_X in G_X. Hence $R \cap T_X = \emptyset$. Let r_1 and r_2 be the rank function of $\mathcal{I}_1$ and $\mathcal{I}_2$, respectively.

Fig 13.3. from (Korte and Vygen, 2007, p. 326)

First we prove $r_2(R) = |X \cap R|$ by contradiction:
If $r_2(R) > |X \cap R|$, there would exist a $y \in R \setminus X$ such that $(X \cap R) \cup \{y\} \in \mathcal{I}_2$. Because $y \in R$ and $R \cap T_X = \emptyset$ we know that $y \notin T_X = \{y \in E \setminus X | X \cup \{y\} \in \mathcal{I}_2\}$ and therefore a circuit $C_2(X, y) \notin \mathcal{I}_2$ exists. Because $\{y\} \in \mathcal{I}_2$ and $(X \cap R) \in \mathcal{I}_2$, there has to exist a $x \in X \setminus R$ with $x \in C_2(X, y)$, even $x \in C_2(X, y) \setminus \{y\}$ because $x \neq y$. But then $(y, x) \in A_X^{(2)} = \{(y, x) | y \in E \setminus X, x \in C_2(X, y) \setminus \{y\}\}$ means there exists an edge in G_X which leaves R because $x \in X \setminus R$. This contradicts the definition of R.

Now we prove $r_1(E \setminus R) = (X \setminus R)$ by contradiction:
If $r_1(E \setminus R) > |X \setminus R|$, there would exist a $y \in (E \setminus R) \setminus (X \setminus R)$ such that $(X \setminus R) \cup \{y\} \in \mathcal{I}_1$. Because $y \notin R$ and $S_X \subseteq R$ we know $y \notin S_X = \{y \in E \setminus X | X \cup \{y\} \in \mathcal{I}_1\}$ and therefore a circuit $C_1(X, y) \notin \mathcal{I}_1$ exists. Because $\{y\} \in \mathcal{I}_1$ and $(X \setminus R) \in \mathcal{I}_1$, there has to exist a $x \in X \cap R$ with $x \in C_1(X, y)$, even $x \in C_1(X, y) \setminus \{y\}$ because $x \neq y$. But then $(x, y) \in A_X^{(1)} := \{(x, y) | y \in E \setminus X, x \in C_1(X, y) \setminus \{y\}\}$ means there exists an edge in G_X which leaves R because $y \notin R$. This contradicts the definition of R.

Altogether we have $|X| = |X \setminus R| + |X \cap R| = r_2(R) + r_1(E \setminus R)$. We know that for all $X' \in \mathcal{I}_1 \cap \mathcal{I}_2$ holds the inequality $|X'| \leq r_2(R) + r_1(E \setminus R) = |X|$[9]. And therefore X is maximum. □

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[9] See (Korte and Vygen, 2007, Proposition 13.29).

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