# The additive collapse

Andreas Baudisch (Berlin)

March 24, 2009

**Summary.** From known examples of theories $T$ obtained by Hrushovski-constructions and of infinite Morley rank, properties are extracted, that allow the collapse to a finite rank substructure. The results are used to give a more model-theoretic proof of the existence of the new uncountably categorical groups in [Bau2].

## 1 Introduction

In 1988 Ehud Hrushovski [Hr1] refuted Zil'bers Conjecture. He constructed strongly minimal theories with non-locally-modular geometries and without any infinite group interpretable in these theories. In [Bau2] uncountably categorical groups are constructed that have non-locally-modular geometries and do not allow the interpretation of infinite fields. The essential Amalgamation Lemma in that paper has a very long "bilinear-combinatorial" proof. The aim of this paper is to find general properties of an $\omega$-stable theory $T$ such that the additive collapse can be done. In [JG] Bruno Poizat gives a proof of Hrushovski's result above. First he constructs a theory of Morley rank $\omega$ already with that kind of geometry he finally wants. In a second step he does the collapse to obtain the desired theory.

By the fusion paper [Hr2] of Ehud Hrushovski new ideas come into the subject. He constructs a strongly minimal fusion of two strongly minimal sets with DMP over a common domain. The new theory has the old theories as reducts, that are "very independent" from each other. Later these ideas are applied to obtain an algebraically closed field with a "black" predicate such that the whole structure has Morley rank 2 ([Po1], [BH]).

Ehud Hrushovski asks in [Hr2] whether the fusion is also possible over a common vectorspace over a finite field. The next question goes in the same direction. In his book [Po0] Bruno Poizat asks whether there exists an algebraically closed field with a "red" predicate for an additive infinite proper subgroup, such that this structure has still finite Morley-rank.

Amador Martin-Pizarro, Martin Ziegler and the author have solved these problems positively in [BMPZ4] and [BMPZ3]. The ideas of these two papers are the basis of the more general additive collaps developed in this paper. It should be mentioned that they are also essential for the construction of a bad field of characteristic 0 in [BHMPW].

In this paper we consider structures $M$ that are expansions of an infinite vectorspace over a finite field $\mathbb{F}_q$. We assume that the language $L$ is at most countable and contains a predicate $R(x)$. We consider seven properties P(I) – P(VII) of the complete theories $T$ of these structures $M$. If $T$ has the properties P(I) – P(IV), then all models $M$ of $T$ are generated their subspace $R(M)$. We have a notion of strongness for subspaces $A \leq R(M)$ and a pregeometry $a \in \mathrm{cl}_d(A)$ on $R(M)$. Both notions are part of $\mathrm{tp}^M(A)$ and $\mathrm{tp}^M(aA)$, respectively. The geometrical closure of a strong subset $A \leq R(M)$ is given by algebraic and prealgebraic extensions. The second kind of extension is given by generic solutions of strongly minimal formulas that behave similar as the code formulas from [BMPZ3]. We move to code formulas. P(I) – P(IV) imply $\omega$-stability and allow us to introduce a notion of difference sequences as in [BMPZ3].

Let $\mathbb{C}$ be a monster model of $T$. We define a class $\mathbb{K}^\mu$ of strong subspaces $U$ of $R(\mathbb{C})$ where a function $\mu$ gives a bound $\mu(\alpha)$ to the length of difference sequences for the code formula $\varphi_\alpha(\bar{x}, \bar{y})$. If we have in addition the properties P(V) and P(VI) we can amalgamate finite subspaces in $\mathbb{K}^\mu$, such that we get a countable strong subspace $R^\mu(\mathbb{C})$ of $R(\mathbb{C})$ in $\mathbb{K}^\mu$. It is rich (B. Poizat's notion of richness) and therefore algebraically closed in $\mathbb{R}(\mathbb{C})$ in the sense of $T$. Let $P^\mu(\mathbb{C})$ be the substructure generated by $R^\mu(\mathbb{C})$. Then $P^\mu(\mathbb{C}) \cap R(\mathbb{C}) = R^\mu(\mathbb{C})$. $P^\mu$ can be defined by one formula over $R^\mu$. This is guaranteed by another property P(VII) of $T$. Let $L^\mu$ be the extension of $L$ by the predicate $P^\mu$. We axiomatize the $L^\mu$-theory of $\mathbb{C}^\mu = (\mathbb{C}, P^\mu(\mathbb{C}))$ and get an $\omega$-stable theory $T^\mu$ where $R^\mu = P^\mu \cap R$ is strongly minimal and $P^\mu$ is of finite Morley-rank. We show that the induced $L^\mu$-structure on $P^\mu$ is the pure $L$-structure. Let $\Gamma(T^\mu)$ be the $L$-theory of this $L$-structure $P^\mu(\mathbb{C}^\mu)$. It is the desired collapse to finite Morley rank. We can present the new uncountably categorical groups, the red fields and the fusion over a vectorspace in this way. Maybe we can only use less $\mu$-functions but still $2^{\aleph_0}$ many as in the original papers. This frame is designed for further concrete applications.

## 2 Group sets

For this chapter $T$ is a countable $\omega$-stable theory where the models of $T$ are expansions of vectorspaces over $\mathbb{F}_q$. We use a version of a result of M. Ziegler in [Z]. We work in $T^{\mathrm{eq}}$.

**Lemma 2.1** *Let $M$ be a model of $T$ as above and $a$, $b$, $c$ be elements of $M$ with $a+b+c=0$ and pairwise independence over some set $B$. Then we have:*

1) *The strong types of the elements $a$, $b$, $c$ over $B$ have the same stabilizer $U$ and $U$ is connected.*

2) *$a$, $b$, and $c$ are generic elements of $\mathrm{acl}(B)$-definable cosets of $U$.*

3) *It follows that $a$, $b$, and $c$ have the same Morley rank over $B$ namely $\mathrm{MR}(U)$. $U$ is definable over $\mathrm{acl}(B)$.*

Let $\mathbb{C}$ be the monster model of $T$. $\deg_M$ is used to denote Morley degree.

**Definition** Let $X$ be a definable subset of $\mathbb{C}^n$ with $\deg_M(X)=1$.
$X$ is called a group set (resp. torsor set) if its generic type is the generic type of a definable subgroup $G$ (resp. coset of a definable subgroup) of $(\mathbb{C}^n,+)$. We say $X$ is groupless, if $X$ is not a torsor set.

**Definition** Two definable sets $X$ and $Y$ of Morley degree 1 are equivalent, if $\mathrm{MR}(X)=\mathrm{MR}(Y)$ and $\mathrm{MR}(X\Delta Y)<\mathrm{MR}(X)$. We write $X\sim Y$.

**Lemma 2.2** *Let $X$, $Y$ be definable sets of Morley degree 1.*

1) *If $X\sim Y$, $X,Y\subseteq\mathbb{C}^n$, and $X$ is a group set (resp. torsor set), then $Y$ is a group set (resp. torsor set).*

2) *If $H$ is in $\mathrm{GL}_n(\mathbb{F}_p)$ and $X\subseteq\mathbb{C}^n$ is a torsor set, then $H(X)+\bar{m}=\{H\bar{x}+\bar{m}:\bar{x}\in X\}$ is a torsor set.*

**Lemma 2.3** *Let $\varphi(\bar{x},\bar{y})$ be a formula such that $\mathbb{C}\models\exists\bar{x}\,\varphi(\bar{x},\bar{b})$ implies that $\varphi(\mathbb{C},\bar{b})$ is a strongly minimal subset of $\mathbb{C}^n$. Then $\{\bar{b}:\varphi(\mathbb{C},\bar{b})$ is a group set$\}$ is definable. Similarly for torsor sets.*

*Proof*. We consider the group case. The following statements are equivalent:

i) $\varphi(\mathbb{C}, \bar{b})$ is a group set.

ii) There exist two generic $\bar{b}$-independent realizations $\bar{a}_1$ and $\bar{a}_2$ of $\varphi(\bar{x}, \bar{b})$ such that $\mathbb{C} \models \varphi(\bar{a}_1 + \bar{a}_2, \bar{b})$.

iii) $\mathbb{C} \models \exists^\infty \bar{x}_1 \in \mathbb{C}^n \, \exists^\infty \bar{x}_2 \in \mathbb{C}^n (\varphi(\bar{x}_1, \bar{b}) \wedge \varphi(\bar{x}_2, \bar{b}) \wedge \varphi(\bar{x}_1 + \bar{x}_2, \bar{b}))$.

The equivalence of i) and ii) follows from Lemma 2.1 as shown in [BMPZ3]. iii) is first order since $\varphi(\bar{x}, \bar{b})$ is strongly minimal. It is clearly equivalent with ii) in the strongly minimal context. $\square$

Note that $X$ is a torsor set if for some (every) $x \in X$ the set $X - x$ is a group set.

**Definition** Given a group set $X$, its invariant group is the set $\mathrm{Inv}(X) = \{H \in \mathrm{GL}_n(\mathbb{F}_p) : H(X) \sim X\}$.

For strongly minimal $\varphi(\bar{x}, \bar{b})$ "$H \in \mathrm{Inv}(\varphi(\bar{x}, \bar{b}))$" is an elementary property of $\bar{b}$. As in [BMPZ3] it follows

**Lemma 2.4** *Let $X$ be a $B$-definable set of Morley degree $1$, and $\bar{e}_0$ and $\bar{e}_1$ be two generic $B$-independent elements in $X$. If $\bar{e}_0 - H\bar{e}_1 \underset{B}{\perp} \bar{e}_0$ for some $H$ in $\mathrm{GL}_n(\mathbb{F}_p)$, then $X$ is a torsor set. Moreover, if $X$ is a group set, then $H$ is in $\mathrm{Inv}(X)$.*

# 3 Starting theories

We consider countable theories $T$. Let $M$, $N$ be models of $T$ and $\mathbb{C}$ be the monster model of $T$. $\langle X \rangle$ is used to denote the substructure generated by $X$. $\langle X \rangle^\ell$ is the linear hull of $X$.

P(I) The models $M$ of $T$ are $\mathbb{F}_q$-vectorspaces with additional structure, where $\mathbb{F}_q$ is a finite field.
Furthermore we have a unary predicate $R(x)$ for a subspace of $M$. For all $M \models T$ we have $\langle R(M) \rangle = M$.

Mainly we consider finite subspaces $A$, $B$, $C$ of $R(M)$. $U$, $V$, $W$ are used for arbitrary subspaces of $R(M)$.

P(II) We have a pregeometry "$a \in \mathrm{cl}_d(A)$" on $R(M)$ and a notion "$A$ is a strong subspace in $R(M)$" (short $A \leq M$). Both notions are invariant under automorphisms of $\mathbb{C}$. $\langle 0 \rangle^\ell \leq M$. For every $B$ there exists a finite algebraic extension that is strong in $M$. Algebraic extensions of strong subspaces are strong. If $M$, $N$ are models of $T$ $A \subseteq R(M)$, $B \subseteq R(N)$, $\mathrm{tp}^M(A) = \mathrm{tp}^N(B)$ and a and $b$ are geometrically independent of $A$ and $B$ respectively, then $\mathrm{tp}^M(a, A) = \mathrm{tp}^N(b, B)$. If furthermore $A \leq M$, then $\langle Aa \rangle^\ell \leq M$. The geometrical dimension $d(\mathbb{C})$ of $R(\mathbb{C})$ is infinite.

We use $d$ to denote the dimension function corresponding to $\mathrm{cl}_d$. Note that P(II) implies the following:

If $A \subseteq R(M)$ and $B \subseteq R(N)$ are the linear hulls of geometrically independent subsets, where $M$, $N$ are models of $T$, then $\mathrm{l.dim}(A) = \mathrm{l.dim}(B)$ implies $\mathrm{tp}(A) = \mathrm{tp}(B)$.

We extend the notions in P(II) to infinite subspaces $U$ of $R(M)$ by the following definitions:

**Definition** $a \in \mathrm{cl}_d(U)$, if $a \in \mathrm{cl}_d(A)$ for some finite subspace $A$ of $U$.

**Definition** $U \leq M$, if for every finite $B \subseteq U$ there is a finite $A \subseteq U$ with $B \subseteq A$ and $A \leq M$.

P(III) There is a set $\mathcal{X}$ of formulas $\varphi(\bar{x}, \bar{y})$ in $L^{\mathrm{eq}}$ such that $\varphi(\bar{x}, \bar{b})$ is either empty or strongly minimal. Furthermore $\varphi(\bar{x}, \bar{b}) \sim \varphi(\bar{x}, \bar{b}')$ implies $\bar{b} = \bar{b}'$. $\mathrm{Length}(\bar{x}) = n_\varphi \geq 2$, $\varphi(\bar{x}, \bar{y})$ implies $x_i \in R$ and the linear independence of $x_1, \ldots, x_{n_\varphi}$. If $\bar{b}$ is in $\mathrm{dcl}^{\mathrm{eq}}(U)$ and $M \models \varphi(\bar{a}, \bar{b})$, then $\bar{a} \in \mathrm{cl}_d(U)$. If furthermore $U \leq M$, then either $\bar{a} \subseteq U$ or $\bar{a}$ is a generic solution over $U$. In the generic case $\langle U\bar{a} \rangle^\ell \leq M$. $\mathcal{X}$ is closed under affine transformations.

In the construction of red fields [BMPZ3] the formulas $\varphi(\bar{x}, \bar{y})$ in $\mathcal{X}$ are of the form $\psi(\bar{x}, \bar{y}) \wedge \bigwedge_{1 \leq i \leq n} R(x_i)$ where $\psi(\bar{x}, \bar{y})$ is a formula in the field language. There we use that $\mathrm{ACF}_q$ is a reduct of $T$ and has the elimination of quantifiers and imaginaries. In the fusion over a vectorspace [BMPZ4] $\varphi(\bar{x}, \bar{y})$ in $\mathcal{X}$ is of the form $\varphi_1(\bar{x}, \bar{y}) \wedge \varphi_2(\bar{x}, \bar{y})$ where $\varphi_i(\bar{x}, \bar{y})$ is a formula of the theory $T_i$. We assume elimination of quantifiers for the theories $T_i$. For the construction of new uncountably categorical groups in this paper we use formulas $\varphi(\bar{x}, \bar{y})$ in $L$ with the property that $\varphi(\bar{x}, \bar{b}) \sim \varphi(\bar{x}, \bar{b}')$ implies $\langle \bar{b} \rangle^\ell = \langle \bar{b}' \rangle^\ell$. Hence we have for these formulas almost a canonical parameter.

We say that a vector space isomorphism $f$ of $A$ onto $B$ is an isomorphism if we can extent it to an $L$-isomorphism of $\langle A\rangle$ onto $\langle B\rangle$.

P(IV) If $A \leq M$, $B \leq M$, and $A \cong B$, then $\operatorname{tp}(A) = \operatorname{tp}(B)$. If $B \leq M$, $A \leq M$ and $B \subseteq A \subseteq \operatorname{cl}_d(B)$, then there is a chain $B = A_0 \subseteq A_1 \subseteq \ldots \subseteq A_n = A$ where $A_i \leq M$ and $A_{i+1} \subseteq \operatorname{acl}(A_i)$ or $A_{i+1}$ is obtained from $A_i$ adding a generic solution of some $\varphi(\bar{x}, \bar{b})$ in $\mathcal{X}$ where $\bar{b} \in \operatorname{dcl}^{\mathrm{eq}}(A_i)$.

Note that by P(III) and the first part of P(IV) $A_{i+1}$ over $A_i$ can be described by a quantifier-free $L$-formula. Let $\downarrow$ be the non-forking independence in $T$. Besides genericity of solutions $\bar{a}$ of $\varphi_\alpha(\bar{x}, \bar{b})$ we introduce $\downarrow^w$-genericity for these solutions. If $\bar{b} \in \operatorname{dcl}^{\mathrm{eq}}(B)$, then in the known examples $\downarrow^w$-genericity of $\bar{a}$ over $B$ means that $\bar{a}$ is linearly independent over $B$ and $\delta(\bar{a}/B) = 0$.

P(V) Let $\varphi(\bar{x}, \bar{y}) \in \mathcal{X}$, $V$ a subspace of $R(M)$, and $\bar{b} \in \operatorname{dcl}^{\mathrm{eq}}(V)$. Then the $\downarrow$-generic type of $\varphi(\bar{x}, \bar{b})$ over $V$ is $\downarrow^w$-generic over $V$ and the $\downarrow^w$-generics of $\varphi(\bar{x}, \bar{b})$ over $V$ have the same isomorphism type over $V$. They are $\downarrow^w$-generic over every $U \subseteq V$ with $\bar{b} \in \operatorname{dcl}^{\mathrm{eq}}(U)$. Furthermore if $\varphi(\bar{x}, \bar{y}) \in \mathcal{X}$, $U \leq M$, $\bar{b} \in \operatorname{dcl}^{\mathrm{eq}}(B)$, and $\bar{e}_0, \bar{e}_1, \ldots$ are solutions of $\varphi(\bar{x}, \bar{b})$ linearly independent over $B$ with $\bar{e}_i \not\subseteq \langle U, B, \bar{e}_0, \ldots, \bar{e}_{i-1}\rangle^\ell$, then there are at most $\operatorname{l.dim}(B/U)$ many $i$ such that $e_i$ is not $\downarrow^w$-generic over $\langle U, B, \bar{e}_0, \ldots, \bar{e}_{i-1}\rangle^\ell$.

In P(V) it is possible to replace $\operatorname{l.dim}(B/U)$ by a fixed function $f(\operatorname{l.dim}(B/U))$ if this is necessary. P(V) implies that a $\downarrow^w$-generic solution of $\varphi(\bar{x}, \bar{b})$ over $V$ is linearly independent over $V$ as a $\downarrow$-generic solution.

P(VI) Assume $C \supseteq B \subseteq A$ are strong subspaces of $R(M)$ linearly independent over $B$ and both minimal strong extensions of $B$ given by generic solutions of formulas in $\mathcal{X}$. If $b \in \operatorname{dcl}^{\mathrm{eq}}(E)$, $E \subseteq A + C$, and there is a solution $\bar{a} \in A + C$ of some $\varphi(x, \bar{b})$ in $\mathcal{X}$ $\downarrow^w$-generic over $C + E$ and over $A + E$, then $\varphi(\bar{x}, \bar{b})$ defines a torsor set. If it defines a group set, then $\bar{b}$ is in $\operatorname{dcl}^{\mathrm{eq}}(B)$.

Note that the assumptions in P(VI) imply that $A + C$ is strong in $R(M)$ and it is the non-forking amalgam of $A$ and $C$ over $B$.

Since we assume $M = \langle R(M)\rangle$ for all $M \models T$, there exists a quantifier-free disjunction $\chi(\bar{x}, y)$ of formulas that describe $\langle R(M)\rangle$ over $R(M)$ such that

(*) $\qquad M = \{b : \text{ there exists } \bar{a} \text{ in } R(M) \text{ with } M \vDash \chi(\bar{a}, b)\}.$

We want that the substructure $P^{\mu}$, that we will construct, will also satisfy (*) for some suitable $\chi$. In the examples that we consider we have either $M = R(M)$ ([BMPZ4]), or (*) for all substructures $H$ with $\mathrm{acl}(R(H)) \cap R(M) = R(H)$ and $H = \langle R(H) \rangle$ ([Bau2]), or some formula in $\mathcal{X}$ that provides the existence of $\chi(\bar{x}, y)$ ([BMPZ3]). Hence we suppose:

P(VII) Either $M = R(M)$ and therefore connected,

or $M$ is connected and there is a quantifier free formula $\theta(\bar{x}, y)$ in $\mathcal{X}$ such that for every $B \subseteq R(M)$ and every tuple $\bar{a}$ of geometrically independent generics over $B$ in $R(M)$ $\quad M \vDash \theta(\bar{a}, b)$ implies that the canonical parameter $b$ is a generic of $M$ over $B$ and for all $\bar{a}$ $b \in \mathrm{dcl}(\bar{a})$,

or for every substructure $H \subseteq M \vDash T$ with $\mathrm{acl}(R(H)) \cap R(M) = R(H)$ and $\langle R(H) \rangle = H$ we have some quantifier free definable function $\eta(\bar{x}) = y$ such that

$$H = \{b : M \vDash \eta(\bar{a}) = b \text{ for some } \bar{a} \text{ in } R(H)\}.$$

**Definition** A countable theory with the properties P(I) – P(VII) is called a starting theory for the red collapse.

We will show that a suitable substructure of $\mathbb{C} \vDash T$ has the wanted theory of finite Morley rank.

Note that P(II) implies that $\mathrm{acl}(A) \leq M$. From P(IV) follows:
If $a \in \mathrm{cl}_d(B)$, then there are $B \subseteq A_0 \subseteq A_1 \subseteq \ldots \subseteq A_n$ such that $A_0 \subseteq \mathrm{acl}(B)$, $a \in A_n$, $A_i \leq M$, $A_{i+1} \subseteq \mathrm{acl}(A_i)$ or there is some $\bar{b} \in \mathrm{dcl}^{\mathrm{eq}}(A_i)$ and some generic solution $\bar{a}$ of some $\varphi(\bar{x}, \bar{b}) \in \mathcal{X}$ with $A_{i+1} = \langle A_i, \bar{a} \rangle^{\ell}$.

In the last case we call $A_{i+1}$ a prealgebraic minimal extension of $A_i$. Our aim is to make it algebraic in the substructure. Note that non-generic solutions of $\varphi(\bar{x}, \bar{b})$ are in $A_i$.

Let $T$ be as above, $M$ and $N$ are $T$-models, $\bar{a}$ is a tuple in $R(M)$ and $f(\bar{a})$ is an isomorphic copy of $\bar{a}$ in $R(N)$ in the language of $\mathbb{F}_q$-vectorspaces. Then we want to show that P(I) – P(IV) implies:

$$\mathrm{tp}^{M}(\bar{a}) = \mathrm{tp}^{N}(f(\bar{a}))$$

if and only if $f$ preserves a geometrical sequence for $\bar{a}$.

First we have to define the notions of geometrical sequence and construction:

**Definition** Assume $B \leq \mathbb{C}$, $B \subseteq A$, and $A \leq \mathbb{C}$. A geometrical sequence of $A$ over $B$ is a sequence $A_0 \subseteq A_1 \subseteq \ldots \subseteq A_m$ where $B = A_0$, $A = A_m$, and all $A_i$ are strong in $M$. Furthermore $A_{i+1}$ is a minimal strong extension of $A_i$ in the following sense:

1. Transcendental Case: $A_{i+1} = \langle A_i, a\rangle^\ell$, and $a$ is geometrically independent from $A_i$.

2. Algebraic Case: $A_{i+1} = \langle A_i, a\rangle^\ell$ where $a$ is in the algebraic closure of $A_i$ linearly independent over $A_i$.

3. Prealgebraic Case: $A_{i+1} = \langle A_i, \bar{c}\rangle^\ell$ where $\bar{c}$ is a solution of some $\varphi(\bar{x}, \bar{b}) \in \mathcal{X}$ generic over $A_i$ where $\bar{b} \in \mathrm{dcl}^{\mathrm{eq}}(A_i)$.

If case 1 does not occur we speak about a geometrical construction. In this case $A \subseteq \mathrm{cl}_d(B)$.

**Definition** The geometrical sequence above is a geometrical sequence for $\bar{a}$, if $A_0$ is the linear hull of geometrically independent elements from $\bar{a}$ and in all transcendental cases $a \in \langle \bar{a}\rangle^\ell$.

To obtain a geometrical sequence for $\bar{a}$ in $R(M)$ there is some $A \leq M$ with $\bar{a} \subseteq A \subseteq \mathrm{acl}(\bar{a})$ by P(II). Choose $A_0 = \langle 0\rangle^\ell$ and $A_i$ for $i \leq i_0$ by transcendental steps with $a \in \langle \bar{a}\rangle^\ell$ such that $\mathrm{l.dim}(A_{i_0}) = d(A_{i_0}) = d(\bar{a})$. By P(II) $A_{i_0} \leq \mathbb{C}$ and $A \subseteq \mathrm{cl}_d(A_{i_0})$. By P(IV) there is a geometrical construction of $A$ over $A_{i_0}$. This gives a geometrical sequence of $\bar{a}$ over $A_0 \subseteq \bar{a}$.

**Definition** Let $A_0 \subseteq A_1 \subseteq \ldots \subseteq A_m \subseteq M \vDash T$ be a geometrical sequence for $\bar{a}$. Let $f$ be an $\mathbb{F}_q$-vectorspace embedding of $\langle \bar{a}\rangle^\ell$ into $R(N)$, where $N \vDash T$, and $\bar{f}$ be an $\mathbb{F}_q$-vectorspace embedding of $A_m$ into $R(N)$ that extends $f$. Then $\bar{f}$ preserves the given geometrical sequence $A_0 \subseteq A_1 \subseteq \ldots \subseteq A_m$ of $\bar{a}$ over $A_0$ if we have $\bar{f}(A_0)$ is $f(A_0)$,
in the transcendental case $\bar{f}(A_{i+1}) = \langle \bar{f}(A_i), \bar{f}(a)\rangle^\ell$ where $\bar{f}(a)$ is geometrically independent from $\bar{f}(A_i)$,
in the algebraic case $\bar{f}(a)$ fulfils the image of the isolating formula in $M$, and
in the prealgebraic case $\bar{f}(\bar{a})$ is a solution of $\varphi(\bar{x}, \bar{f}(\bar{b}))$ generic over $\bar{f}(A_i)$ where $\bar{f}(\bar{b}) \in \mathrm{dcl}^{\mathrm{eq}}(\bar{f}(A_i))$ is isolated by the image of the corresponding formula for $M$.

**Lemma 3.1** *Let $T$ be a theory with* P(I) – P(IV). *We consider models $M$ and $N$ of $T$. Let $A_0 \subseteq \ldots \subseteq A_m \subseteq R(M)$ be a geometrical sequence over $A_0$ as defined above and $g$ be an $\mathbb{F}_q$-vectorspace embedding of $A_m$ into $R(N)$ that preserves the geometrical sequence.*
*If* $\mathrm{tp}^M(A_0) = \mathrm{tp}^N(g(A_0))$, *then* $\mathrm{tp}^M(A_m) = \mathrm{tp}^N(g(A_m))$.

*Proof*. We show $\mathrm{tp}^M(A_i) = \mathrm{tp}^N(g(A_i))$ by induction on $i \leq m$.
We have $\mathrm{tp}^M(A_0) = \mathrm{tp}^N(g(A_0))$ by assumption. Assume $\mathrm{tp}^M(A_i) = \mathrm{tp}^N(g(A_i))$. By P(II) $g(A_i)$ is strong in $N$. For $A_{i+1}$ we have the three cases in the definition.

*Case* 1 Transcendental Case
$A_{i+1} = \langle A_i, a\rangle^\ell$ where $a$ is geometrically independent over $A_i$. Then $g(A_{i+1}) = \langle g(A_i), g(a)\rangle^\ell$ where $g(a)$ is geometrically independent over $g(A_i)$. By P(II) $\mathrm{tp}^M(A_{i+1}) = \mathrm{tp}^N(g(A_{i+1}))$.
*Case* 2 Algebraic Case
$A_{i+1} = \langle A_i, a\rangle^\ell$, $a$ algebraic over $A_i$. By definition $g(a)$ is algebraically isolated over $g(A_i)$ by the $g$-image of an isolating formula for $a$ over $A_i$. Hence $\mathrm{tp}^M(A_{i+1}) = \mathrm{tp}^N(g(A_{i+1}))$.
*Case* 3 Prealgebraic Case
$A_{i+1} = \langle A_i, \bar{a}\rangle^\ell$ where $\bar{a}$ is a solution of some $\varphi(\bar{x}, \bar{b}) \in \mathcal{X}$ generic over $A_i$ and $\bar{b} \in \mathrm{dcl}^{\mathrm{eq}}(A_i)$. By definition of preservation $g(\bar{a})$ is a solution of $\varphi(\bar{x}, g(\bar{b}))$. By induction $\mathrm{tp}^M(\bar{b}) = \mathrm{tp}^N(g(\bar{b}))$. Hence $\varphi(\bar{x}, g(\bar{b}))$ is strongly minimal. Since $g(A_i) \leq N$ and $g(\bar{a}) \not\subseteq g(A_i)$ the solution $g(a)$ is generic over $g(A_i)$. We get $\mathrm{tp}^M(\langle A_i, \bar{a}\rangle^\ell) = \mathrm{tp}^N(\langle g(A_i), g(\bar{a})\rangle^\ell)$. □

**Lemma 3.2** *Let $A_0 \subseteq A_1 \subseteq \ldots \subseteq A_m$ be a geometrical sequence of $\bar{a}$ in $\mathbb{C}$. Assume $\bar{a} \in M \preceq \mathbb{C}$. Then there is a geometrical sequence $f(A_0) \subseteq \ldots \subseteq f(A_m)$ of $\bar{a}$ in $M$ such that $f$ preserves the geometrical construction.*

*Proof*. By definition $A_0 \subseteq \bar{a} \subseteq M$ and in each transcendental case $A_{i+1} = \langle A_i, a\rangle^\ell$ we have $a \in \bar{a} \subseteq M$. We define $f(a) = a$ for $a$ in $\bar{a}$. Then $f(A_0) = A_0$. We define $f(A_i)$ by induction on $i$ such that $f$ preserves the geometrical sequence $A_0 \subseteq A_0 \subseteq \ldots \subseteq A_i$, and $\mathrm{tp}^M(f(A_i)\bar{a}) = \mathrm{tp}^{\mathbb{C}}(A_i\bar{a})$. We have $f(A_0) = A_0$ is part of $\bar{a}$. Now we prove the induction step.

1. Transcendental Case $A_{i+1} = \langle A_i a\rangle^\ell$ and $a$ is geometrically independent from $A_i$. Then $a \in \bar{a}$ and $a$ is geometrically independent from $f(A_i)$ since $\mathrm{cl}_d(A_i) = \mathrm{cl}_d(f(A_i))$. Then $f(A_{i+1}) = \langle f(A_i)\bar{a}\rangle^\ell$ fulfils the assertion.

2. Algebraic Case $A_{i+1} = \langle A_i a\rangle^\ell$ and $a$ is algebraic over $A_i$. If $a \in \langle A_i \bar{a}\rangle^\ell$, then $f(a)$ is already defined. Otherwise there is an algebraic formula $\psi(x, \bar{d})$ isolating $a$ over $\langle A_i \bar{a}\rangle^\ell$. Let $f(a) = c$ for some $c \in M$ with $M \vDash \psi(c; f(\bar{d}))$.
3. Prealgebraic Case $A_{i+1} = \langle A_i, \bar{e}\rangle$ where $\bar{e}$ is a solution of some $\varphi_\alpha(\bar{x}, \bar{b})$ in $\mathcal{X}$ generic over $A_i$. If $\bar{e}$ is algebraic over $\langle A_i \bar{a}\rangle^\ell$ we proceed as above in the algebraic case. Otherwise $f(\bar{e})$ is any solution of $\varphi_\alpha(\bar{x}, f(\bar{b}))$ in $M$ generic over $\langle f(A_i)\bar{a}\rangle^\ell$. Such a solution exists by P(II) and P(III). $\square$

**Lemma 3.3** *Let $T$ be a theory with* P(I) – P(IV). *Assume $M, N \vDash T$, $\bar{a}$ is in $M$, and $f(\bar{a})$ is an $\mathbb{F}_q$-vectorspace-isomorphic copy of $\bar{a}$ in $N$. Then the following are equivalent:*

i) $\mathrm{tp}^M(\bar{a}) = \mathrm{tp}^N(f(\bar{a}))$.

ii) *Given a geometrical sequence for $\bar{a}$ we can extend $f$ in such a way that the extension $\bar{f}$ preserves the geometrical sequence.*

iii) *Some geometrical sequence for $\bar{a}$ is preserved by an extension $\bar{f}$ of $f$.*

*Proof*. Lemma 3.1 gives us iii) $\Rightarrow$ i), ii) $\Rightarrow$ iii) is trivial.
Finally we show i) $\Rightarrow$ ii). We assume that $\mathrm{tp}^M(\bar{a}) = \mathrm{tp}^N(f(\bar{a}))$. To show ii) let $A_0 \subseteq A_1 \subseteq \ldots \subseteq A_m$ be a geometrical sequence for $\bar{a}$. We consider $M$ and $N$ as elementary substructures of the monster model $\mathbb{C}$. Then we can extend $f$ to an automorphism $g$ of $\mathbb{C}$. We get that $g$ preserves the given geometrical sequence in $\mathbb{C}$. By Lemma 3.2 we obtain the desired geometrical sequence in $N$. $\square$

**Lemma 3.4** *If $\bar{a} \in \mathrm{cl}_d(A)$, then there is a geometrical sequence for $\langle A, \bar{a}\rangle^\ell$ inside* $\mathrm{acl}(A, \bar{a})$.

*Proof*. Choose $A_0 \subseteq A$ with $\mathrm{l.dim}(A_0) = d(A_0) = d(A)$. Then $A_0 \leq M$ by P(II). Again by P(II) there is some $A'$ with $\langle A, \bar{a}\rangle^\ell \subseteq A' \subseteq \mathrm{acl}(A, \bar{a})$ and $A' \leq M$. By P(IV) there is a geometrical construction of $A'$ over $A_0$. $\square$

**Lemma 3.5** *If $c \notin \mathrm{cl}_d(B)$, $a \in \mathrm{cl}_d(B)$, and $a \in \mathrm{acl}(Bc)$, then $a \in \mathrm{acl}(B)$.*

*Proof*. We assume $c_0 \notin \mathrm{cl}_d(B)$, $a_0 \in \mathrm{cl}_d(B) \setminus \mathrm{acl}(B)$ and $a_0 \in \mathrm{acl}(Bc_0)$ and show a contradiction.
Assume $\mathbb{C} \vDash \psi(a_0, c_0, \bar{b}) \wedge \exists^{\leq n} y \psi(y, c_0, \bar{b})$ where $\bar{b}$ is in $B$. Then

$$x \notin \mathrm{cl}_d(B) \cup \{\psi(a_0, x, \bar{b}) \wedge \exists^{\leq n} y \psi(y, x, \bar{b}\}$$

is consistent. But by P(II) there is a unique type $p(x)$ with $x \notin \mathrm{cl}_d(B)$ over $\mathrm{cl}_d(B)$.

(+) Hence every element $d \notin \mathrm{cl}_d(B)$ fulfils $p(x)$ and we have $\psi(a_0, d, \bar{b})$.

Let $M \preceq \mathbb{C}$ be a model that contains $Bc_0$ and $a_0$. Since $\mathrm{acl}(B) \cap \langle c_0, a_0\rangle^\ell = \langle 0\rangle^\ell$ there are $c_1$ and $a_1$ such that $\langle c_1, a_1\rangle^\ell \cap M = \langle 0\rangle^\ell$ with $\mathrm{tp}(c_1a_1/B) = \mathrm{tp}(c_0a_0/B)$ and $c_1$ realizes $p$. If we apply (+), then we get $\models \psi(a_0, c_1, \bar{b})$. If we continue in this way we get more than $n$ solutions of $\psi(y, c_{n+1}, \bar{b})$ and $\exists^{\leq n} y\, \psi(y, c_{n+1}, \bar{b})$, a contradiction. □

**Lemma 3.6** *Assume $U \subseteq R(M)$, $\mathrm{acl}(U) = U$, $A \leq M$, and $d(A/U) = d(A/U \cap A)$. Then*

i) *$A \cap U \leq M$, and*

ii) *$U + A \leq M$.*

*Proof*. Let $M$ be sufficiently saturated. Note that $\mathrm{acl}(U) = U$ implies $U \leq M$. First we show that ii) is a consequence of i). Using i), P(II), and P(IV) we get a geometrical sequence $A \cap U = B_0 \subseteq B_1 \subseteq \ldots \subseteq B_k = A$. We show by induction on $i$ that $U + B_i \leq M$. We have $U + B_0 = U \leq M$. If $B_{i+1} = \langle B_i, b\rangle^\ell$ where $b$ is algebraic over $B_i$, then either $b \in U + B_i$ or $\langle UB_ib\rangle^\ell \leq M$ by P(II). In the prealgebraic case the assertion follows from $U + B_i \leq M$ and P(III). In the transcendental case $B_{i+1} = \langle B_ib\rangle$ with $b \notin \mathrm{cl}_d(B_i)$ we have $b \notin \mathrm{cl}_d(B_i + U)$ since $d(A/U) = d(A/B_0)$. By P(II) $B_{i+1} + U \leq M$.
To show i) let $A_0 \subseteq A \cap U$ be such that $\mathrm{l.dim}(A_0) = d(A_0) = d(A \cap U)$. By P(IV) we get a geometrical sequence $A_0 \subseteq A_1 \subseteq \ldots \subseteq A_m = A$. By the choice of $A_0$ and by $d(A/U) = d(A/U \cap A)$ we have that

(++) $A_{i+1} = \langle A_ic\rangle^\ell$ with $c \notin \mathrm{cl}_d(A_i)$ implies $c \notin \mathrm{cl}_d(A_i + U)$.

We show by induction on $\mathrm{l.dim}(A)$ that we can choose the geometrical sequence for $A$ over $A_0$ in such a way that there is some $i_0$ with $A_{i_0} = A \cap U$.
We start with $\mathrm{l.dim}(A) = 0$. To prove the induction step we assume that we have $A_0 \subseteq A_1 \subseteq \ldots \subseteq A_m$ and $i_0$ such that $A_{m-1} \cap U = A_{i_0} \leq M$. Assume $m > 0$.

1. Algebraic Case: $A_m = \langle A_{m-1}, a\rangle^\ell$ where $a$ is in the algebraic closure of $A_{m-1}$ linearly independent from $A_{m-1}$.

If $A_m \cap U \neq A_{i_0}$, then we can choose $a$ in such a way that $a \in U$. Assume $i_0 \leq m-2$, since otherwise we are done.

If $a \in \mathrm{acl}(A_{m-2})$, then we apply the induction to $A_0 \subseteq \ldots \subseteq A_{m-2} \subseteq \langle A_{m-2}, a\rangle^\ell$. We get $A_0 = A'_0 \subseteq \ldots \subseteq A'_{j_0} \subseteq \ldots \subseteq A'_{m'-1} = \langle A_{m-2}, a\rangle^\ell$ where $A'_{m'-1} \cap U = A'_{j_0} \leq M$. Then $A'_0 \subseteq \ldots \subseteq A'_{m'-1} \subseteq A_m$ is the desired sequence with $A_m \cap U = A'_{j_0} \leq M$.

Now we assume $a \in \mathrm{acl}(A_{m-1}) \setminus \mathrm{acl}(A_{m-2})$.

First we assume $A_{m-1} = \langle A_{m-2} c\rangle^\ell$, where $c \notin \mathrm{cl}_d(A_{m-2})$. Since $a \in \mathrm{acl}(\langle A_{m-2}, c\rangle^\ell) \setminus \mathrm{acl}(A_{m-2})$ we have $a \notin \mathrm{cl}_d(A_{m-2})$ by Lemma 3.5. Hence by the Exchange-Property for $\mathrm{cl}_d$ we get $c \in \mathrm{cl}_d(A_{m-2}a) \subseteq \mathrm{cl}_d(A_{m-2} + U)$ a contradiction to $(++)$.

Finally consider $A_{m-1} = \langle A_{m-2}, \bar{c}\rangle^\ell$ where $\bar{c}$ is a solution of some $\varphi(\bar{x}, \bar{b}) \in \mathcal{X}$ where $\bar{b} \in \mathrm{dcl}^{\mathrm{eq}}(A_{m-2})$ and $\bar{c}$ is a generic solution over $A_{m-2}$. Hence $a \in \mathrm{acl}(A_{m-2}, \bar{c}) \setminus \mathrm{acl}(A_{m-2})$. By the Exchange Lemma for acl in strongly minimal sets $\bar{c} \in \mathrm{acl}(A_{m-2}, a)$. By induction and ii) $U + A_i \leq M$ for $i \leq m-2$. By P(III) $\bar{c} \subseteq U + A_{m-2}$. Since $\bar{c}$ is linearly independent over $A_{m-2}$ we get $A_{m-1} \cap U \neq A_{i_0}$ a contradiction to our induction assumption.

2. Prealgebraic Case: $A_m = \langle A_{m-1}, \bar{a}\rangle^\ell$ where $\bar{a}$ is a solution of $\varphi(\bar{x}, \bar{b}) \in \mathcal{X}$ generic over $A_{m-1}$ where $\bar{b} \in \mathrm{dcl}^{\mathrm{eq}}(A_{m-1})$. By induction and ii) we have again $U + A_{m-1} \leq M$. If $A_m \cap U \neq A_{i_0}$, then some element in $\langle \bar{a}\rangle^\ell \setminus \langle 0\rangle^\ell$ is in $U + A_{m-1}$. By $U + A_{m-1} \leq M$ and P(III) we get $\bar{a} \subseteq U + A_{m-1}$. Since $\mathcal{X}$ is closed under affine transformations we can assume w.l.o.g. that $\bar{a}$ is in $U$. If $i_0 = m-1$ we are done.

Assume $i_0 < m-1$. There is some $s$ with $i_0 < s < m-1$ such that $A_{m-1} \subseteq \mathrm{acl}(A_s)$ and $A_s$ is a transcendental or a prealgebraic extension of $A_{s-1}$. Note that $A_{m-1} \subseteq \mathrm{acl}(A_{i_0})$ is impossible since $\mathrm{acl}(U) = U$ would imply $A_{m-1} \subseteq U$.

First we assume that $A_s = \langle A_{s-1} c\rangle$ and $c \notin \mathrm{cl}_d(A_{s-1})$ and $A_{m-1} \subseteq \mathrm{acl}(A_s)$. Then $\bar{a} \in \mathrm{cl}_d(A_{s-1})$ since otherwise $\bar{a} \in \mathrm{cl}_d(A_{s-1}c) \setminus \mathrm{cl}_d(A_{s-1})$ and therefore $c \in \mathrm{cl}_d(A_{s-1}\bar{a}) \subseteq \mathrm{cl}_d(A_{s-1} + U)$, a contradiction to $(++)$.

In the next step we show that $\bar{b} \in \mathrm{dcl}^{\mathrm{eq}}(\mathrm{cl}_d(A_{s-1}))$. Let $f$ be an automorphism that fixes $\mathrm{cl}_d(A_{s-1})$ pointwise. We show $f(\bar{b}) = \bar{b}$. By the following argument we can restrict us to the case where $c$ and $f(c)$ are geometrically independent over $A_{s-1}$. If $c$ and $f(c)$ are not geometrically independent over $A_{s-1}$, then we choose $f_1$ such that $f_1(c) \notin \mathrm{cl}_d(A_{s-1}, c)$ and consider $f_1$ and $f_2 = f f_1^{-1}$. Then $c$ and $f_1(c)$ are geometrically independent over $A_{s-1}$ and also $f(c)$ and $f_1(c)$. Furthermore $f_2(f_1(c)) = f(c)$. Hence we assume w.l.o.g. that $c$ and $f(c)$ are geometrically independent over $A_{s-1}$. Then $\langle A_{s-1}, c, f(c)\rangle^\ell \leq M$. We have $f(\bar{a}) = \bar{a}$ since we have shown that $\bar{a} \in \mathrm{cl}_d(A_{s-1})$. Since $\bar{a} \not\subseteq \langle A_{s-1}, c, f(c)\rangle^\ell$ it is a common solution of $\varphi(\bar{x}, \bar{b})$ and $\varphi(\bar{x}, f(\bar{b}))$ generic over $\langle A_{s-1}, c, f(c)\rangle^\ell$ by P(III). Hence $f(\bar{b}) = \bar{b}$ by P(III). Finally we show $\bar{b} \in \mathrm{dcl}^{\mathrm{eq}}(A_{s-1})$. Let $f$ be an automorphism of $\mathbb{C}$ that fixes $A_{s-1}$ pointwise. Since $\bar{b} \in \mathrm{dcl}^{\mathrm{eq}}(A_{m-1})$ and $A_{m-1} \subseteq \mathrm{acl}(A_{s-1}, c)$, we have that $\bar{b} \in \mathrm{dcl}^{\mathrm{eq}}(A_{s-1}, c)$. Let $g(x)$ be a function definable with parameters in $A_{s-1}$ such that $g(c) = \bar{b}$. Since $\mathrm{tp}(c/\mathrm{cl}_d(A_{s-1})) = \mathrm{tp}(f(c)/\mathrm{cl}_d(A_{s-1}))$ and $\bar{b} \in \mathrm{dcl}^{\mathrm{eq}}(\mathrm{cl}_d(A_{s-1}))$, as shown above, we get $g(c) = \bar{b} = g(f(c)) = f(\bar{b})$, as desired.

Now we can apply the induction to $A_0 \subseteq \ldots \subseteq A_{s-1} \subseteq \langle A_{s-1}, \bar{a}\rangle^\ell$ and can use this to prove the assertion.

Next we assume $A_s = \langle A_{s-1}, \bar{c}\rangle^\ell$ is a prealgebraic extension and $A_{m-1} \subseteq \mathrm{acl}(A_s)$. W.l.o.g. we can assume that $A_{m-1} \cap \mathrm{acl}(A_{s-1}) = A_{s-1}$. Since $U + A_{s-1} \leq M$, $\bar{c}$ is a solution of some $\psi(\bar{x}, \bar{d})$ in $\mathcal{X}$ generic over $U + A_{s-1}$, where $\bar{d} \in \mathrm{dcl}^{\mathrm{eq}}(A_{s-1})$. By assumption $A_{m-1} \cap (U + A_{s-1}) = A_{s-1}$. Let $\{f_i(\bar{c}) : i < \omega\}$ be a Morley sequence of $\mathrm{tp}(\bar{c}/U + A_{s-1})$. We can speak about a Morley-sequence, if we use the independence in the strongly minimal set $\psi(\bar{x}, \bar{d})$. Note $A_{m-1} \subseteq \mathrm{acl}(A_{s-1}\bar{c})$. We can consider $f_i$ as an automorphism that fixes $U + A_{s-1}$ pointwise. Assume $f_0 = \mathrm{id}$. Furthermore $A_{s-1} + \bigoplus_{i<\omega} f_i(\bar{c}) \leq M$ and $\bar{a}$ is a solution of all $\varphi(\bar{x}, f_i(\bar{b}))$ for $i < \omega$ linearly independent over this strong subspace and therefore generic over this space. It follows $f_i(\bar{b}) = \bar{b}$. Since $f_i(\bar{b}) = \bar{b}$ is in $\mathrm{acl}^{\mathrm{eq}}(A_{s-1}, f_i(\bar{c}))$ we get $f_i(\bar{c}) \in \mathrm{acl}(A_{s-1}, \bar{b})$ by the Exchange Lemma for $\psi(\bar{x}, \bar{d})$. Then $\bar{b} \in \mathrm{acl}^{\mathrm{eq}}(A_{s-1}, \bar{c})$ implies $f_i(\bar{c}) \in \mathrm{acl}^{\mathrm{eq}}(A_{s-1}, \bar{c})$. This contradicts the construction of our Morley-sequence.

3. Transcendental Case: $A_m = \langle A_{m-1}, a\rangle^\ell$ where $a \notin \mathrm{cl}_d(A_{m-1})$. By (++) we get $a \notin \mathrm{cl}_d(A_{m-1} + U)$. Hence $A_m \cap U = A_{i_0}$ as desired. $\square$

**Lemma 3.7** *Let $T$ be a countable theory that satisfies the conditions* P(I) – P(IV). *Then $T$ is $\omega$-stable. Furthermore the subspace $R(x)$ is connected.*

*Proof*. Since $\langle R(M)\rangle = M$ for $M \models T$ it is sufficient to count $\mathrm{tp}(\bar{a}/M)$ where $\bar{a}$ is in $R(\mathbb{C})$ and $M$ countable. We choose a geometrical sequence $A_0 \subseteq A_1 \subseteq \ldots \subseteq A_m$ with $d(A_m/M) = d(A_m/M \cap A_m)$ and $\bar{a} \subseteq A_m$. By Lemma 3.6 $A_m \cap M \leq \mathbb{C}$. Hence w.l.o.g. $A_0 = A_m \cap M$. By Lemma 3.1 $\mathrm{tp}(A_m/A_0)$ is given by the geometrical sequence. The same remains true if we replace $A_0$ by a larger strong subspace in $R(M)$. Hence $\mathrm{tp}(A_m/M)$ is uniquely determined by the geometrical sequence $A_0 \subseteq A_1 \subseteq \ldots \subseteq A_m$. If there is a preservation map between two such geometrical sequences over $A_0$, then they present the same type over $M$. There are only countably many $A_0$ in $M$ and countably many geometrical sequences $A_0 \subseteq \ldots \subseteq A_m$ with $A_0 = A_m \cap M$ and $d(A_m/M) = d(A_m/A_0)$ up to preservation. Hence there are only countably many types over $M$.

Note that $a \notin \mathrm{cl}_d(M)$ gives the only generic 1-type in $R(\mathbb{C})$ over $M$. Hence $R(\mathbb{C})$ is connected. $\square$

**Lemma 3.8** *Assume $A_0 \leq \mathbb{C}$ and $\bar{a} \in \mathrm{cl}_d(A_0)$. Let $A_0 \subseteq A_1 \subseteq \ldots \subseteq A_m$ be a geometrical construction for $\bar{a}$ over $A_0$. Then there is a geometrical construction for $\bar{a}$ over $A_0$ inside of $A_m \cap \mathrm{acl}(A_0 \cup \{\bar{a}\})$.*

*Proof*. Let $U$ be $\mathrm{acl}(A_0 \cup \{\bar{a}\})$. By definition $A_m \leq \mathbb{C}$ and $d(A_m/U) = 0 = d(A_m/U \cap A_m)$ since $A_0 \subseteq U \cap A_m$. By Lemma 3.6 $A_m \cap U = D \leq \mathbb{C}$. By P(IV) there is the desired geometrical construction of $D$ over $A_0$. $\square$

Note that by Lemma 3.4 there is a geometrical construction for $\bar{a}$ over $A_0$ inside $\mathrm{acl}(A_0 \cup \{\bar{a}\})$.

**Lemma 3.9** *Let $A_0 \subseteq A_1 \subseteq \ldots \subseteq A_m$ be a geometrical sequence in $\mathbb{C}$. If $A_0 \subseteq U$, $U \leq \mathbb{C}$, $U \cap A_m = A_0$, and $d(A_m/A_0) = d(A_m/U)$. Then $A_i + U$ is a geometrical sequence. Hence $U + A_m$ is strong in $\mathbb{C}$.*

*Proof*. We use induction on $i$ and start with $A_0 + U = U \leq \mathbb{C}$. If we have the transcendental case $A_{i+1} = \langle A_i a \rangle^{\ell}$ with $a \notin \mathrm{cl}_d(A_i)$, then $a \notin \mathrm{cl}_d(A_i + U)$ and $A_{i+1} + U \leq \mathbb{C}$, as desired. In the algebraic case $\langle A_i a \rangle^{\ell} + U = A_{i+1} + U = A_i + U$ or $a$ is algebraic over $A_i + U \leq \mathbb{C}$. Hence $A_{i+1} + U \leq \mathbb{C}$. The prealgebraic case uses P(III) and the induction hypothesis $A_i + U \leq \mathbb{C}$. $\square$

**Lemma 3.10** *Assume $U \leq \mathbb{C}$ and $B$ finite. Then there is some $V$ with $U + B \subseteq V$, $V \leq \mathbb{C}$, $\mathrm{l.dim}(V/U)$ finite and $d(V/U) = d(U + B/U)$.*

*Proof*. Choose $B_0 \subseteq B$ such that $\mathrm{l.dim}(B_0/U) = d(B_0/U) = d(B/U)$. By P(II) $U + B_0 \leq \mathbb{C}$. Extend $B_0$ to $A_0$ with $A_0 = (A_0 \cap U) + B_0$, $d(A_0) = \mathrm{l.dim}(A_0)$ and $B \subseteq \mathrm{cl}_d(A_0)$. Note $A_0 \leq \mathbb{C}$ and $U + A_0 \leq \mathbb{C}$ by P(II). Choose a geometrical construction $A_0 \subseteq A_1 \subseteq \ldots \subseteq A_m$ such that $B \subseteq A_m$. There are only algebraic and prealgebraic steps. As in the proof of Lemma 3.9 we can show by induction, that $A_i + U \leq \mathbb{C}$. Then $A_m + U$ is the desired subspace $V$. $\square$

Similarly as in the finite case we define

**Definition** Let $D \subseteq D'$ be strong subspaces of $R(\mathbb{C})$ with $\mathrm{l.dim}(D'/D)$ finite. A geometrical sequence for $D'$ over $D$ is a sequence $D = D_0 \subseteq D_1 \subseteq \ldots \subseteq D_m = D'$ where $D_i \leq \mathbb{C}$ and for $D_{i+1}$ over $D_i$ we have one of the following cases:

1. *Transcendental minimal extension* $D_{i+1} = \langle D_i, a\rangle^\ell$ and $a \notin \mathrm{cl}_d(D_i)$.
2. *Algebraic minimal extension* $D_{i+1} = \langle D_i, a\rangle^\ell$ and $a \in \mathrm{acl}(D_i)$.
3. *Prealgebraic minimal extension* $D_{i+1} = \langle D_i, \bar{a}\rangle^\ell$, where $\bar{a}$ is a solution of some $\varphi(\bar{x}, \bar{b})$ in $\mathcal{X}$ generic over $D_i$ and $\bar{b} \in \mathrm{dcl}^{\mathrm{eq}}(D_i)$.

**Lemma 3.11** *Let $D \subseteq D'$ be strong subspaces of $R(\mathbb{C})$ where* $\mathrm{l.dim}(D'/D)$ *is finite. Then there is a geometrical sequence for $D'$ over $D$.*

*Proof..* Choose $A \leq D'$, $A \leq \mathbb{C}$ such that $D + A = D'$ and $d(D'/D) = d(A/A \cap D)$. By Lemma 3.6 $A_0 = \mathrm{acl}(D) \cap A$ is strong. By P(II) and P(IV) there is a geometrical sequence $A_0 \subseteq A_1 \subseteq \ldots \subseteq A_m = A$. Note $d(A_m/A_0) = d(A_m/D)$. By P(II) there is a geometrical sequence for $D + A_0$ over $D$. By Lemma 3.9 there is a geometrical sequence $D + A_0 \subseteq D + A_1 \subseteq \ldots \subseteq D + A$. We combine the two sequences to get the desired one. $\square$

**Lemma 3.12** *Assume $T$ satisfies* P(I) – P(IV). *Geometrical independence implies non-forking independence in $R(\mathbb{C})$.*

# 4 Codes and difference sequences

In this chapter we assume that $T$ satisfies P(I) – P(IV) as defined in Chapter 3. Let $\mathbb{C}$ be the monster model. We work in $T^{\mathrm{eq}}$. Many notions and proofs in this chapter are taken from [BMPZ3]. But we work in a different context. In [BMPZ3] $T$ is the theory of an algebraically closed field of characteristic $p > 0$. It is a reduct of the final theory. For the construction of red fields of finite Morley rank in this paper the considered theory $T$ is already the theory of such a field with a (red) additive subgroup. It is obtained by an amalgamation procedure (see [Po2]) and has infinite Morley rank.

**Lemma 4.1** a) *Let $\varphi(\bar{x}, \bar{y})$ and $\psi(\bar{x}_0, \ldots, \bar{x}_\mu, \bar{y})$ be formulas where $\bar{x}$ and $\bar{x}_i$ are in the home sort. Assume that $\varphi(\bar{x}, \bar{b})$ is strongly minimal where $\bar{b}$ is in $\mathbb{C}^{\mathrm{eq}}$. Then we can express that any Morley sequence $\bar{a}_0, \ldots, \bar{a}_\mu$ of $\varphi(\bar{x}, \bar{b})$ fulfils $\models \psi(\bar{a}_0, \ldots, \bar{a}_\mu, \bar{b})$.*

b) *$X \sim Y$ for strongly minimal sets can be expressed.*

*Proof.* b) follows from a) and

$$\exists^\infty \bar{x}_0 \, \exists^\infty \bar{x}_1 \ldots \exists^\infty \bar{x}_\mu \Big( \bigwedge_{i \le \mu} \varphi(\bar{x}_i, \bar{b}) \wedge \psi(\bar{x}_0, \ldots, \bar{x}_\mu, \bar{b}) \Big)$$

is the desired formula in a). □

**Definition** If $X$ is a strongly minimal subset of $\mathbb{C}^n$ and $X \sim \varphi(\bar{x}, \bar{b})$ where $\bar{b} \in \mathbb{C}^{\mathrm{eq}}$, then we say that $X$ is encoded by $\varphi(\bar{x}, \bar{y})$.

We define codes similarly as in [BMPZ3]. This is a modification of E. Hrushovski's definition [Hr2] to the vectorspace case.

**Definition** $\varphi_\alpha(\bar{x}, \bar{y})$ is a code formula or short a code, if it fulfils the following conditions:

a) Length$(\bar{x}) = n_\alpha \ge 2$, and $\varphi_\alpha(\bar{x}, \bar{y})$ implies $R(x_i)$ for all $x_i$.

b) The set $\varphi_\alpha(\bar{x}, \bar{b})$ is either empty or strongly minimal.

c) $n_\alpha$ is the linear dimension for all solutions.

d) $\varphi_\alpha(\bar{x}, \bar{b}) \sim \varphi_\alpha(\bar{x}, \bar{b}')$ implies $\bar{b} = \bar{b}'$.

e) If some non-empty $\varphi_\alpha(\bar{x}, \bar{b})$ is groupless, then all $\varphi_\alpha(\bar{x}, \bar{b}')$ are.

f) $\varphi_\alpha(\bar{x} + \bar{m}, \bar{b})$ is encoded by $\varphi_\alpha$ for all $\bar{m}$.

g) For all $H$ in $\mathrm{GL}_{n_\alpha}(\mathbb{F}_q)$ the set $\varphi_\alpha(H\bar{x}, \bar{b})$ is encoded by $\varphi_\alpha$.

By d) $\bar{b}$ is the canonical parameter of the generic type of Morley rank 1 determined by $\varphi_\alpha(\bar{x}, \bar{b})$.

**Lemma 4.2** *There is a set $\mathcal{C}$ of codes $\varphi_\alpha(\bar{x}, \bar{y})$ that encodes the strongly minimal sets that are encoded by the formulas in $\mathcal{X}$.*

*Proof*: The proof is a copy of the proof in [BMPZ3]. But in [BMPZ3] we work in $\mathrm{ACF}_q$. To get the red fields in this paper we start already with coloured fields. The formulas in $\mathcal{X}$ have the properties a)–d). Using Lemma 2.3 we can assume w.l.o.g. that the formulas in $\mathcal{X}$ satisfy a)–e). Since $\mathcal{X}$ is closed under affine transformations by compactness there are finitely many $\varphi_1, \dots, \varphi_r$ in $\mathcal{X}$ that encode all possible affine transformations of some given $\varphi(\bar{x}, \bar{b})$. Moreover we know that either all or none encode groupless sets by Lemma 2.2.
Choose a sequence $w_1, \dots, w_r$ of different definable elements in $T^{\mathrm{eq}}$. Define

$$\begin{aligned} \theta_i^1(\bar{b}) &= \text{"No } \varphi_j \ (j < i) \text{ encodes } \varphi_i(\bar{x}, \bar{b})\text{"} \\ \theta_i^2(\bar{b}) &= \text{"}\varphi_i(\bar{x}, \bar{b}) \text{ is equivalent to some } \varphi(H(\bar{x}) + \bar{m}, \bar{b}')\text{"} \\ \varphi_i'(\bar{x}, \bar{y}) &= \varphi_i(\bar{x}, \bar{y}) \wedge \theta_i^1(\bar{y}) \wedge \theta_i^2(\bar{y}) \end{aligned}$$

Finally let $\varphi_\alpha(\bar{x}, y_1, \bar{y}) = \bigvee_{i=1}^{r} (\varphi_i'(\bar{x}, \bar{y}) \wedge y_1 = w_i)$. $\varphi_\alpha$ has the properties a)–e). To show f) and g) let $\bar{b}$, $\bar{m}$, $\bar{H}$ be given. By construction $\varphi_\alpha(\bar{x}, b_1, \bar{b})$ is equivalent to some $\varphi(H'\bar{x} + \bar{m}', \bar{b}')$. Hence

$$\varphi_\alpha(H\bar{x} + \bar{m}, \bar{b}) \sim \varphi((H'H)\bar{x} + H'\bar{m} + \bar{m}', \bar{b}')$$

and the right side is encoded by $\varphi_\alpha$ by construction. $\square$

**Theorem 4.3** *There is a set $\mathcal{C}$ of codes such that for every $\varphi(\bar{x}, \bar{b})$ in $\mathcal{X}$ there is a unique $\varphi_\alpha(\bar{x}, \bar{c})$ in $\mathcal{C}$ such that $\varphi(\mathbb{C}, \bar{b}) \sim \varphi_\alpha(\mathbb{C}, \bar{c})$.*

*Proof*: (as in [BMPZ3])
Let $\alpha_i$ be a list of all codes from Lemma 4.2. Again define:

$$\theta_i(\bar{b}) = \text{"No } \varphi_{\alpha_j} \ (j < i) \text{ encodes } \varphi_{\alpha_i}(\bar{x}, \bar{b})\text{" and } \varphi_{\alpha_i}'(\bar{x}, \bar{y}) = \varphi_{\alpha_i}(\bar{x}, \bar{y}) \wedge \theta_i(\bar{y}).$$

$\varphi_{\alpha_i}'$ satisfies a)–e) we have to show f) and g). By construction $\varphi_{\alpha_i}'(H\bar{x} + \bar{m}, \bar{b})$ is encoded by $\varphi_{\alpha_i}$. We need only to show that no $\varphi_{\alpha_j}$ with $j < i$ encodes it. Suppose that

$$\varphi_{\alpha_i}(H\bar{x} + \bar{m}, \bar{b}) \sim \varphi_{\alpha_j}(\bar{x}, \bar{b}').$$

Then

$$\varphi_{\alpha_i}(\bar{x}, \bar{b}) \sim \varphi_{\alpha_j}(H^{-1}\bar{x} - H^{-1}\bar{m}, \bar{b}') \sim \varphi_{\alpha_j}(\bar{x}, \bar{b}'')$$

for some $\bar{b}''$. This contradicts the definition of $\varphi_\alpha'$. Hence $\mathcal{C} = \{\alpha_i' : i < \omega\}$ has the desired properties. $\square$

A set of codes as in Theorem 4.3 is called a set of good codes.

**Corollary 4.4** *In the definition of* P(I) – P(VII) *we can replace* $\mathcal{X}$ *by a set* $\mathcal{C}$ *of good codes.*

For each $\alpha \in \mathcal{C}$ we choose a natural number $m_\alpha$ such that the existence of $m_\alpha$ common solutions of $\varphi_\alpha(\bar{x}, \bar{b})$ and $\varphi_\alpha(\bar{x}, \bar{b}')$ implies $\varphi_\alpha(\bar{x}, \bar{b}) \sim \varphi_\alpha(\bar{x}, \bar{b}')$. This is possible by the strong minimality of $\varphi_\alpha(\bar{x}, \bar{y})$.

**Theorem 4.5** *For each* $\alpha \in \mathcal{C}$ *and* $\lambda \geq m_\alpha$ *there is a formula* $\psi_\alpha(\bar{x}_0, \ldots, \bar{x}_\lambda)$ *with the following properties:*

a) *For any initial segment* $\{\bar{e}_0, \ldots, \bar{e}_\lambda, \bar{f}\}$ *of a Morley sequence of* $\varphi_\alpha(\bar{x}, \bar{b})$

$$\psi_\alpha(\bar{e}_0 - \bar{f}, \ldots \bar{e}_\lambda - \bar{f})$$

*holds.*

b) *For each realization* $(\bar{e}_0, \ldots, \bar{e}_\lambda)$ *of* $\psi_\alpha$ *there is a unique* $\bar{b}$ *with* $\models \varphi_\alpha(\bar{e}_i, \bar{b})$ *for* $0 \leq i \leq \lambda$. *Moreover* $\bar{b} \in \mathrm{dcl}^{\mathrm{eq}}(\bar{e}_{i_1}, \ldots, \bar{e}_{i_{m_\alpha}})$ *for any* $i_1 < \ldots < i_{m_\alpha}$. (*We call* $\bar{b}$ *the canonical parameter of the sequence* $\bar{e}_0, \ldots, \bar{e}_\lambda$).

c) *Each realization of* $\psi_\alpha$ *is* $\mathbb{F}_q$*-linear independent.*

d) *If* $\models \psi_\alpha(\bar{e}_0, \ldots, \bar{e}_\lambda)$, *then for* $i \in \{0, \ldots, \lambda\}$ :

$$\models \psi_\alpha(\bar{e}_0 - \bar{e}_i, \ldots, \bar{e}_{i-1} - \bar{e}_i, -\bar{e}_i, \bar{e}_{i+1} - \bar{e}_i, \ldots, \bar{e}_\lambda - \bar{e}_i).$$

e) *Given a realization* $(\bar{e}_0, \ldots, \bar{e}_\lambda)$ *of* $\psi_\alpha$ *with canonical parameter* $\bar{b}$ *as in* b), *we have the following:*
*Suppose* $\alpha$ *is groupless:*

1) *If* $\bar{e}_i$ *is a generic solution of* $\varphi(\bar{x}, \bar{b})$, *then* $\bar{e}_i - H\bar{e}_j \not\mathop{\smile}\limits_{\bar{b}} \bar{e}_i$ *for all* $H \in \mathrm{GL}_{n_\alpha}(\mathbb{F}_q)$ *and* $j \neq i$.

*Suppose* $\alpha$ *is a coset code, then:*

2) $\varphi_\alpha(\bar{x}, \bar{b})$ *is a group-set.*
3) $\psi_\alpha(\bar{e}_0, \ldots, \bar{e}_{i-1}, \bar{e}_i - \bar{e}_j, \bar{e}_{i+1}, \ldots, \bar{e}_\lambda)$ *for* $j \neq i$.
4) $\psi_\alpha(\bar{e}_0, \ldots, \bar{e}_{i-1}, H\bar{e}_i, \bar{e}_{i+1}, \ldots, \bar{e}_\lambda)$ *for all* $H \in \mathrm{Inv}(\varphi_\alpha(\bar{x}, \bar{b}))$.

5) *Moreover, if* $\bar{e}_i$ *is generic in* $\varphi_\alpha(\bar{x},\bar{b})$*, then* $\bar{e}_i - H\bar{e}_j \mathop{\not\downarrow}_{\bar{b}} \bar{e}_i$ *for all* $j \neq i$ *and* $H \in \mathrm{GL}_{n_\alpha}(\mathbb{F}_q) \setminus \mathrm{Inv}(\varphi_\alpha(\bar{x},\bar{b}))$.

*Proof*: (Copy of the corresponding proof in [BMPZ3] but in another theory.) We consider the following partial type

$$\Sigma(\bar{e}_0,\ldots,\bar{e}_\lambda) = \text{"There is some } \bar{b}' \text{ and some Morley sequence } \bar{a}_0,\ldots,\bar{a}_\lambda,\bar{f} \text{ of } \varphi_\alpha(\bar{x},\bar{b}') \text{ with } \bar{e}_i = \bar{a}_i - \bar{f}.\text{"}$$

*Claim.* $\Sigma$ has the properties a) – e).

*Proof of the claim.* a) is clear.
Given a realization $\bar{e}_0,\ldots,\bar{e}_\lambda$ of $\Sigma$, there are some $\bar{b}'$ and $\bar{a}_0,\ldots,\bar{a}_\lambda,\bar{f}$ as above. Hence $\{\bar{e}_i\}_{0\le i\le\lambda}$ is a Morley sequence of $\varphi_\alpha(\bar{x}+\bar{f},\bar{b}')$. Then $\varphi_\alpha(\bar{x}+\bar{f},\bar{b}') \sim \varphi_\alpha(\bar{x},\bar{b})$ for some $\bar{b}$ by f) in the definition of codes. Since $\bar{b}$ is the canonical parameter of the generic type determined by $\varphi_\alpha(\bar{x},\bar{b})$, the sequence $\{\bar{e}_i\}_{0\le i\le\lambda}$ is a Morley sequence for $\varphi_\alpha(\bar{x},\bar{b})$. Given another $\bar{b}^*$ which satisfies $\varphi_\alpha(\bar{e}_i,\bar{y})$ for $m_\alpha$ many $i$'s, it follows that $\varphi_\alpha(\bar{x},\bar{b}^*) \sim \varphi_\alpha(\bar{x},\bar{b})$ by the choice of $m_\alpha$. By d) in the code-definition $\bar{b}^* = \bar{b}$. Hence b) is true for $\Sigma$.
The linear independence in c) is clear.
Since $\bar{a}_0,\ldots,\bar{a}_{i-1},\bar{f},\bar{a}_{i+1},\ldots,\bar{a}_\lambda,\bar{a}_i$ is again a Morley sequence for $\varphi_\alpha(\bar{x},\bar{b}')$ we have

$$(\bar{a}_0-\bar{a}_i,\ldots,\bar{a}_{i-1}-\bar{a}_i,\bar{f}-\bar{a}_i,\bar{a}_{i+1}-\bar{a}_i,\ldots,\bar{a}_\lambda-\bar{a}_i) \models \Sigma$$

and hence

$$(\bar{e}_0-\bar{e}_i,\ldots,\bar{e}_{i-1}-\bar{e}_i,-\bar{e}_i,\bar{e}_{i+1}-\bar{e}_i,\ldots,\bar{e}_\lambda-\bar{e}_i) \models \Sigma.$$

We get d).
To prove e) we assume first that $\alpha$ is groupless. That means $\varphi_\alpha(\bar{x},\bar{b})$ is not a torsor set. By Lemma 2.4 the assertion follows.
Otherwise $X = \varphi_\alpha(\mathbb{C},\bar{b}')$ is a torsor set. Hence $X - \bar{f} \sim \varphi_\alpha(\bar{x},\bar{b})$ is a group set since $\bar{f}$ is in $X$.
We extend the Morley sequence $\{\bar{e}_i : 0 \le i \le \lambda\}$ by an element $\bar{d}$. Then

$$\bar{e}_0+\bar{d},\ldots,\bar{e}_{i-1}+\bar{d},\bar{e}_i-\bar{e}_j+\bar{d},\bar{e}_{i+1}+\bar{d},\ldots,\bar{e}_\lambda+\bar{d},\bar{d}$$

is again a Morley sequence for $\varphi_\alpha(\bar{x},\bar{b})$. Hence

$$\Sigma(\bar{e}_0,\ldots,\bar{e}_i-\bar{e}_j,\ldots,\bar{e}_\lambda).$$

Similarly we get e4).

e5) follows again by Lemma 2.4. □(Claim)

Using compactness we get a finite part $\psi'_\alpha$ of $\Sigma$ that implies a), b), c), e1), e2), e5).

If $\alpha$ is groupless consider the following operations:

$$V_i(\bar{x}_0, \ldots, \bar{x}_\lambda) = (\bar{x}_0 - \bar{x}_i, \ldots, \bar{x}_{i-1} - \bar{x}_i, -\bar{x}_i, \bar{x}_{i+1} - \bar{x}_i, \ldots, x_\lambda - \bar{x}_i)$$

and $V$ be the subgroup generated by these operations. $V$ is finite. Then

$$\psi_\alpha(\bar{x}_0, \ldots, \bar{x}_\lambda) = \bigwedge_{v \in V} \psi'_\alpha(V(\bar{x}_0, \ldots, \bar{x}_\lambda))$$

satisfies d) and is also part of $\Sigma$.

If $\alpha$ is an coset code, property d) follows from e3) and e4). Hence it is sufficient that $\psi_\alpha$ satisfies e3) and e4). Let $\mathcal{W}(\bar{x}_0, \ldots, \bar{x}_\lambda)$ be the subgroup of $\mathrm{GL}_{n_{\alpha(\lambda+1)}}(\mathbb{F}_q)$ generated by the operations mentioned in e3) and e4). Again $\mathcal{W}$ is finite, and depends on $\mathrm{Inv}(\varphi_\alpha(\bar{x}, \bar{b}))$. Note that $\lambda \geq m_\alpha$, hence $\bar{b}$ remains constant in b) after applying these operations. Set therefore:

$$\psi_\alpha(\bar{x}_0, \ldots, \bar{x}_\lambda) = \bigwedge_{W \in \mathcal{W}(\bar{x}_0, \ldots, \bar{x}_\lambda)} \psi'_\alpha(W(\bar{x}_0, \ldots, \bar{x}_\lambda)),$$

which has the required properties. □

**Definition** Let $\alpha$, $\lambda$ and $\psi_\alpha$ be as above. A realization of $\psi_\alpha$ is called a difference sequence for $\alpha$. Moreover, given a realization $\bar{e}_0, \ldots, \bar{e}_\lambda$ of $\psi_\alpha$, we denote by a derived difference sequence one obtained by composition of the following operations:

$$\bar{e}_0 - \bar{e}_i, \ldots, \bar{e}_{i-1} - \bar{e}_i, -\bar{e}_i, \bar{e}_{i+1} - \bar{e}_i, \ldots, \bar{e}_\lambda - \bar{e}_i.$$

If $\nu \leq \lambda$ and we use the operations above only for $i \leq \nu$, then we speak about a $\nu$-derived sequence.

**Corollary 4.6** *A permutation of a difference sequence is a difference sequence.*

*Proof*. Note that all permutations of a difference sequence are obtained by the operation in d) of Theorem 4.5. □

**Corollary 4.7** *If $D \leq \mathbb{C}$ and $D'$ is a prealgebraic minimal extension of $D$, then there is a unique good code $\alpha$ such that there is a unique $\bar{b}$ in $\mathrm{dcl}^{\mathrm{eq}}(D)$ and a generic solution $\bar{a}$ of $\varphi_\alpha(\bar{x}, \bar{b})$ that generates $D'$ over $D$.*

*Proof*. This follows from Theorem 4.3 and poperties f) and g) of the codes. □

# 5 Bounds for difference sequences

$T$ is again a starting theory for a red collaps as described in Chapter 3. As shown in Corollary 4.4 we can assume that $\mathcal{X}$ is a set $\mathcal{C}$ of good codes as given by Theorem 4.3. We work in $T^{\mathrm{eq}}$. $A$, $B$, $C$, $D$ are subspaces of $R(\mathbb{C})$.

**Lemma 5.1** *For every code formula $\varphi_\alpha(\bar{x}, \bar{y})$ and every natural number $r$ there is some $\lambda(r, \alpha) = \lambda > 0$ such that for every $D \leq \mathbb{C}$, and every difference sequence $\bar{e}_0, \ldots, \bar{e}_\mu$ for $\varphi_\alpha(\bar{x}, \bar{y})$ with canonical parameter $\bar{b}$ and $\mu \geq \lambda$ either*

i) *the canonical parameter of some $\lambda$-derived sequence of $\bar{e}_0, \ldots, \bar{e}_\mu$ lies in* $\mathrm{dcl}^{\mathrm{eq}}(D)$

*or*

ii) *for every $n_\alpha$-tuple $\bar{m}$ the sequence contains a subsequence $\bar{e}_{i_0}, \ldots, \bar{e}_{i_r-1}$ such that $m_\alpha \leq i_j$ and $e_{i_j}$ is $\mathop{\smile\!\!\!\!\!\!|}^{w}$-generic over $\langle \bar{e}_{i_0}, \ldots, \bar{e}_{i_{j-1}} \rangle^\ell + D + B$, where $B = \langle \bar{e}_0, \ldots, \bar{e}_{m_\alpha - 1}, \bar{m} \rangle^\ell$.*

*Proof.* If assertion i) is not true, then every coset of $\mathbb{C}^{n_\alpha}/D^{n_\alpha}$ contains at most $m_\alpha$-many elements $\bar{e}_i$ of the difference sequence under consideration with $i \leq \lambda$. Otherwise we could substract one of these elements $\bar{e}_j$ $(j \leq \lambda)$ and would get $\bar{e}_{i_0} - \bar{e}_j, \ldots, \bar{e}_{i_{m_{\alpha-1}}} - \bar{e}_j$ in $D$ for some $i_0, \ldots, i_{m_{\alpha-1}}$ different from $j$. Hence the canonical parameter of the corresponding derived sequence would be in $\mathrm{dcl}^{\mathrm{eq}}(D)$ by property b) of a difference sequence. Now we have to choose $\lambda(r, \alpha)$ such that ii) is true.
Let $s = \mathrm{l.dim}(\bar{e}_0, \ldots, \bar{e}_\lambda / \langle D + B \rangle^\ell)$. Then $\mathrm{l.dim}(\bar{e}_0, \ldots, \bar{e}_\lambda / D) \leq s + (m_\alpha + 1)n_\alpha$. By the considerations above we get

$$\lambda + 1 \leq m_\alpha q^{(s + (m_\alpha + 1)n_\alpha)n_\alpha}.$$

Let $X$ be the set of all $i \leq \lambda$ such that

$$\mathrm{l.dim}(\bar{e}_i / D + B + \langle \bar{e}_j : j < i \rangle^\ell) > 0.$$

Then $s \leq |X| \cdot n_\alpha$ and hence

$$\lambda + 1 \leq m_\alpha q^{(|X| \cdot n_\alpha + (m_\alpha + 1)n_\alpha)n_\alpha}.$$

If we choose $\lambda(r, \alpha)$ large enough, then we get $|X| \geq r + (m_\alpha + 1)n_\alpha$. Since $\mathrm{l.dim}(B) \leq (m_\alpha + 1)n_\alpha$ P(V) provides us a subsequence $\bar{e}_{i_0}, \ldots, \bar{e}_{i_{r-1}}$ such that $\bar{e}_{i_j}$ is $\mathop{\smile\!\!\!\!\!\!|}^{\omega}$-generic over $D + B + \langle \bar{e}_{i_0}, \ldots, \bar{e}_{i_{j-1}} \rangle^\ell$. $\square$

Now we consider all finite-to-one functions $\mu^*$ and $\mu$ defined on the good codes $\alpha \in \mathcal{C}$ with values in $\mathbb{N}$. We assume that the following inequalities hold:

- $\mu(\alpha) \geq m_\alpha$,
- $\mu^*(\alpha) \geq \max(\lambda(m_\alpha + 1, \alpha) + 1, n_\alpha + 1)$,
- $\mu(\alpha) \geq \lambda(\mu^*(\alpha), \alpha) + 1$.
- $\mu^*(\alpha) > r$, if in $\theta(\bar{x}, y) \in \mathcal{X}$ from P(VII) $\bar{x}$ is an $r$-tuple.

For the definition above we fix a function $\lambda(r, \alpha)$ given by Lemma 5.1 and we assume that it is monotonous in the first argument.

Finally we will get for each such function $\mu$ as above a countable "generic" subspace $R^\mu(\mathbb{C})$ of $R(\mathbb{C})$ such that

$$R^\mu(\mathbb{C}) \leq \mathbb{C}, \quad \mathrm{acl}(R^\mu(\mathbb{C})) \cap R(\mathbb{C}) = R^\mu(\mathbb{C}).$$

We will extend the language $L$ by a new predicate $P^\mu$ and consider the structure $\langle \mathbb{C}, P^\mu(\mathbb{C}) \rangle$ in the new language $L^\mu$, where $P^\mu(\mathbb{C}) = \langle R^\mu(\mathbb{C}) \rangle$. $P^\mu(\mathbb{C})$ will be the desired $L$-structure of finite Morley rank. We will get $R^\mu(\mathbb{C})$ by amalgamation in the class $\mathbb{K}^\mu$ of strong subspaces of $R(\mathbb{C})$ defined below.

**Definition** Let $\mathbb{K}^\mu$ be the class of all strong subspaces $U$ of $R(\mathbb{C})$, such that for every good code $\alpha$ there is no difference sequence for $\alpha$ of length $\mu(\alpha) + 1$ in $U$. $\mathbb{K}^\mu_{\mathrm{fin}}$ are the finite spaces in $\mathbb{K}^\mu$.

Note that difference sequences are given by realizations of the formulas $\psi_\alpha(\bar{x}_0, \ldots, \bar{x}_{\mu(\alpha)})$ in Theorem 4.5. Their realizations are contained in $R(\mathbb{C})$.

Let $D \subseteq D'$ be strong subspaces of $\mathbb{C}$ with $\mathrm{l.dim}(D'/D)$ finite. By Lemma 3.11 there is a geometrical sequence for $D'$ over $D$. In the next lemmas we will investigate the minimal steps in this sequence, especially the prealgebraic minimal steps for the case that $D \in \mathbb{K}^\mu$ but $D' \notin \mathbb{K}^\mu$.

**Lemma 5.2** *Assume $D \leq \mathbb{C}$, $D \in \mathbb{K}^\mu$, $D'$ is a prealgebraic minimal extension of $D$ and $D'$ is not $\mathbb{K}^\mu$. Let $\bar{e}_0, \ldots, \bar{e}_{\mu(\alpha)}$ be a difference sequence for a good code $\alpha$ in $D'$, such that its canonical parameter $\bar{c}$ is in $\mathrm{dcl}^{\mathrm{eq}}(D)$. Then we find a difference sequence $\bar{d}_0, \ldots, \bar{d}_{\mu(\alpha)}$ for $\alpha$ in $D'$ with the same canonical parameter such that $\bar{d}_0, \ldots, \bar{d}_{\mu(\alpha)-1}$ are in $D$, and $\bar{d}_{\mu(\alpha)}$ is a $D$-generic realization of $\varphi_\alpha(\bar{x}, \bar{c})$*

*that generates $D'$ over $D$.*
*If we cannot find the new sequence by a permutation of the old one, then $\alpha$ is a group code and the new sequence is obtained using operations as $\bar{e}_j$ is replaced by some $H\bar{e}_j - \bar{e}_i$ where $H$ is in $\mathrm{Inv}(\varphi_\alpha(\bar{x}, \bar{c}))$. $\alpha$ is the unique good code that describes $D'$ over $D$.*

*Proof.* Since $D \in \mathbb{K}^\mu$, there is some $\bar{e}_i$ not completely in $D$. Since $D \leq \mathbb{C}$ by P(III) $\bar{e}_i$ is $D$-generic and generates $D'$ over $D$. If there is some other $\bar{e}_j$ not completely in $D$, then again $\bar{e}_j$ is $D$-generic and generates $D'$ over $D$. Hence $\bar{e}_i = H\bar{e}_j - \bar{m}_j$ where $H$ is in $\mathrm{GL}_{n_\alpha}(\mathbb{F}_p)$ and $\bar{m}_j$ is in $D$. Then $H\bar{e}_j - \bar{e}_i$ is in $D$. Since $\bar{e}_j$ is $D$-generic, we have

$$\bar{e}_j \underset{\bar{c}}{\perp} H\bar{e}_j - \bar{e}_i.$$

By the properties of a difference sequence it follows that $\alpha$ is a group code and $H$ is in $\mathrm{Inv}(\varphi_\alpha(\bar{x}, \bar{c}))$. If we replace $\bar{e}_j$ by $H\bar{e}_j - \bar{e}_i$ we obtain again a difference sequence with the same canonical parameter and this sequence has one more element in $M$. We can iterate the argument to obtain the assertion.
Finally every strongly minimal set that gives us $D'$ over $D$ determines a unique code by Theorem 4.3. All such generic solutions of code formulas $\varphi_\alpha$ can be transformed into each other by elements of $\mathrm{GL}_{n_\alpha}(\mathbb{F}_q)$ and translations. By the properties of good code we use only one formula. $\square$

**Corollary 5.3** *Let $D$ be in $\mathbb{K}^\mu$ and $D \leq D'$ be a minimal extension. If $D'$ has linear dimension one over $D$, then $D'$ is in $\mathbb{K}^\mu$. Otherwise, in the prealgebraic case, $D'$ is in $\mathbb{K}^\mu$ if and only if none of the following two conditions holds:*

a) *There is a code $\alpha \in \mathcal{C}$ and a difference sequence $\bar{e}_0, \ldots \bar{e}_{\mu(\alpha)}$ for $\alpha$ in $D'$ such that*

   i) *$\bar{e}_0, \ldots, \bar{e}_{\mu(\alpha)-1}$ are contained in $D$.*
   ii) *$D' = \langle D\bar{e}_{\mu(\alpha)}\rangle^\ell$.*
   iii) *In this case $\alpha$ is the unique good code that describes $D'$ over $D$.*

b) *There exists a code $\alpha \in \mathcal{C}$ and a difference sequence for $\alpha$ in $D'$ of length $\mu(\alpha)+1$ with canonical parameter $\bar{b}$ and with a subsequence $\bar{e}_0, \ldots, \bar{e}_{\mu^*(\alpha)-1}$ of length $\mu^*(\alpha)$ such that $\bar{e}_i$ is $\perp^w$-generic over $D+B+\langle\bar{e}_0, \ldots, \bar{e}_{i-1}\rangle^\ell$ where $B$ is generated by the first $m_\alpha$ elements of the given difference sequence.*

*Proof*. Consider first the case where $\mathrm{l.dim}(D'/D) = 1$. Assume that $D'$ is not in $\mathbb{K}^\mu$. That means there is a difference sequence $\bar{e}_0, \ldots, \bar{e}_{\mu(\alpha)}$. If the canonical parameter $\bar{b}$ lies in $\mathrm{dcl}^{\mathrm{eq}}(D)$, then all $\bar{e}_i$ would be in $D$, since no $\bar{e}_i$ is linearly independent over $D$. This contradicts $D \in \mathbb{K}^\mu$.
Otherwise by Lemma 5.1 some $\bar{e}_j$ is a realization of $\varphi_\alpha(\bar{x}, \bar{b})$ linearly independent over $D$ since $\mu(\alpha) \geq \lambda(1, \alpha)$. Again we have a contradiction.

Finally we assume that $D'$ is minimal prealgebraic over $D$. Again we assume that there is a difference sequence $\bar{e}_0, \ldots, \bar{e}_{\mu(\alpha)}$ in $D'$ for some good code formula $\varphi_\alpha(\bar{x}, \bar{b})$ where $\bar{b}$ is the canonical parameter. If $\bar{b}$ lies in $\mathrm{dcl}^{\mathrm{eq}}(D)$, then by Lemma 5.2 we get case a). Otherwise since $\mu(\alpha) \geq \lambda(\mu^*(\alpha), \alpha)$ our sequence contains a subsequence of length $\mu^*(\alpha)$ as described in b) by Lemma 5.1. □

**Corollary 5.4** *Assume there is a formula $\theta(\bar{x}, y) \in \mathcal{X}$ given by* P(VII)*, $D$ is in $\mathbb{K}^\mu$, $d \in \mathrm{dcl}^{\mathrm{eq}}(D)$ and $\theta(\bar{x}, d)$ has no solution in $D$. Let $D' = \langle D\bar{a} \rangle$ where $\bar{a}$ is a generic solution of $\theta(\bar{x}, d)$. Then $D'$ is in $\mathbb{K}^\mu$.*

*Proof*. Note that $\bar{x}$ in $\theta(\bar{x}, y)$ has length $r$ and $\mu^*(\alpha) > r$ for all $\alpha$. Hence $\mathrm{l.dim}(D'/D) = r$. Assume $D' \notin \mathbb{K}^\mu$. Case a) of Corollary 5.3 is not fulfilled, since the unique code that describes $D'$ is given by $\theta(\bar{x}, y)$. Hence Case b) provides us more than $r$ solutions of some $\varphi_\alpha(\bar{z}, \bar{y})$ linearly independent over $D$. This is a contradiction. □

# 6 Amalgamation in $\mathbb{K}^\mu$

Let $T$ be a starting theory for a collapse as above. Again we work in $T^{\mathrm{eq}}$.

**Lemma 6.1** *Let $B \subseteq A$ and $B \subseteq C$ all be strong subspaces of $\mathbb{C}$ such that $A$ and $C$ are linearly independent over $B$. Assume that $A$ and $C$ are prealgebraic extensions of $B$ and that $\bar{e}_0, \ldots, \bar{e}_{\mu(\alpha)}$ is a difference sequence for a good code $\alpha$ in $A + C$. Then there is a derived difference sequence of the above sequence with the canonical parameter in $\mathrm{dcl}^{\mathrm{eq}}(C)$ or in $\mathrm{dcl}^{\mathrm{eq}}(A)$.*

*Proof*. We assume that the assertion of the lemma is not true. Let $E = \langle e_0, \ldots, e_{m_\alpha - 1} \rangle^\ell$. By Lemma 5.1 we get a subsequence $\bar{e}_{i_0}, \ldots, \bar{e}_{i_{\mu^*(\alpha)}}$, such that $\bar{e}_{i_j}$ is $\mathop{\smile\hskip-0.9em\vert}^w$-generic over $\langle \bar{e}_{i_0}, \ldots, \bar{e}_{i_{j-1}} \rangle^\ell + C + E$. Since $\mu^*(\alpha) \geq \lambda(m_\alpha + 1, \alpha) + 1$ we get a subsequence of this sequence of length $m_\alpha + 1$ such that every element is $\mathop{\smile\hskip-0.9em\vert}^w$-generic over $C + E$ and over $A + E$. Again we have applied Lemma 5.1. By

P(VI) $\varphi_\alpha(x, y)$ defines a torsor set and by the properties of a difference sequence a group set. Hence by P(VI) $\bar{b} \in \mathrm{dcl}^{\mathrm{eq}}(B)$, a contradiction to the assumption. $\square$

Note that all subspaces in $\mathbb{K}^\mu$ are strong subspaces of $\mathbb{C}$. By Lemma 3.3 $\mathrm{tp}^{\mathbb{C}}(A)$ for $A \leq R(\mathbb{C})$ is given by any geometrical sequence for $A$. On the other hand for strong $A$ $\mathrm{tp}^{\mathbb{C}}(A)$ is given by the isomorphism type of $\langle A \rangle$ (P(IV)). The partial elementary maps $B \overset{\equiv}{\hookrightarrow} A$ of strong subspaces of $R(\mathbb{C})$ are the strong embeddings $\langle B \rangle \hookrightarrow \langle A \rangle$.

**Theorem 6.2** *Assume that* $T$ *satisfies* P(I) – P(VI). *The class* $\mathbb{K}^\mu_{\mathrm{fin}}$ *has the amalgamation property with respect to partial elementary maps.*

*Proof*. We assume that $B \subseteq C$, $B \subseteq A$ are all strong subspaces of $R(\mathbb{C})$ in $\mathbb{K}^\mu_{\mathrm{fin}}$. We need to show that there is an extension $D$ of $B$ in $\mathbb{K}^\mu$ and partial elementary maps $f : A \hookrightarrow D$ and $g : C \hookrightarrow D$ extending the inclusion of $B$ in $D$ such that $\mathrm{tp}(A/B) = \mathrm{tp}(f(A)/B)$ and $\mathrm{tp}(C/B) = \mathrm{tp}(g(C)/B)$. Splitting $A$ and $C$ into chains of minimal extensions in $\mathbb{K}^\mu$ we can assume w.l.o.g. that $A$ and $C$ are minimal extensions. Let $D^+$ be a non-forking amalgam of $A$ and $C$ over $B$. W.l.o.g. $C$ and $A$ are linearly independent over $B$ and $D^+$ is $A + C$.

*Case* 1: $\mathrm{l.dim}(C/B) = 1$ or $\mathrm{l.dim}(A/B) = 1$.
By Corollary 5.3 the amalgam $D^+$ is in $\mathbb{K}^\mu$.

*Case* 2: Both extensions $C/B$ and $A/B$ are prealgebraic.
We assume that $D^+$ is not in $\mathbb{K}^\mu$ and show in this case that $C$ and $A$ have the same type over $B$.

There is a good code $\alpha$ with a difference sequence $\bar{e}_0, \ldots, \bar{e}_{\mu(\alpha)}$ in $D^+$. By Lemma 6.1 and symmetry we may assume w.l.o.g. that its canonical parameter $\bar{b}$ lies in $\mathrm{dcl}^{\mathrm{eq}}(C)$. By Lemma 5.2 we may assume that $\bar{e}_0, \ldots, \bar{e}_{\mu(\alpha)-1}$ are in $C$ and $\bar{e}_{\mu(\alpha)}$ is an $C$-generic realization of $\varphi_\alpha(\bar{x}, \bar{b})$ which generates $D^+$ over $C$.

Assume that $A = \langle B\bar{a} \rangle^\ell$ where $\bar{a}$ is a $B$-generic solution of some $\varphi_\gamma(\bar{x}, \bar{c})$ with canonical parameter $\bar{c}$ in $\mathrm{dcl}^{\mathrm{eq}}(B)$. Since $C$ is strong, $\bar{a}$ is $C$-generic. We consider three subcases:

*Case* 2.1: $\bar{e}_{\mu(\alpha)} \in A$.

Then $\bar{e}_{\mu(\alpha)}$ is generic over $B\bar{b} \subseteq C$. On the other hand $A = \langle B\bar{a} \rangle^\ell$ and $\bar{a}$ is a $B$-generic solution of some $\varphi_\gamma(\bar{x}, \bar{c})$, where $\bar{c}$ is in $\mathrm{dcl}^{\mathrm{eq}}(B)$. Then $\langle B, e_{\mu(\alpha)} \rangle^\ell = A$, since $\langle C, \bar{e}_{\mu(\alpha)} \rangle^\ell = \langle C, \bar{a} \rangle^\ell = D^+$. We have $\bar{e}_{\mu(\alpha)} = H(\bar{a}) + \bar{e}$, where $\bar{e} \in B$. We get that $\bar{e}_{\mu(\alpha)}$ is a solution of $\varphi_\alpha(\bar{x}, \bar{b})$ and $\varphi_\gamma(\bar{x}, \bar{d})$ where $\bar{d} \in \mathrm{dcl}^{\mathrm{eq}}(B)$ generic over

$C$. Here we use the properties f) and g) of codes. The new parameter $\bar{d}$ is again in $\mathrm{dcl}^{\mathrm{eq}}(B)$ since $\bar{e} \in B$. By Theorem 4.3 we have $\alpha = \gamma$ and by P(III) we get $\bar{b} = \bar{d} \in \mathrm{dcl}^{\mathrm{eq}}(B)$.
By minimality of $A$ over $B$ we have that $A = \langle B, \bar{e}_{\mu(\alpha)} \rangle^{\ell}$. Since $A$ is in $\mathbb{K}^{\mu}$ there exists an $\bar{e}_i$ which lies in $C$ but not in $B$. Since $B$ is strong $\bar{e}_i$ is $B$-generic. Hence $\langle B\bar{e}_i \rangle^{\ell} = C$ by minimality and

$$\mathrm{tp}(A/B) = \mathrm{tp}(\langle B, \bar{e}_{\mu(\alpha)} \rangle^{\ell}/B) = \mathrm{tp}(\langle B, \bar{e}_i \rangle^{\ell}/B) = \mathrm{tp}(C/B).$$

*Case* 2.2: $\bar{e}_{\mu(\alpha)} \notin A$ and the canonical parameter for some $(\mu(\alpha) - 1)$-derived difference sequence is in $\mathrm{dcl}^{\mathrm{eq}}(B)$.
Hence w.l.o.g. $\bar{b} \in \mathrm{dcl}^{\mathrm{eq}}(B)$. Then $\bar{e}_{\mu(\alpha)}$ is $A$-generic and $C$-generic. By P(V) $\bar{e}_{\mu(\alpha)}$ is $\downarrow^{w}$-generic over $A$ and $C$. By P(VI) $\varphi_{\alpha}(\bar{x}, \bar{b})$ defines a torsor set. Since we consider a difference sequence, $\varphi_{\alpha}(\bar{x}, \bar{b})$ defines a group set. Since $\langle C, \bar{e}_{\mu(\alpha)} \rangle^{\ell} = \langle C, \bar{a} \rangle^{\ell}$ we can choose $\bar{a}$ such that $\bar{a} = \bar{e}_{\mu(\alpha)} + \bar{m}$, where $\bar{m} \in C$. As above $\gamma = \alpha$. Since $\bar{e}_{\mu(\alpha)}$ is generic over $A$ and $C$ we have that $\bar{a}$, $-\bar{m}$, and $-\bar{e}_{\mu(\alpha)}$ are pairwise $\downarrow$-independent over $B$. By Lemma 2.1 they are generic elements in $\mathrm{acl}(B)$-definable cosets of the $B$-definable connected group given by $\varphi_{\alpha}(\bar{x}, \bar{b})$. Since $\bar{e}_{\mu(\alpha)}$ is a generic element of this group, $a$ and $m$ are generic elements of the same coset. Therefore $\mathrm{tp}(\bar{m}/B) = \mathrm{tp}(\bar{a}/B)$ as desired and $C = \langle B, \bar{m} \rangle^{\ell}$ by minimality.

*Case* 2.3: Neither Case 2.1 nor 2.2
Again we have $\bar{e}_{\mu(\alpha)} \notin A$. Since $\langle C, \bar{a} \rangle^{\ell} = \langle C, \bar{e}_{\mu(\alpha)} \rangle^{\ell}$ w.l.o.g. $\bar{e}_{\mu(\alpha)} = \bar{a} + \bar{m}$ where $\bar{m} \in C$. By application of Lemma 5.1 to $B \leq C$ and the choice of $\mu(\alpha)$ there exists a subsequence of elements $\bar{e}_i$ in $C$ that are $\downarrow^{w}$-generic over $B + \langle \bar{e}_0, \ldots, \bar{e}_{m_{\alpha}-1}, \bar{m} \rangle^{\ell}$. Hence $\bar{e}_i$ and $\bar{e}_{\mu(\alpha)}$ are solutions of $\varphi_{\alpha}(\bar{x}, \bar{b})$ $\downarrow^{w}$-generic over $B + \langle \bar{e}_0, \ldots, \bar{e}_{m_{\alpha}-1}, \bar{m} \rangle^{\ell}$ and therefore isomorphic over this subspace by P(V). Hence $\bar{e}_i - \bar{m}$ and $\bar{a} = \bar{e}_{\mu(\alpha)} - \bar{m}$ are isomorphic over $B$ and fulfil $\varphi_{\gamma}(\bar{x}, \bar{c})$ by P(IV). This gives an embedding $h$ of $A$ in $C$ over $B$ with $\mathrm{tp}(h(\bar{a})/B) = \mathrm{tp}(\bar{a}/B)$. By minimality $C = h(A)$. $\square$

Remember that subspaces in $\mathbb{K}^{\mu}$ are strong and in $R(\mathbb{C})$.

**Definition** Let $D$ be a subspace of $R(\mathbb{C})$. $D$ is called rich if it is in $\mathbb{K}^{\mu}$ and if for every finite $B \subseteq A$ in $K^{\mu}$ with $B \subseteq D$, there exists an $A'$ with $B \subseteq A' \subseteq D$ and $\mathrm{tp}(A'/B) = \mathrm{tp}(A/B)$. By P(II) $A' \leq \mathbb{C}$. Richness is a property of the elementary type of $D$ in $\mathbb{C}$. Hence, it makes sense in every model $M \models T$. We call a substructure $V$ of $\mathbb{C}$ rich, if $\langle R(V) \rangle = V$ and $R(V)$ is rich.

Now we can use Theorem 6.2 to produce a countable rich structure via a Fraïssé-style-argument:

**Corollary 6.3** *There is a unique* (*up to automorphisms*) *countable rich subspace of* $R(\mathbb{C})$.

**Corollary 6.4** *Let* $D$ *be a rich subspace of* $R(\mathbb{C})$*. Then*

a) $\operatorname{acl}(D) \cap R(\mathbb{C}) = D$.

b) $d(D) \geq \aleph_0$.

*Proof.* a) Let $A$ be a finite subspace of $D$ and $a \in \operatorname{acl}(A)$. W.l.o.g. we assume that $A$ is a strong subspace of $\mathbb{C}$ since $D$ is a strong subspace. By property P(II) and Corollary 5.3 every extension $A'$ of $A$ by an element algebraic over $A$ is strong and in $\mathbb{K}^\mu$. By richness follows the assertion.
b) Let $U \leq \mathbb{C}$ be a maximal subspace of $D$ linearly generated by geometrically independent elements. Then $U$ is strong in $\mathbb{C}$ and $U$ is in $\mathbb{K}^\mu$ (Axiom P(II), Corollary 5.3). $U$ cannot be finite since in this case an extension $U'$ of $U$ by an geometrically independent element would be in $\mathbb{K}^\mu$ and had to be realized in $D$. $\square$

We extend our language $L$ by a predicate $P^\mu$. Let $L^\mu$ be the extended language. We are interested in the $L^\mu$-structure $(M_0, \langle D \rangle)$ where $M_0 \preceq \mathbb{C}$ is a model $T$, $D$ is a countable rich subspace of $R(M_0)$, and $d(R(M_0)/D) \geq \aleph_0$. The interpretation of $P^\mu$ is $\langle D \rangle$. We use $\operatorname{acl}^L$ and often acl only for the algebraic closure in the $L$-reducts. $\operatorname{acl}^\mu$ is the algebraic closure in the full $L^\mu$-structure.

**Lemma 6.5** *We consider* $L^\mu$*-structures* $\langle M_0, D \rangle$ *as above.*

i) *There is a formula* $\chi(\bar{x}, y) = \exists z_1 z_2 \in R^\mu(\eta(\bar{x}, z_1, z_2) = y)$ *where* $\eta(\bar{x}, z_1, z_2) = y$ *is a quantifier-free* $L$*-definable function such that*

$$\langle D \rangle = \{e : \langle M_0, D \rangle \models \chi(\bar{d}, e),\ \bar{d} \in D\}.$$

ii) $\langle D \rangle = \{e : \langle D \rangle \models_L \exists z_1 z_2 \in R\,(\eta(\bar{d}, z_1, z_2) = e), \bar{d} \in D\}$.

*Proof*. ii) follows from i). By P(VII) we have three cases:
If $M_0 = R(M_0)$, then $\langle D\rangle = D$ since $\mathrm{acl}(D) = D$ by Corollary 6.4 a). In the third case of P(VII) we use again $\mathrm{acl}(D) \cap R(\mathbb{C}) = D$. Then $R(\langle D\rangle) = D$. Hence P(VII) provides the desired formula. It remains the second case of P(VII). We define

$$\chi(\bar{x}_1, \bar{x}_2, y) \equiv \exists z_1 z_2 \in R^\mu(\theta(\bar{x}_1, z_1) \wedge \theta(\bar{x}_2, z_2) \wedge z_1 + z_2 = y),$$

where $\theta(\bar{x}, z)$ is given by P(VII). For $\bar{d}_1$, $\bar{d}_2$ in $D$ $\chi(y, \bar{d}_1, \bar{d}_2)$ defines a unique element in $\langle D\rangle$. If $e$ is in $\langle D\rangle$, then there exists $B \leq M_0$ such that $B \subseteq D$ and $e \in \langle B\rangle$. By Corollary 6.4 there is $\bar{d}_1$ in $D$ a sequence of generics of $R(\mathbb{C})$ geometrically independent over $B$. By P(VII) there is a unique $e_1 \in \langle D\rangle$ such that $M \models \theta(\bar{d}_1, e_1)$ and $e_1$ is generic over $\langle B\rangle$ in $M$. Then $e - e_1$ is an element of $\langle D\rangle$ generic over $\langle B\rangle$. There is a solution $\bar{d}_2$ of $\theta(\bar{x}, e - e_1)$ in $D$. The noexistence in $D$ would imply that there is an extension $A$ of $B$ given by a solution $\bar{a}$ of $\theta(\bar{x}, e - e_1)$. Then $A$ is in $\mathbb{K}^\mu$ by Corollary 5.4. By richness of $D$ there is a solution in $D$ in contrast to the assumption. $\square$

Let $D$ be a rich subspace of $M_0 \overset{L}{\preceq} \mathbb{C}$ such that $d(R(M_0)/D) \geq \aleph_0$. Then $M = (M \restriction L, \langle D\rangle)$ is an $L^\mu$-structure, where $M \restriction L = M_0$ and we interpret $P^\mu$ as $\langle D\rangle$. By Lemma 6.5 $P^\mu(M)$ is definable over $P^\mu(M) \cap R(M) = R^\mu(M)$. Hence $M$ fulfils the conditions of the next definition.

**Definition** We call an $L^\mu$-structure $M = (M \restriction L, P^\mu(M))$ rich, if $M \restriction L \models T$, $P^\mu(M) \cap R(M) = R^\mu(M)$ is rich. $P^\mu(M)$ is defined over $R^\mu(M)$ by $\chi$ in Lemma 6.5, and $d(R(M)/R^\mu(M)) \geq \aleph_0$.

Corollary 6.3 provides us a rich $L^\mu$-structure.

**Lemma 6.6** *Let $M$ be a $L^\mu$-structure where $M \restriction L \models T$, $R(M) \cap P^\mu(M) = R^\mu(M) \in \mathbb{K}^\mu$, and $\varphi_\alpha(\bar{x}, \bar{b})$ a code formula. Then $\varphi_\alpha(\bar{x}, \bar{b})$ has only finitely many solutions in $R^\mu(M)$.*

*Proof*. Choose a finite strong subspace $B \leq M$ such that $\bar{b} \in \mathrm{dcl}^{\mathrm{eq}}(B)$. Each coset in $\mathbb{C}^{n_\alpha}/B^{n_\alpha}$ contains only $q^{n_\alpha \cdot \mathrm{ldim}(B)}$ elements. As in the proof of 5.1 there is a number $s = s(\alpha, \mathrm{ldim}(B))$ such that every sequence $\bar{e}_0, \ldots, \bar{e}_s$ of solutions of $\varphi_\alpha(\bar{x}, \bar{b})$ contains a subsequence $\bar{e}_{i_0}\bar{e}_{i_1} \ldots \bar{e}_{i_{\mu(\alpha)+1}}$ with $e_{i_j} \notin \langle B\bar{e}_{i_0}, \ldots \bar{e}_{i_{j-1}}\rangle^\ell$. By the property P(III) and induction $\langle B\bar{e}_{i_0}, \ldots, \bar{e}_{i_{j-1}}\rangle^\ell$ is strong and $e_{i_j}$ is generic over $B\bar{e}_{i_0}\ldots\bar{e}_{i_{j-1}}$. Hence $\bar{e}_{i_0}-\bar{e}_{i_{\mu(\alpha)+1}},\ldots,\bar{e}_{i_{\mu(\alpha)}}-\bar{e}_{i_{\mu(\alpha)+1}}$ is a Morley-sequence and therefore a difference sequence of length $\mu(\alpha)+1$. If the starting sequence $\bar{e}_0,\ldots,\bar{e}_s$ was in $R^\mu(M)$, then the canonical parameter of this difference sequence is in $\mathrm{dcl}^{\mathrm{eq}}(R^\mu(M))$ and we get a contradiction. □

**Definition** Let $M$ be a rich $L^\mu$-structure and $A$ be a subspace of $R(M)$. $A$ satisfies the condition $(*)$ if

$(*)$ $\quad A \leq M$, and $d(A/A\cap R^\mu(M)) = d(A/R^\mu(M))$.

The condition $d(A/A\cap R^\mu(M)) = d(A/R^\mu(M))$ says that the union of every geometrical basis of $A\cap R^\mu(M)$ and every geometrical basis of $A$ over $R^\mu(M)$ is a geometrical basis of $A$. This implies

$$\mathrm{cl}_d(R^\mu(M))\cap A = \mathrm{cl}_d(A\cap R^\mu(M))\cap A.$$

Note that Lemma 3.6 implies for $A$ with property $(*)$ that

$$A\cap R^\mu(M)\leq M \quad \text{and} \quad A\cap \mathrm{cl}_d(R^\mu(M))\leq M.$$

**Lemma 6.7** *Let $M$ be a rich $L^\mu$-structure.*

i) *Every $\bar{a}\subseteq R(M)$ is contained in a finite subspace $A$ that satisfies $(*)$.*

ii) *If $A$ has property $(*)$ then there is a geometrical sequence*

$$A_0\subseteq\ldots\subseteq A_{i_0}\subseteq\ldots\subseteq A_{i_1}\subseteq\ldots\subseteq A_m = A$$

*such that*

$$A_{i_0} = A\cap R^\mu(M),\quad A_{i_1} = A\cap \mathrm{cl}_d(R^\mu(M)).$$

*Proof*. i) Choose step by step a geometrical independent set $XY$ such that $X\subseteq R^\mu(M)$, $Y$ is geometrically independent over $R^\mu(M)$ and $\bar{a}\subseteq \mathrm{cl}_d(XY)$. Then any $A\leq M$ with $\langle XY,\bar{a}\rangle^\ell\subseteq A\subseteq \mathrm{cl}_d(XY)$ fulfils $(*)$.
ii) By Lemma 3.6 $A\cap R^\mu(M)$ and $A\cap\mathrm{cl}_d(R^\mu(M))$ are strong in $M$. Then P(II) and P(IV) provide the desired geometrical sequence. □

**Theorem 6.8** *Let $M$ and $N$ be rich $L^\mu$-structures. Assume $A\leq R(M)$ and $f(A)\leq R(N)$ satisfy $(*)$ where $f$ is an $\mathbb{F}_q$-vectorspace isomorphism of $A$ onto $f(A)$, $\mathrm{tp}^M_L(A)=\mathrm{tp}^N_L(f(A))$, and $f(A\cap R^\mu(M)) = f(A)\cap R^\mu(N)$. Then $(M,A)$ and $(M,f(A))$ are $L^\mu_{\infty,\omega}$-equivalent.*

Note: $f$ can be extended to an $L$-isomorphism of $\langle A\rangle$ onto $\langle f(A)\rangle$ if an only if $\mathrm{tp}_L^M(A) = \mathrm{tp}_L^N(f(A))$. This follows from P(IV) since $A \leq R(M)$ and $f(A) \leq R(N)$.

**Corollary 6.9** *The $L^\mu$-theory $T^\mu$ of the rich $L^\mu$-structures is complete.*

**Corollary 6.10** *Let $M$ and $N$ be rich $L^\mu$-structures, $\bar a \in R^\mu(M)$ and $\bar b \in R^\mu(N)$. If $\mathrm{tp}_L^M(\bar a) = \mathrm{tp}_L^N(\bar b)$, then $(M, \bar a)$ and $(N, \bar b)$ are $L^\mu_{\infty,\omega}$-equivalent.*

*Proof*. Adding elements from the algebraic closures we can assume w.l.o.g. that $\langle \bar a\rangle^\ell$ and $\langle \bar b\rangle^\ell$ are strong subspaces. Then they fulfil $(*)$. $\square$

*Proof of Theorem* 6.8. We show that the conditions in the theorem describe a winning strategy for the Ehrenfeucht-Fraïssé-game between $(M, A)$ and $(N, f(A))$: Since $M = \langle R(M)\rangle$ and $N = \langle R(N)\rangle$ we can assume w.l.o.g. that the players choose only elements in $R(M)$ and $R(N)$. The situation is completely symmetric. Hence we can assume that player I has choosen some element $a$ in $R(M)$. We show that there are $A \cup \{a\} \subseteq D \leq R(M)$ and $g$ extending $f$ such that:

$(**)$ $\quad$ $D$ and $g(D)$ fulfil $(*)$, $\mathrm{tp}_L^M(D) = \mathrm{tp}_L^N(g(D))$, and
$$g(D \cap R^\mu(M)) = g(D) \cap R^\mu(N).$$

$(**)$ describes again the winning strategy of player II in our Fraïssé-Ehrenfeucht-game for $(M, A)$ and $(N, f(A))$.

Case 1: $a \in R^\mu(M)$.

1.1 $a \notin \mathrm{cl}_d(A)$. Since $N$ is a rich $L^\mu$-structure there is some $b \in R^\mu(N) \setminus \mathrm{cl}_d(f(A))$. Let $D = \langle A, a\rangle^\ell$ and $g = f$ on $A$ and $g(a) = b$. $(**)$ is true.

1.2 $a \in \mathrm{cl}_d(A)$. By Lemma 3.6 $C_0 = A \cap R^\mu(M) \leq M$. By $(*)$ we have $d(A/C_0) = d(A/R^\mu(M))$ and therefore $a \in \mathrm{cl}_d(C_0)$. By Lemma 6.4 $\mathrm{acl}(R^\mu(M)) \cap R(M) = R^\mu(M)$. Hence by P(II) there is some finite $C \leq R^\mu(M)$ with $\langle C_0, a\rangle^\ell \subseteq C \subseteq \mathrm{acl}(C_0, a) \subseteq R^\mu(M)$. By P(IV) there is a geometrical construction $C_0 \subseteq C_1 \subseteq \ldots \subseteq C_m = C \subseteq R^\mu(M)$. By richness of $N$ we get $g$ extending $f$ to $A + C$ such that $g \restriction C$ preserves the geometrical construction above. By Lemma 3.9 $A = A + C_0 \subseteq A + C_1 \subseteq \ldots \subseteq A + C_m \leq M$ and $g(A) = g(A + C_0) \subseteq g(A + C_1) \subseteq \ldots \subseteq g(A + C_m) \leq N$ are geometrical constructions. They are preserved by $g$. By Lemma 3.3 $\mathrm{tp}_L^M(A + C) = \mathrm{tp}_L^N(g(A + C))$. $A + C$ and $g(A + C)$ fulfil $(*)$. Furthermore $(A + C) \cap R^\mu(M) = C$ and $g(A + C) \cap R^\mu(N) = g(C)$. Hence $A + C$ and $g$ fulfil $(**)$.

Case 2: $a \in \mathrm{cl}_d(R^\mu(M))$

Using Case 1 we add elements of $R^\mu(M)$ to $A$ such that we can assume w.l.o.g. that $a \in \mathrm{cl}_d(R^\mu(M) \cap A)$. Let $C$ be a strong subspace of $R(M)$ in $\mathrm{cl}_d(R^\mu(M) \cap A)$ such that $\{a\} \cup (\mathrm{cl}_d(R^\mu(M)) \cap A) \subseteq C$. By Lemma 3.6 $C \cap R^\mu(M) \leq M$. Furthermore $C \cap \mathrm{cl}_d(R^\mu(M)) \supseteq A \cap \mathrm{cl}_d(R^\mu(M))$ by construction. Let $C_0$ be $C \cap R^\mu(M)$. Using Case 1 we can assume w.l.o.g. that $C_0$ is a subspace of $A$. By P(IV) there is a geometrical construction $C_0 \subseteq C_1 \subseteq \ldots \subseteq C_m = C$. By Lemma 3.9 $A = A + C_0 \subseteq A + C_1 \subseteq \ldots \subseteq A + C_m$ is a geometrical construction and all $A + C_i \leq M$. We show the assertion for each $A + C_i$ by induction on $i$. By Lemma 3.6 $A \cap \mathrm{cl}_d(R^\mu(M))$ is strong. Hence we can assume w.l.o.g. $A \cap \mathrm{cl}_d(R^\mu(M)) = A \cap \mathrm{cl}_d(C_0) = C_{i_0}$ and $A + C_{i+1} \neq A + C_i$ for $i \geq i_0$. The induction starts with the trivial case $i_0$. We assume there is $g$ with $(**)$ for $A + C_i$. That means we have property $(*)$ for $A + C_i$ and $g(A + C_i)$, where $g$ is an extension of $f$ that preserves $\mathrm{tp}_L(A + C_i)$ and satisfies

$$f(C_0) = g(C_0) = g(R^\mu(M) \cap (A + C_i)) = R^\mu(N) \cap g(A + C_i).$$

First we assume that $\langle C_i c\rangle^\ell = C_{i+1}$ $(i \geq i_0)$ where $c$ is isolated over $C_i$ by an algebraic $L$-formula $\psi(x, \bar{d})$ with $\bar{d}$ in $C_i$. By construction $c \notin R^\mu(M) + (A + C_i)$. We can assume that $\psi(x, \bar{d})$ isolates $c$ over $A + C_i$ with respect to the $L$-theory. Now we choose $g(c)$ as a solution of $\psi(x, g(\bar{d}))$. This formula is algebraic and isolates $g(c)$ over $g(A + C_i)$ by induction. $A + C_{i+1}$ and $g(A + C_{i+1})$ satisfy $(*)$ and they have the same $L$-type. It remains to show that $g(c) \notin R^\mu(N) + g(A + C_i)$. $g(c) \in R^\mu(N) + g(A + C_i)$ would imply $g(c) \in R^\mu(N) + g(C_i)$ since

$$\mathrm{cl}_d(R^\mu(N)) \cap g(A + C_i) = g(C_i).$$

Hence w.l.o.g. $g(c) \in R^\mu(N)$. Otherwise we can change $c$. $c \notin \mathrm{acl}(C_0)$, since otherwise $c \in R^\mu(M)$. Hence there is some $s > 0$ such that $c \in \mathrm{acl}(C_s) \setminus \mathrm{acl}(C_{s-1})$. Then $C_s = \langle C_{s-1}\bar{e}\rangle^\ell$ and $\bar{e}$ is a solution of some $\varphi_\alpha(\bar{x}, \bar{b})$ with $\bar{b} \in \mathrm{dcl}^{\mathrm{eq}}(C_{s-1})$ and $\bar{e}$ is generic over $C_{s-1}$. Since $C_i \cap R^\mu(M) = C_0$ we have $\bar{e} \not\subseteq R^\mu(M) + C_{s-1}$ and therefore $\bar{e}$ is generic over $R^\mu(M) + C_{s-1}$. Since $g(c)$ fulfils $g(\mathrm{tp}^L(c/A + C_i))$, we have $g(c) \in \mathrm{acl}(g(C_{s-1}, g(\bar{e}))) \setminus \mathrm{acl}(g(C_{s-1}))$. By the Exchange Property for strongly minimal sets we get $g(\bar{e}) \in \mathrm{acl}(g(C_{s-1}), g(c)) \subseteq \mathrm{acl}(g(C_{s-1}) + R^\mu(N))$. $g(\bar{e})$ is linearly independent over $R^\mu(N) + g(C_{s-1}))$. Hence by P(III) $g(\bar{e})$ is generic over $R^\mu(N) + g(C_{s-1})$. This is the desired contradiction to $g(\bar{e}) \in \mathrm{acl}(g(C_{s-1}) + R^\mu(N))$.

Now we assume $C_{i+1} = \langle C_i, \bar{c}\rangle^\ell$ where $\bar{c}$ is a solution of some $\psi_\alpha(\bar{x}, \bar{d}) \in \mathcal{X}$ where $\bar{d} \in \mathrm{dcl}^{\mathrm{eq}}(C_i)$ and $\bar{c}$ is generic over $C_i$. By assumption $\bar{c} \not\subseteq A + C_i$. Hence

$\bar{c}$ is generic over $A + C_i$ by P(III). Since $(A + C) \cap R^\mu(M) = A \cap R^\mu(M)$ we have $\bar{c} \not\subseteq R^\mu(M) + (A + C_i)$. Since by Lemma 3.9 $R^\mu(M) + A + C_i \leq M$ $\bar{c}$ is also generic over this space. Now we consider $\psi_\alpha(\bar{x}, g(\bar{d}))$. This formula is again strongly minimal. By Lemma 6.6 there are only finitely many solutions of $\psi_\alpha(\bar{x}, \bar{e})$ in $R^\mu(N)$ for every parameter $\bar{e}$ and therefore also in every coset of this subspace. Hence we have infinitely many solutions of $\psi_\alpha(c, g(\bar{d}))$ generic over $R^\mu(N) + g(A + C_i) \leq N$. Let $g(\bar{c})$ be one of them and define $g(C_{i+1}) = \langle g(c) C_i \rangle^\ell$ accordingly. Then $A + C_{i+1}$ and $g(A + C_{i+1})$ satisfy again $(*)$ and have the same $L$-type and by the choice of $g(\bar{c})$ we have

$$g(R^\mu(M) \cap (A + C_{i+1})) = R^\mu(N) \cap g(A + C_{i+1}).$$

Case 3: $\quad a \in \mathrm{cl}_d(A)$
Using induction we consider an algebraic or prealgebraic extension $\langle A\bar{c} \rangle^\ell$ of $A$. Using Case 2 we can assume that $\bar{c} \notin \mathrm{cl}_d(R^\mu(M))$. Note that this implies

$$\langle \bar{c} \rangle^\ell \cap \mathrm{cl}_d(R^\mu(M)) = \langle 0 \rangle^\ell.$$

Then we choose $g(\bar{c})$ such that $\mathrm{tp}_L^M(A\bar{c}) = \mathrm{tp}_L^N(g(A)g(\bar{c}))$. This is possible, since $\bar{c}$ is isolated over $A$. Hence by P(II) $g(\bar{c})$ has the same geometrical behaviour over $g(A)$ as $\bar{c}$ over $A$. By the conditions $(**)$ of the game

$$d(g(A)/R^\mu(N)) = d(g(A)/R^\mu(N) \cap g(A)).$$

Since $\bar{c} \in \mathrm{cl}_d(A) \setminus \mathrm{cl}_d(A \cap R^\mu(M))$ we get $g(\bar{c}) \in \mathrm{cl}_d(A) \setminus \mathrm{cl}_d(A \cap R^\mu(N))$. But $g(\bar{c}) \in \mathrm{cl}_d(R^\mu(N))$ would contradict the above equation. Hence $\langle A\bar{c} \rangle^\ell$ and $g$ ensure the conditions of the game.

Case 4: $\quad a \notin \mathrm{cl}_d(A)$
If $a \in \mathrm{cl}_d(A + R^\mu(M))$ then we use the cases before to play the game. We add the necessary element from $R^\mu(M)$ to $A$. Otherwise $a \notin \mathrm{cl}_d(A + R^\mu(M))$. Since $N$ is rich there is some $g(a) \notin \mathrm{cl}_d(g(A) + R^\mu(N))$. Again the conditions $(**)$ of the game are fulfilled. □

**Corollary 6.11** *Let $M$ be a rich $L^\mu$-structure. The code formulas $\varphi_\alpha(\bar{x}, \bar{b})$ with $\bar{b}$ in $P^\mu(M)^{\mathrm{eq}}$ are minimal.*

*Proof*. By Lemma 6.6 there are only finitely many solutions in $R^\mu(M)$. Let $B \leq M$ be a strong subspace of $R^\mu(M)$ such that $\bar{b} \in \mathrm{dcl}^{\mathrm{eq}}(B)$. We show that any two solutions $\bar{a}$, $\bar{c}$ that are not in $B$ have the same $L^\mu$-type over $B$. $\bar{a}$ and $\bar{c}$ are solutions of $\varphi_\alpha(\bar{x}, \bar{b})$ generic over $B$. Then $\langle Ba\rangle^\ell$ and $\langle Bc\rangle^\ell$ with $f(B) = B$ and $f(a) = c$ fulfil the conditions in Theorem 6.8. Hence $\mathrm{tp}_{L^\mu}(a/B) = \mathrm{tp}_{L^\mu}(c/B)$. $\square$

**Lemma 6.12** *For every good code $\alpha$ there is a $L^\mu$-sentence $\chi_\alpha$ such that for all $L^\mu$-structures $M$ where $M \restriction L \models T$, $R(M) \cap P^\mu(M) = R^\mu(M) \in \mathbb{K}^\mu$ and $\langle R^\mu(M)\rangle = P^\mu(M)$:*
*$M \models \chi_\alpha$ if and only if every minimal prealgebraic extension of $R^\mu(M)$ given by $\varphi_\alpha(\bar{x}, \bar{b})$ with $\bar{b} \in P^\mu(M)^{\mathrm{eq}}$ is not in $\mathbb{K}^\mu$.*

*Proof*. Let $\bar{a}$ be a solution of $\varphi_\alpha(\bar{x}, \bar{b})$ not in $R^\mu(M)$. By P(III) $\bar{a}$ is a generic solution. If $\langle R^\mu(M)\bar{a}\rangle^\ell$ is not in $\mathbb{K}^\mu$ we have the cases a) or b) of Corollary 5.3. In case a) $\langle R^\mu(M)\bar{a}\rangle^\ell$ contains a difference sequence for $\varphi_\alpha(\bar{x}, \bar{b})$ of length $\mu(\alpha) + 1$ for $\alpha$. In case b) there is a difference sequence of length $\mu(\beta)+1$ for a good code $\beta$, which contains a subsequence of length $\mu^*(\beta)$ linearly independent over $R^\mu(M)$. Hence $\mu^*(\beta)n_\beta \leq n_\alpha$ in this case. Since $\mu^*$ is finite-to-one, only a finite set $C_\alpha$ of codes $\beta$ can occur. Let $C'_\alpha = C_\alpha \cup \{\alpha\}$. Then $R^\mu(M)$ has no prealgebraic minimal extensions in $\mathbb{K}^\mu$ given by $\alpha$ if and only if

$$\begin{aligned} M \models\ & \forall \bar{b} \in P^\mu \bigvee_{\beta \in C'_\alpha} \exists \bar{y}_0 \ldots \bar{y}_{\mu(\beta)} \in R^\mu(M)[\exists \bar{x}\varphi_\alpha(\bar{x}, \bar{b}) \longrightarrow \\ & \exists^\infty \bar{x}(\varphi_\alpha(\bar{x}, \bar{b}) \wedge \exists \bar{z}_0 \ldots \bar{z}_{\mu(\beta)} \in \langle \bar{x}\rangle\, \psi_\beta(\bar{y}_0 + \bar{z}_0, \ldots \bar{y}_{\mu(\beta)} + \bar{z}_{\mu(\beta)}))]. \end{aligned}$$

W.l.o.g. $\varphi_\alpha(\bar{x}, \bar{y})$ is in $L$. Note that the formula after $\exists^\infty \bar{x}$ is in $L$ and $\varphi_\alpha(\bar{x}, \bar{b})$ is strongly minimal in $M \restriction L$. In this way we express "for $\bar{x}$ generic over $R^\mu(M)$", since by Lemma 6.6 $\varphi_\alpha(\bar{x}, \bar{b})$ has only finitely many solutions in $R^\mu(M)$. $\square$

# 7 Axiomatization of $T^\mu$

We assume that $T$ satisfies P(I) – P(VII). Let $T^\mu$ be the theory of rich $L^\mu$-structures. By Corollary 6.9 $T^\mu$ is a complete theory. We use the notation $R^\mu = P^\mu \cap R$. Using the Amalgamation Theorem 6.2 we get a countable rich subspace $D$ of $R(\mathbb{C})$. Let $P^\mu$ be $\langle D\rangle$. This subspace is $L$-definable over $D$ (Lemma 6.5). Then $(\mathbb{C}, P^\mu(\mathbb{C}^\mu))$ is a rich $L^\mu$-structure. We call it our standard model. The following is true in $T^\mu$ and can be expressed in $L^\mu$. Let $M$ be a model of $T^\mu$:

$T^\mu$ 1) $M \upharpoonright L$ is a model of $T$.

$T^\mu$ 2) $\mathrm{acl}^L(R^\mu(M)) \cap R(M) = R^\mu(M)$ and $P^\mu(M) = \langle R^\mu(M)\rangle$ described by Lemma 6.5. $d(R^\mu(M))$ and $d(R(M)/R^\mu(M))$ are infinite for $\omega$-saturated models.

$T^\mu$ 3) $R^\mu(M)$ is in $\mathbb{K}^\mu$.

$T^\mu$ 4) If $\bar{b}$ is in $\mathrm{dcl}^{\mathrm{eq}}(R^\mu(M)))$ and $\bar{a}$ is a solution of $\varphi_\alpha(\bar{x}, \bar{b})$ in $R(M)$ generic over $R^\mu(M)$ for some code formula $\varphi_\alpha(\bar{x}, \bar{b})$, then $R^\mu(M) + \langle \bar{a}\rangle^\ell$ is not in $K^\mu$.

These sets of axioms are elementary. For $T^\mu$ 1) this is clear. Using Lemma 6.5 there is no problem to express $T^\mu$ 2). For $T^\mu$ 3) note that $R^\mu(M)$ is strong since it is closed under $\mathrm{acl}^L$ in $R(M)$. The absense of difference sequences for $\varphi_\alpha(\bar{x}, \bar{y})$ of length $\mu(\alpha) + 1$ in $R^\mu(M)$ can be expressed by Theorem 4.5. For $T^\mu$ 4) we use Lemma 6.12. Finally for all axioms we find $L^\mu$-formulas.
It is clear that rich $L^\mu$-structures satisfy $T^\mu$ 1), $T^\mu$ 2) and $T^\mu$ 3). That they also satisfy $T^\mu$ 4) is part of the following theorem:

**Theorem 7.1** *An $L^\mu$-structure $M$ that satisfies $T^\mu$ 1), $T^\mu$ 2) and $T^\mu$ 3) is rich if and only if it is an $\omega$-saturated model of $T^\mu$.*

*Proof.* First assume that $M = (M \upharpoonright L, P^\mu(M))$ is an $\omega$-saturated model of $T^\mu$. We show that $M$ is rich. Let $B \subseteq A$ be in $\mathbb{K}^\mu$. Then $B$ and $A$ are strong in $R(\mathbb{C})$. Assume $B \leq R^\mu(M)$. W.l.o.g. $A$ is a minimal strong extension of $B$. There are three cases:

i) If $A = \langle Ba\rangle^\ell$ and $a$ is algebraic over $B$, then $A$ is in $R^\mu(M)$ by $T^\mu$ 2).

ii) Let $A$ be a minimal prealgebraic extension of $B$: $A = \langle B\bar{a}\rangle^\ell$ where $\bar{a}$ is the generic solution of some code formula $\varphi_\alpha(\bar{x}, \bar{b})$ where $\bar{b}$ is in $\mathrm{dcl}^{\mathrm{eq}}(B)$. There is a solution $\bar{a}$ in $R(M)$ generic over the strong subspace $R^\mu(M)$. By Axiom $T^\mu$ 4) this free amalgam $R^\mu(M) \oplus \langle \bar{a}\rangle^\ell$ over $B$ is not in $\mathbb{K}^\mu$. By Theorem 6.2 and P(III) there is a partial $L$-elementary map of $\bar{a}$ over $B$ into $R^\mu(M)$ as desired.

iii) $A$ is a minimal transcendental extension. Then Axiom $T^\mu$ 2) ensures the assertion.

Now let $M$ be a rich $L^\mu$-structure. $M$ satisfies $T^\mu\,1) - T^\mu\,3)$. We show $T^\mu\,4)$. Choose a strong subspace $B$ in $R^\mu(M)$ such that $\bar{b} \in \mathrm{dcl}^{\mathrm{eq}}(B)$. Assume there is a solution $\bar{a}$ of $\varphi_\alpha(\bar{x}, \bar{b})$ generic over $R^\mu(M)$ such that $R^\mu(M) + \langle \bar{a} \rangle^\ell$ is in $K^\mu$. Since $M$ is rich there is a partial elementary copy $A_0 \supseteq B$ of $\langle B\bar{a} \rangle$ over $B$ in $R^\mu(M)$. Since $B \leq M \quad A_0 \leq M$. In the next step we get a copy $A_1$ of $\langle A_0\bar{a} \rangle^\ell$ over $A_0$ inside $R^\mu(M)$. We can continue this process as long as we want and get a contradiction to the fact that $R^\mu(M)$ is in $\mathbb{K}^\mu$. By Corollary 6.3 there exists a rich $L^\mu$-structure. Hence $T^\mu$ is consistent and we have an $\omega$-saturated model $N$ of $T^\mu$. As shown above $N$ is a rich $L^\mu$-structure. By Theorem 6.8 $M$ and $N$ are $L^\mu_{\infty,\omega}$-equivalent. Hence $M$ is an $\omega$-saturated model of $T^\mu$. $\square$

**Corollary 7.2** *The deductive closure of* $T^\mu\,1)$ – $T^\mu\,4)$ *is the complete theory* $T^\mu$.

*Proof*. This follows from Theorem 7.1 and Corollary 6.9. $\square$

Let $\mathbb{C}^\mu$ be the monster model of $T^\mu$ where we work in.

**Lemma 7.3** *Let* $M \preceq \mathbb{C}^\mu$ *be a model of* $T^\mu$.

i) $R^\mu(\mathbb{C}^\mu)$ *and* $M$ *are geometrically independent over* $R^\mu(M)$.

ii) *In* $R^\mu(\mathbb{C}^\mu)\quad \mathrm{cl}_d(X)$ *is part of* $\mathrm{acl}^\mu(X)$.

iii) $R^\mu(x)$ *is strongly minimal.*

iv) $P^\mu(x)$ *is of finite Morley rank.*

*Proof*. i) If $\bar{a}$ in $M$ is geometrically dependent over $R^\mu(M)$, then there is a geometrically construction over a proper subspace $A'$ of $\langle \bar{a} \rangle^\ell$ and $R^\mu(M)$ that contains an element of $\langle \bar{a} \rangle^\ell \setminus A'$. Hence "$\bar{a}$ is geometrically independent over $R^\mu$" is part of the $L^\mu$-type of $\bar{a}$. Since $\mathrm{tp}^M(\bar{a}) = \mathrm{tp}^{C^\mu}(\bar{a})$ it follows the assertion.

ii) W.l.o.g. we assume $B \leq R(\mathbb{C}^\mu)$, $B \subseteq R^\mu(\mathbb{C}^\mu)$ and $a \in \mathrm{cl}_d(B) \cap R^\mu(\mathbb{C}^\mu)$. Let $A \leq R(\mathbb{C}^\mu)$ be a geometrical construction over $B$ that contains $a$ and has only algebraic and prealgebraic steps. By Lemma 3.4 we can assume w.l.o.g. that $A \subseteq R^\mu(\mathbb{C})$. By Lemma 6.6 $A \subseteq \mathrm{acl}^\mu(B)$.

iii) To show the strong minimality of $R^\mu(x)$, we consider again some $\omega$ saturated $M \preceq \mathbb{C}^\mu \models T^\mu$ and $a, c \in R^\mu(\mathbb{C}^\mu) \setminus M$. By ii) $a$ and $c$ are not in $\mathrm{cl}_d(R^\mu(M))$. By i) they are both not in $\mathrm{cl}_d(M)$. By Lemma 6.7 every finite subspace of $R(M)$ is contained in some $A \subseteq R(M)$ that satisfies $(*)$. If we define $f = \mathrm{id}$ on $A$

and $f(a) = c$, then $\langle Aa \rangle^{\ell}$ and $f$ satisfy the conditions of Theorem 6.8. Hence $\mathrm{tp}_{L^\mu}(A, a) = \mathrm{tp}_{L^\mu}(A, c)$ and therefore $\mathrm{tp}_{L^\mu}(a/M) = \mathrm{tp}_{L^\mu}(c/M)$ as desired.

iv) Since $P^\mu(\mathbb{C}^\mu) = \langle R^\mu(\mathbb{C}^\mu) \rangle$ and $R^\mu(\mathbb{C}^\mu)$ is strongly minimal, $P^\mu(x)$ has finite Morley rank. $\square$

**Theorem 7.4** *$T^\mu$ is $\omega$-stable.*

*Proof*. Let $M$ be a countable elementary submodel of $\mathbb{C}^\mu$. We show that there are only countably many types $\mathrm{tp}(\bar{a}/M)$ where $\bar{a}$ is a finite tuple in $\mathbb{C}^\mu$ . W.l.o.g. we can restrict us to $\bar{a} \subseteq R(\mathbb{C}^\mu)$. Furthermore we will consider finite subspaces $\bar{a} \subseteq A \subseteq R(\mathbb{C})$ with certain properties only. For a given $\bar{a} \subseteq R(\mathbb{C}^\mu)$ it is easy to find a set XYZW of geometrically independent elements (short geo. basis) such that the following is true:

(0) $\bar{a} \subseteq \mathrm{cl}_d(XYZW)$

(1) $X \subseteq R^\mu(M)$

(2) $Y \subseteq R(M)$ is geometrically independent over $R^\mu(M)$.

(3) $Z \subseteq R^\mu(\mathbb{C}^\mu)$ (short $R^\mu$) is geometrically independent over $R^\mu(M)$.

(4) $W$ is geometrically independent over $R(M) + R^\mu$.

By Lemma 7.3 i) $Y$ is geometrically independent over $R^\mu$ and $Z$ over $M$. Now we choose any $A$ such that $XYZW \subseteq A \subseteq \mathrm{cl}_d(XYZW)$, $\bar{a} \subseteq A$ and $A \leq \mathbb{C}^\mu$. Then

$$\begin{aligned}
&A \cap R^\mu(M) \subseteq A \cap \mathrm{cl}_d(R^\mu(M)) \subseteq \mathrm{cl}_d(X),\\
&A \cap R(M) \subseteq A \cap \mathrm{cl}_d(R(M)) \subseteq \mathrm{cl}_d(XY),\\
&A \cap R^\mu \subseteq A \cap \mathrm{cl}_d(R^\mu) \subseteq \mathrm{cl}_d(XZ),\\
&A \cap (R(M) + R^\mu) \subseteq A \cap \mathrm{cl}_d(R(M) + R^\mu) \subseteq \mathrm{cl}_d(XYZ).
\end{aligned}$$

By Lemma 3.6 the eight intersections above are strong in $\mathbb{C}^\mu$. Note that $A \cap (R(M) + R^\mu)$ contains the free sum of $A \cap R(M)$ and $A \cap R^\mu$ over $A \cap R^\mu(M)$. Let $\bar{D}$ be a geometrical construction for $A$ over $XYZW$ that starts with a geometrical construction of $A \cap R^\mu(M)$ over $X$, then extends this to $A \cap R(M)$ and $A \cap R^\mu$.

Now we consider $XYZ'W' \subseteq A' \subseteq \mathrm{cl}_d(XYZ'W')$ with $A' \cap R(M) = A \cap R(M)$, $XYZ'W'$ satisfy the properties (1)–(4) and there is a vectorspace isomorphism $f$

of $A$ onto $A'$ that extends the identity on $A \cap R(M)$, preserves the geometrical construction $\bar{D}$, and

$$f(Z) = Z', \quad f(W) = W', \quad f(R^\mu \cap A) = R^\mu \cap A'.$$

Then $A$ and $A'$ satisfy the conditions in Theorem 6.8: $A$ and $A'$ have $(*)$ and $f(A \cap R^\mu) = f(A) \cap R^\mu$. By Lemma 3.3 $\mathrm{tp}_L(A) = \mathrm{tp}_L(A')$. Hence by Theorem 6.8 $\mathrm{tp}_{L^\mu}(A) = \mathrm{tp}_{L^\mu}(A')$.
For any subspace $E \subseteq R(M)$ we can enlarge $X$ to $X_E$, $Y$ to $Y_E$, $A$ to $A_E$, $A'$ to $A'_E$, $\bar{D}$ to $\bar{D}_E$ and $f$ to $f_E$ such that the conditions above remain true $E \subseteq M \cap A_E = M \cap A'_E$, $A_E = A + (M \cap A_E)$ and $A'_E = A' + (M \cap A'_E)$. Then again $\mathrm{tp}_{L^\mu}(A_E) = \mathrm{tp}_{L^\mu}(A'_E)$.
Since $E$ was arbitrary we have shown that $\mathrm{tp}_{L^\mu}(A/M) = \mathrm{tp}_{L^\mu}(A'/M)$ if $A$ and $A'$ are given as above. The conditions above define an equivalence relation for subspaces $A$ with only countably many classes. hence $T^\mu$ is $\omega$-stable. □

Let $T_i$ $(i = 0, 1)$ be complete $L_i$-theories. Let $\Delta$ be an interpretation of the theory $T_0$ in the theory $T_1$. In [Bau1] is defined that $\Delta$ is an interpretation of $T_0$ in $T_1$ without new information, if for every $M \models T_1$ every subset $X$ of $\Delta(M)$ defined in $M$ by a $L_1$-formula without parameters is definable by a $L_0$-formula without parameters. If $T_1$ is stable, then we have the same for formulas with parameters. In [Bau1] the following result of Lascar is published:

**Lemma 7.5 (Lascar)** *If $T_1$ is stable and $\Delta$ is an interpetation of $T_0$ in $T_1$ without new information, then for every model $N$ of $T_0$ there is some model $M \models T_1$ such that $\Delta(M) \cong N$.*

**Definition** If $M$ is a model of $T^\mu$, then let $\Gamma(M)$ be the $L$-substructure of $M$ with domain $P^\mu(M)$. Let $\Gamma(T^\mu)$ be the complete $L$-theory of all $\Gamma(M)$ where $M \models T^\mu$.

$\Gamma$ defined above is an interpetation. We get:

**Theorem 7.6** *Let $T$ be a theory with* P(I) – P(VII). *$\Gamma(T^\mu)$ is uncountably categorical, $R(x)$ is a strongly minimal formula in this theory. The pregeometry of $R$ is given by $\mathrm{acl} = \mathrm{cl}_d$. For models $N$ of $\Gamma(T^\mu)$ we have $N = \langle R(N) \rangle$.*

*Proof.* $R^\mu(x)$ is strongly minimal for $T^\mu$ by Lemma 7.3 iii). Hence $R(x)$ is strongly minimal in $\Gamma(T^\mu)$. Since $\Gamma(M) = \langle R(\Gamma(M)) \rangle$ we have that $\Gamma(T^\mu)$ is uncountably categorical. Since $\mathrm{cl}_d$ contains acl we get $\mathrm{acl} = \mathrm{cl}_d$ by Lemma 6.6. □

**Theorem 7.7** *Let $T$ be a theory with* P(I) – P(VII). *Every subset of $P^\mu(\mathbb{C}^\mu)^n$ defined in $\mathbb{C}^\mu$ is $L$-definable in $\Gamma(\mathbb{C}^\mu)$. Hence $\Gamma$ is an interpretation without new information and every model of $\Gamma(T^\mu)$ has the form $\Gamma(M)$ with $M \vDash T^\mu$.*

*Proof*. Let $M$ be a $\omega$-saturated model of $T^\mu$. Since $T^\mu$ is $\omega$-stable $P^\mu$ is stably embedded in $M$. Hence it is sufficient to consider $\emptyset - L^\mu$-definable sets $X$. By Lemma 6.5 we can assume that $X \subseteq R^\mu(\mathbb{C}^\mu)^n$. We have to show the following: If $\bar{a}$ and $\bar{b}$ are tuples in $R^\mu(M)$ with $\mathrm{tp}^{\Gamma(M)}(\bar{a}) = \mathrm{tp}^{\Gamma(M)}(\bar{b})$ then $\mathrm{tp}^M(\bar{a}) = \mathrm{tp}^M(\bar{b})$. $\mathrm{tp}^{\Gamma(M)}(\bar{a}) = \mathrm{tp}^{\Gamma(M)}(\bar{b})$ is equivalent to $\mathrm{tp}^{*R^\mu(M)}(\bar{a}) = \mathrm{tp}^{*R^\mu(M)}(\bar{b})$. $\mathrm{tp}^{*R^\mu(M)}(a)$ is used to denote the subset of all formulas of $\mathrm{tp}^M(\bar{a})$ with quantifiers that are restricted to $R^\mu$. By Corollary 6.10 it is sufficient to show that

$$\mathrm{tp}^{M\restriction L}(\bar{a}) = \mathrm{tp}^{M\restriction L}(\bar{b}).$$

For this we use Lemma 3.3. Let $A_0 \subseteq A_1 \subseteq \ldots \subseteq A_m$ be a geometrical construction for $\bar{a}$ over $A_0$ where $A_0 \subseteq \langle \bar{a} \rangle^\ell$ is the linear hull of geometrically independent elements and $\bar{a} \subseteq A_m \subseteq \mathrm{cl}_d(A_0)$. We can choose $A_m \subseteq \mathrm{acl}^L(\bar{a})$ and therefore $A_m \subseteq R^\mu(M)$. Now we use that there are quantifier free formulas that describe the geometrical construction (P(IV)).
Assume $\bar{x}_0$ are variables for a vector basis $\bar{a}_0$ for $A_0$.
$\bar{x}_1$ are variables for some $\bar{a}_1$ such that $\bar{a}_0\bar{a}_1$ is a vector basis for $\langle \bar{a} \rangle^\ell$. W.l.o.g. $\bar{a} = \bar{a}_0\bar{a}_1$.
$\bar{y}$ are variables for some $\bar{c}$ such that $\bar{a}_0\bar{a}_1\bar{c}$ is a vector basis for $A_m$.
$\mathrm{tp}^{*R^\mu(M)}(\bar{a}, \bar{c})$ contains a description of the geometrical construction and the information about the geometrical independence of the subset $\bar{a}_0$ of $\bar{a}$. There is a quantifier free formula $\varphi(\bar{x}_0, \bar{x}_1, \bar{y})$ that describes the algebraic and prealgebraic extensions of $A_0$ necessary to obtain $A_m$.
The geometrical independence of the elements of $\bar{a}_0$ can be described by formulas $\neg\exists \bar{z}(\bar{z} \in R^\mu \wedge \psi(\bar{x}_0, \bar{z}))$, where $\psi(\bar{x}_0, \bar{z})$ is quantifier free and describes a possible geometrical construction over a proper subset of $\bar{a}_0$ that uses only algebraic and prealgebraic steps. Note we can restrict us to $\bar{z} \in R^\mu$ since such a geometrical construction would exist inside $\mathrm{acl}^L(R^\mu(M)) \cap R(M) = R^\mu(M)$. These formulas are all in $\mathrm{tp}^{*R^\mu(M)}(\bar{a}) = \mathrm{tp}^{*R^\mu(M)}(\bar{b})$. They ensure $\langle \bar{a}_0 \rangle \leq M$ and $M \vDash \varphi(\bar{a}_0, \bar{a}_1, \bar{c})$ describes the geometrical construction over $A_0$. By Lemma 3.3 these facts fix $\mathrm{tp}^{M\restriction L}(\bar{a}_0\bar{a}_1\bar{c})$.
By $\omega$-saturation there is some $\bar{d}$ in $R^\mu(M)$ such that $\mathrm{tp}^{*R^\mu(M)}(\bar{a}, \bar{c}) = \mathrm{tp}^{*R^\mu(M)}(\bar{b}, \bar{d})$. Hence there is some vectorspace isomorphism $f$ of $\langle \bar{a}\bar{c} \rangle^\ell$ onto $\langle \bar{b}\bar{d} \rangle^\ell$ that preserves the geometrical construction. By Lemma 3.3 $\mathrm{tp}^{M\restriction L}(\bar{a}\bar{c}) = \mathrm{tp}^{M\restriction L}(\bar{b}\bar{d})$ as desired.
□

# 8 A new uncountably categorical group

We consider 2-nilpotent graded $\mathbb{F}_q$-Lie algebras $M$. If we say this we mean the following:
Let $\mathbb{F}_q$ be the finite field with $q$ elements. $M$ is an $\mathbb{F}_q$-vectorspace $M_1 \oplus M_2$ with a Lie-multiplication $[\ ,\ ]$ such that $[M_1, M_1] \subseteq M_2$, $[M_1, M_2] = 0$ and $[M_2, M_2] = 0$. Furthermore we assume that $\langle M_1 \rangle^M = M$, where $\langle X \rangle^M$ is the Lie subalgebra generated by $X$. That means $\langle [M_1, M_1] \rangle^\ell = M_2$.

We use an elementary language $L$ that is an extension of the language of $\mathbb{F}_q$-vectorspaces by $[\ ,\ ]$ for the Lie multiplication, $R = R_1$ and $R_2$ for $M_1$ and $M_2$, respectively. Note that a free algebra $F(M_1)$ over $M_1$ is given by $(F(M_1))_2 = \Lambda^2 M_1$ where $\Lambda^2 M_1$ is the exterior square over $M_1$. Then $M \cong F(M_1)/_{N(M)}$ where $N(M)$ is a subspace of $\Lambda^2 M_1$.

If $H_1$ is a subspace of $M_1$, then

$$H = \langle H_1 \rangle^M \cong F(H_1)/_{N(M) \cap \Lambda^2 H_1},$$

since there is a canonical embedding of $F(H_1)$ into $F(M_1)$.

**Definition** We define $\delta(H) = \mathrm{l.dim}(H_1) - \mathrm{l.dim}(N(H))$ where $N(H) = N(M) \cap \Lambda^2 H_1$.

This is the approach in [Bau2]. We follow the ideas of the first four chapters in this paper to get a theory $T$ with P(I) – P(VII). Omitted proofs are in that paper. We use $A$, $B$, $C$ to denote finite subspaces of $M_1$ where $M$ is as above. Let $U$, $V$ be arbitrary subspaces of $M_1$. If we write $\delta(E)$ for $E \subseteq M_1$, then this is $\delta(\langle E \rangle)$.

**Definition** We say $B \leq U$ for $B \subseteq U \subseteq M_1$ ($B$ is self-sufficient or strong in $U$), if $\delta(B) \leq \delta(A)$ for all $B \subseteq A \subseteq U$.
We define $B \underset{n}{\leq} U$ ($B$ is $n$-strong in $U$) if we consider only $A$ with $\mathrm{l.dim}(A/B) \leq n$.
$V \leq U$ if for every $B \subseteq V$ there is some $A$ such that $B \subseteq A \subseteq V$ and $A \leq U$.
We also use $A \leq M$ and $U \leq M$ instead of $A \leq M_1$ and $U \leq M_1$.

**Lemma 8.1** $\delta(A + B) \leq \delta(A) + \delta(B) - \delta(A \cap B)$.

**Assumption** We consider only $M$ with $\langle a \rangle \leq M$ for all $a \in M$.

That means $\delta(A) \geq 1$ for all $A \neq \langle 0 \rangle$ in $M$. Hence we can define

**Definition** $d(A) = \min\{\delta(B) : A \subseteq B \subseteq M\}$. $a \in \mathrm{cl}_d(A_1)$, if $d(A) = d(A \cup \{a\})$. We also use $d(H_1) = d(H)$.

**Lemma 8.2** *For $\delta$ defined above the following is true:*

i) *The intersection of strong subspaces is strong.*

ii) $\mathrm{cl}_d$ *defines a pregeometry on the subspaces of $M_1$ with dimension function $d$.*

iii) *Strongness is transitiv.*

iv) *If $V \leq U$, then $X \cap V \leq X \cap U$ for every subspace $X$.*

By i) we can define $\mathrm{CSS}(A)$ as the intersection of all $B$ that are strong in $M$ and contain $A$. Then $\mathrm{CSS}(A) \subseteq \mathrm{acl}^M(A)$.
Let $\mathbb{K}$ be the class of all 2-nilpotent graded $\mathbb{F}_q$-Lie algebras $M$ with $M = \langle M_1 \rangle$ such that

i) $[a, b] \neq 0$ for linearly independent $a$, $b$ in $M_1$.

ii) $\langle a \rangle^\ell \leq M_1$ for all $a \in M_1$.

Note that i) implies $\delta(A) = \mathrm{l.dim}(A)$ for $A \subseteq M$ with $\mathrm{l.dim}(A) \leq 3$.
In [Bau2] a class is considered where i) is replaced by $d(A) = \mathrm{l.dim}(A)$ for $\mathrm{l.dim}(A) \leq 3$.
If $H$ and $K$ are 2-nilpotent graded $\mathbb{F}_q$-Lie algebras as above with a common subalgebra $E$, then the free amalgam $H \underset{E}{*} K$ of $H$ and $K$ over $E$ is the Lie algebra $M$ such that $M_1 = H_1 \underset{E_1}{\oplus} K_1$ and $M_2 = \Lambda^2 M_1/N$ where $N = N(H) + N(E)$. Note that $\Lambda^2 H_1$, $\Lambda^2 E_1$, and $\Lambda^2 K_1$ are naturally embedded in $\Lambda^2 M_1$ and therefore also $N(H)$, $N(E)$ and $N(K)$.
If $B \leq M$, $B \subseteq A$, and $A \leq M$, then $A$ is a minimal strong extension of $B$ if there is no $A'$ with $B \subsetneq A' \subsetneq A$ and $A' \leq M$. There are three possibilities of minimal strong extensions:

a) Transcendental Case: $\mathrm{l.dim}(A) = \mathrm{l.dim}(B) + 1$ and $\delta(A) = \delta(B) + 1$.

b) Algebraic Case: $\mathrm{l.dim}(A) = \mathrm{l.dim}(B) + 1$ and $\delta(A) = \delta(B)$.
In this case $A_1 = B_1 \oplus \langle a \rangle^\ell$ and $N(A) = N(B) \oplus \langle [a, b] + \psi \rangle^\ell$ where $\psi \in \Lambda^2(B_1)$ and $b \in B_1$. By property i) in the definition of $\mathbb{K}$ $\langle a \rangle^\ell$ is uniquely determined modulo $\langle b \rangle^\ell$. We call $a$ a $b$-divisor of $\psi$.

c) Prealgebraic Case: $\mathrm{l.dim}(A) > \mathrm{l.dim}(B) + 1$ and $\delta(A) = \delta(B)$. In this case $B \subsetneq A' \subsetneq A$ implies $\delta(B) < \delta(A')$.

As in [Bau2] we obtain:

**Theorem 8.3** i) $\mathbb{K}$ *has the amalgamation with respect to strong embeddings.*

ii) *If* $B \underset{\mathrm{l.dim}(A/B)+n}{\leq} U$ *and* $B \leq A$ *for* $A$, $B$, $U$ *in* $\mathbb{K}$, *then there is an amalgam* $D$ *of* $\langle A\rangle$ *and* $\langle U\rangle$ *over* $\langle B\rangle$ *in* $\mathbb{K}$ *such that* $U \leq D$ *and* $A \underset{n}{\leq} D$.

*Proof*. As above $A$, $B$ are finite and $U$ can be infinite. We prove ii). i) follows from ii). Consider $B = A_0 \subseteq A_1 \subseteq \ldots \subseteq A_n = A$ where all $A_i$ are strong in $A$ and minimal over $A_{i-1}$ with this property. Using induction it is sufficient to assume that $A$ is a minimal strong extension of $B$. First we assume that $B \leq A = \langle B, a\rangle^\ell$ is given by $[a, b] + \psi = 0$, where $\psi \in \Lambda^2(B_1)$ and $b \in B_1$ (algebraic case) and $[x, b] + \psi = 0$ has already a solution $c$ in $U$. By $A$ in $\mathbb{K}$ $c$ is not in $B$. Then we embed $A$ over $B$ into $U$. This is possible because of $B \underset{1}{\leq} U$. Note: If $B \underset{n+1}{\leq} U$ then the image of $A$ is $n$-strong in $D$.

In all other cases let $D$ be the free amalgam of $U$ and $A$ over $B$. We have to show i) and ii) in the definition of $\mathbb{K}$. Let $E$ be a finite subspace of $D$. We assume the non-trivial situation $E \neq \langle 0\rangle$, $E \not\subseteq U$ and $E \not\subseteq A$.

a) Transcendental Case: $\mathbb{K}$i) is clear. Furthermore $E = E \cap U + \langle e\rangle^\ell$ with $e \notin U$ and $\delta(E) = \delta(E \cap U) + 1 \geq \max\{1, \delta(E \cap U)\}$, since $N(E) = N(E \cap U)$.

b) Algebraic Case: $A_1 = B_1 \oplus \langle a\rangle^\ell$, $N(A) = N(B) + \langle [a, b] + \psi\rangle^\ell$ where $b \in B_1$ and $\psi \in \Lambda^2(B_1)$.

First we show $\mathbb{K}$i). Assume $[c, d] = 0$ for $c, d \in D_1$. Let $X_u X_b ba$ be a vector basis of $D_1$ where $X_b b$ is a basis of $B_1$, $X_b ba$ a basis of $A_1$ and $X_u X_b b$ a basis of $U_1$. Let $c = u_c + w_c + r_c b + s_c a$ and $d = u_d + w_d + r_d b + s_d a$ where $u_c, u_d \in \langle X_u\rangle^\ell$, $w_c, w_d \in \langle X_b\rangle^\ell$ $r_c, s_c, r_d, s_d \in \mathbb{F}_q$. W.l.o.g. $\{s_c, s_d\} \subseteq \{0, 1\}$ and $s_c = 1$ or $s_d = 1$. $a$ must be involved since $U$ is in $\mathbb{K}$. Let us work in $\Lambda^2 D_1$. Then

$$(c \wedge d) = (u_c + w_c + r_c b \wedge u_d + w_d + r_d b) + (s_d(u_c + w_c + r_c b) - s_c(u_d + w_d + r_d b) \wedge a).$$

(∗) Every element of $N(D)$ has the form $t((a \wedge b) + \psi) + \Phi$ where $\Phi \in N(U)$ and $t \in \mathbb{F}_q$.

Hence $s_d = 1$ implies $u_c = 0$, and $s_c = 1$ implies $u_d = 0$. $s_d = s_c = 1$ is impossible since then $u_c = u_d = 0$ and $c, d \in A$. Hence w.l.o.g. $s_d = 1$, $u_d \neq 0$, and $s_c = 0$. Then we get

$$c \wedge d = ((w_c + r_c b) \wedge a) + ((w_c + r_c b) \wedge (u_d + w_d + r_d b)).$$

By $(*)$ we get $w_c = 0$. If also $r_c = 0$ then $c = 0$. Hence w.l.o.g. $c = b$ and $c \wedge d = (b \wedge a) + (b \wedge (u_d + w_d))$. If $c \wedge d \in N(D)$, then $b \wedge (u_d + w_d) - \psi \in N(U)$ and $-(u_d + w_d)$ is a solution of the equation that defines the considerd algebraic case. This contradicts our assumption.

To show $\mathbb{K}$ii) let $E$ be as above. Then $\mathrm{l.dim}(N(E)) \leq \mathrm{l.dim}(N(E \cap U)) + 1$ and therefore $\delta(E) \geq \delta(E \cap U) > 0$ or $E \cap U = \langle 0 \rangle$, $E = \langle e \rangle$ and $\delta(e) = 1$.

c) Prealgebraic Case: $A_1 = B_1 \oplus \langle \bar{a} \rangle^\ell$.
Again we show $\mathbb{K}$i) first. We work with a vector basis $X_u X_b \bar{a}$ where $X_b$ is a basis for $B_1$, $X_u X_b$ for $U_1$, and $X_b \bar{a}$ for $A$. We have

$$c = u_c + w_c + a_c \quad \text{ and } \; d = u_d + w_d + a_d$$

with $u_c, u_d \in \langle X_u \rangle^\ell$, $w_c, w_d \in \langle X_b \rangle$ and $a_c$ and $a_d$ in $\langle \bar{a} \rangle^\ell$. In $\Lambda^2 D_1$ we have

$$c \wedge d = (u_c + w_c \wedge u_d + w_d) - (u_d + w_d) \wedge a_c + (u_c + w_c) \wedge a_d.$$

Since $U \in \mathbb{K}$ we can assume $a_c \neq 0$ or $a_d \neq 0$. If $\mathrm{l.dim}(\langle a_c, a_d \rangle^\ell) = 1$, then w.l.o.g. $a_c = a_d$ or $a_c \neq 0$ and $a_d = 0$. In the first case $c \wedge (d - c) = 0$. Since $c$ and $d$ are linearly independent, this implies an equation $a_c \wedge b + \psi \in N(A)$, a contradiction to the prealgebraic case. Hence we can assume that we are in the case $a_c \neq 0$ and $a_d = 0$. Then $u_d = 0$ and $w_d \neq 0$. We have that $N(A)$ contains an element $w_d \wedge a_c + \psi$ with $\psi \in \Lambda^2(B_1)$. But then $\delta(a_c/B) = 0$ a contradiction to $\mathrm{l.dim}(\bar{a}) \geq 2$ and the minimality of $A$ over $B$. Now we assume that $a_c$ and $a_d$ are linearly independent. Again $u_d = 0 = u_c$, since $N(U) + N(A)$ cannot produce elements that contain $u_d \wedge a_c$ or $u_c \wedge a_d$. But then $c$ and $d$ are in $A$ and $[c, d] = 0$ is impossible since $A \in \mathbb{K}$.

It remains to show $\mathbb{K}$ii). Again we consider $E$ in $D_1$ with $E \not\subseteq U$, $E \not\subseteq A$. Let $\bar{c}$ with $c_i = a'_i + d_i$ $(1 \leq i \leq m)$ be a vector basis of $E$ over $E \cap U$ where $a'_i \in \langle \bar{a} \rangle^\ell$ and $d_i \in U$. Then $a'_1, \ldots, a'_m$ are linearly independent over $B_1$ and therefore over $U_1$. Since $N(D) = N(U) + N(A)$ $\;N(E)/N(E \cap U)$ has a basis $\psi_i(a) + \psi_i(u)$ $(1 \leq i \leq \ell)$ where $\psi_i(a) \in N(A)$ and $\psi_i(u) \in$

$N(U)$. Then $\psi_i(a) \in \Lambda^2(B_1 \oplus \langle a'_1, \ldots, a'_m \rangle^\ell)$ and $\psi_1(a), \ldots, \psi_\ell(a)$ have to be linearly independent over $\Lambda^2 B_1$. Since $B \leq A$ we get $\ell \leq m$. Hence $\delta(E) = \delta(E \cap U) + m - \ell \geq \delta(E \cap U) > 0$, if $E \cap U \neq \langle 0 \rangle$. Now assume $E \cap U = \langle 0 \rangle$. If $E \cap A \neq \langle 0 \rangle$, then we get similarly that $\delta(E) \geq \delta(E \cap A)$, since $\mathrm{l.dim}(E/B) \leq \mathrm{l.dim}(A/B)$ and $B$ is $\mathrm{l.dim}(A/B) + n$ strong in $U$. If $E \cap U = \langle 0 \rangle$ and $E \cap A = \langle 0 \rangle$, then $\delta(E) = \mathrm{l.dim}(E)$. The proof of $\mathbb{K}$ ii) is finished.

By the considerations above we have $U \leq D$. Similarly we get for $E \supseteq A$ with $\mathrm{l.dim}(E/A) \leq n \quad \delta(E) \geq \delta(A)$ since $B \underset{\mathrm{l.dim}(A/B)+n}{\leq} U$. Hence $A \underset{n}{\leq} D$. $\square$

By the usual procedure we get a countable Fraïssé-Hrushovski-Limit $M_{\mathrm{G}}$ in $\mathbb{K}$:

**Theorem 8.4** *There is a countable structure $M_{\mathrm{G}}$ in $\mathbb{K}$ that satisfies the following condition:*

(rich) *If $B \leq A$ are in $\mathbb{K}$ and there is a strong embedding $f$ of $B$ in $M_{\mathrm{G}}$, then it is possible to extend $f$ to a strong embedding $\bar{f}$ of $A$ in $M_{\mathrm{G}}$.*

*$M_{\mathrm{G}}$ is uniquely determined up to isomorphisms.*

**Definition** A structure $M$ in $\mathbb{K}$ that satisfies the condition (rich) is called a rich $\mathbb{K}$-structure.

Also the following result is easily proved by standard methods.

**Theorem 8.5** *Let $M$ and $N$ be rich $\mathbb{K}$-structures, $\langle \bar{a} \rangle \leq M$, $\langle \bar{b} \rangle \leq N$ and $\langle \bar{a} \rangle \cong \langle \bar{b} \rangle$. Then $(M, \bar{a}) \equiv_{L_{\infty,\omega}} (N, \bar{b})$.*

By Theorem 8.5 all rich $\mathbb{K}$-structures have the same elementary theory $T$. To axiomatize $T$ we write the following sets of $L$-sentences:

T 1) $M$ is a 2-nilpotent graded $\mathbb{F}_q$-Lie algebra.

T 2) $\forall xy \in R_1$("$x$ and $y$ are linearly independent"$\rightarrow [x, y] \neq 0)$
$\forall xz \exists y (x \in R_1 \wedge x \neq 0 \wedge z \in R_2 \rightarrow [x, y] = z)$.

T 3) $\langle a \rangle \leq M$ for all $a \in M$.

T 4) If $B \underset{\mathrm{l.dim}(A/B)+n}{\leq} M$, $B \leq A$ are in $\mathbb{K}$, then there is an $n$-strong embedding of $A$ in $M$.

It is easily seen how to formulate this axioms in $L$. The strong form of $\langle M_1\rangle = M$ in T 2) follows from the richness.

**Theorem 8.6** i) *A rich* $\mathbb{K}$*-structure satisfies* T 1)–T 4)*.*

ii) *Let* $M$ *be a model of* T 1), T 2) *and* T 3)*. Then* $M$ *is a rich* $\mathbb{K}$*-structure if and only if* $M$ *is a* $\omega$*-saturated model of* T 1)–T 4)*.*

*Proof*. Let $M$ be a rich $\mathbb{K}$-structure.

i) First we show

$$M \vDash \forall xz\exists y(x \in R_1 \wedge x \neq 0 \wedge z \in R_2 \rightarrow [xy] = z).$$

Take $0 \neq b \in M_1$ and $0 \neq w \in M_2$. Since $M \in \mathbb{K}$ there is some strong subspace $B_1$ of $M$ such that $\langle B_1\rangle^M$ contains $b$ and $w$. If there is some element $c \in B_1$ such that $[b, c] = w$, then we are done.

Otherwise we define $A_1 = B_1 \oplus \langle a\rangle^\ell$ and $A_2 = \Lambda^2 A_1 /_{N(B) + \langle (b \wedge a) - \hat{w}\rangle^\ell}$ where $\hat{w}$ is a preimage of $w$ in $\Lambda^2 B_1$. We show that $A \in \mathbb{K}$. $\mathbb{K}$ii) is clear. It remains to show $\mathbb{K}$i). If $[b_1 + r_1 a, b_2 + r_2 a] = 0$, where $b_i \in B_1$ and $b_1 + r_1 a$ and $b_2 + r_2 a$ are linearly independent, then $[b_1, b_2] + [r_1 b_2 - r_2 b_1, a] = 0$. $r_1 = 0 = r_2$ is impossible since $\langle B_1\rangle^M$ satisfies $\mathbb{K}$i). If $r_1 b_2 = r_2 b_1$, then $r_2(b_1 + r_1 a) = r_1(b_2 + r_2 a)$, a contradiction to the linear independence.

Otherwise we assume w.lo.g. $r_2 \neq 0$. Then w.l.o.g. $0 \neq r_1 b_2 - r_2 b_1 = b$ and $w = [b_1, b_2]$. It follows

$$\left[\frac{1}{-r_2} b, b_2\right] = [b_1, b_2] = w$$

a contradiction to the assumption that $[b, x] - w = 0$ has no solution in $\langle B_1\rangle^M$.

Since $A$ is in $\mathbb{K}$ and $M$ is rich there is a strong image of $A$ over $B$ in $M$. We have $[b, a'] - w = 0$ for some $a'$ in $M$ as desired. Now we know that $M$ is a model of T 1), T 2) and T 3). To show T 4) assume $B \underset{\mathrm{l.dim}(A/B)+n}{\leq} M$ and $B \leq A$ be $\mathbb{K}$. Then $B \subseteq C \leq M$. By Theorem 8.3ii) there is an amalgam $D \geq C$ of $A$ and $C$ over $B$. We have $A \underset{n}{\leq} D$. By the condition (rich) there is a strong embedding of $D$ over $C$ in $M$. This gives the desired embedding of $A$ over $B$ in $M$. The image of $A$ is $n$-strong in $M$.

ii) Let $M$ be an $\omega$-saturated model of T 1)–T 4). Then $M$ is a $\mathbb{K}$-structure and we have to show that $M$ is rich. Assume $B \leq M$ and $B \leq A$ are in $\mathbb{K}$. W.l.o.g. $A$ is minimal over $B$. If $A$ is algebraic, we get the strong embedding by T 2). In the prealgebraic case we use T 4). Since $B \leq M$ the image of $A$ in $M$ is strong. In the transcendental case we get some $a \in M_1 \setminus \mathrm{cl}_d(B)$ by T 4) and $\omega$-saturation. It remains to show that rich models $M$ of T 1)–T 3) are $\omega$-saturated models of T 1)–T 4). Since the $\omega$-saturated models $N$ of T 1)–T 4) are rich as shown above we get by Theorem 8.4 that $M \equiv_{L_{\infty,\omega}} N$. Therefore also the rich models $M$ are $\omega$-saturated. $\square$

Let $T$ be the theory T 1)–T 4). To show that $T$ satisfies P(I) – P(VII) let $R$ be $R_1$. Strongness $\leq$ and the pregeometry $\mathrm{cl}_d$ are given by the $\delta$-function defined above. The Lemmas 8.1 and 8.2 provide us that $a \in \mathrm{cl}_d(A)$ defines a pregeometry. By the definitions it is clear that "$a \in \mathrm{cl}_d(A)$" is part of $\mathrm{tp}(aA)$ and "$A \leq R(M)$" is part of $\mathrm{tp}(A)$. $\langle 0\rangle^\ell \leq M$ is true in all models of $T$.
Furthermore $\mathrm{CSS}(A)$ is strong in $M$ and is part of the algebraic closure of $A$. If $A \leq M$ and $A \subseteq C \subseteq \mathrm{acl}(A)$, then $\delta(C/A) = 0$. Then w.l.o.g. $C = \mathrm{CSS}(C)$ and therefore $C$ is strong in $M$. It follows $A = A_0 \subseteq A_1 \subseteq \ldots \subseteq A_n = C$ where $A_{i+1}$ is a strong minimal extension of $A_i$. But then $A_{i+1}$ is a minimal strong algebraic extension of $A_i$ for all $i$, as described above. By Theorem 8.6 the geometrical dimension of every $\omega$-saturated model of $T$ is infinite.
To prove the rest of P(II) assume $A \subseteq M$ and $B \subseteq N$ with $\mathrm{tp}^M(A) = \mathrm{tp}^N(B)$. W.l.o.g. $M$ and $N$ are $\omega$-saturated and therefore rich (Theorem 8.6). Furthermore w.l.o.g. $A \leq M$ and $B \leq N$. If $a$ is geometrically independent over $A$ and $b$ is geometrically independent over $B$, then $\langle aA\rangle \cong \langle bB\rangle$, $\langle A, a\rangle \leq M$, $\langle B, b\rangle \leq N$ and by Theorem 8.5 $\mathrm{tp}^M(a\,\hat{}\,A) = \mathrm{tp}^N(b\,\hat{}\,B)$. Hence we know that P(I) and P(II) are satisfied.

To continue we have to define $\mathcal{X}$. We consider minimal prealgebraic extensions $\langle B\bar{a}\rangle = A$ in a model $M \models T$. If $\bar{a} = (a_0, \ldots, a_{n-1})$ has linear dimension $n$ over $B$, then there are $n$ "relations" $\Phi_i$ $(i < n)$ in $\Lambda^2 A$ linearly independent over $\Lambda^2 B$, that are a basis of $N(A)$ over $N(B)$:

$$(+) \qquad \Phi_i = \sum_{\ell<j<n} r^i_{\ell j}(a_\ell \wedge a_j) + \sum_{j<n} (b^i_j \wedge a_j) + \psi_i$$

where $r^i_{\ell j} \in \mathbb{F}_p$, $\;b^i_j \in B_1$ and $\psi_i \in \Lambda^2 B_1$.

**Remark**

- If we choose another basis of $N(A)$ over $N(B)$, then $\langle b^i_j : i, j < n\rangle^\ell$ and $\langle \psi_0, \ldots, \psi_{n-1}\rangle^\ell \subseteq \Lambda^2 B_1$ in (+) do not change.

- Let $H$ be a vectorspace automorphism of $\langle \bar{a}\rangle^\ell$. The representation (+) of $\langle H(\bar{a})B\rangle$ over $B$ uses the same spaces $\langle b^i_j : i, j < n\rangle^\ell$ and $\langle \psi_0, \ldots, \psi_{n-1}\rangle^\ell$.

To describe $\langle B\bar{a}\rangle$ over $B$ as above, let $\varphi(\bar{x}; \bar{y}, \bar{z}) = \varphi_1(\bar{x}; \bar{y}, \bar{z}) \wedge \varphi_2(\bar{y}, \bar{z})$. $\bar{y}$ is used for an enumeration $\bar{b}$ of the $b^i_j$'s above and $\bar{z}$ for the images $\hat{\psi}_i$ of the $\psi_i$'s in $M$. $\varphi_2(\bar{y}, \bar{z})$ describes the isomorphism-type of $\langle \bar{b}\rangle$ and of $\hat{\psi}_0, \ldots, \hat{\psi}_{n-1}$ over $\langle \bar{b}\rangle_2$. $\varphi_1(\bar{x}, \bar{y}, \bar{z})$ says that $\bar{a}$ (represented by $\bar{x}$) is linearly independent over $\bar{b}$ and describes $N(\langle B\bar{a}\rangle)$ over $N(B)$ by the equations corresponding to (+) and suitable unequations. That means $(\langle \bar{a}, \bar{b}\rangle)_2 + \langle \hat{\psi}_0, \ldots, \hat{\psi}_{n-1}\rangle^\ell$ is described. All formulas can be chosen quantifier free.

**Definition** Let $\mathcal{X}^{\text{home}}$ be the set of formulas $\varphi(\bar{x}, \bar{y}, \bar{z})$ above.

**Remark**

- If $M \models T$, $D \leq M$, $\bar{d}$, $\bar{e}$ in $D$ with $M \models \varphi_2(\bar{d}, \bar{e})$ and $\bar{a}$ is a solution of $\varphi(\bar{x}, \bar{d}, \bar{e})$, linearly independent over $D_1$, then $\bar{a}$ is a generic solution and defines a minimal prealgebraic extension of $D$. $\varphi(\bar{x}, \bar{d}, \bar{e})$ is strongly minimal.

- A subset of $M$ that is an affine transformation of a set defined by a formula in $\mathcal{X}^{\text{home}}$ is again encoded by a formula in $\mathcal{X}^{\text{home}}$.

**Lemma 8.7** *Let $\varphi(\bar{x}; \bar{y}, \bar{z})$ be in $\mathcal{X}^{\text{home}}$, $M \models T$, and $f \in \mathrm{Aut}(M)$ that fixes the generic type of $\varphi(\bar{x}; \bar{b}, \bar{c})$ where $M \models \varphi_2(\bar{b}, \bar{c})$. Then $f$ fixes the vectorspaces $\langle \bar{b}\rangle^\ell$ and $\langle \bar{c}\rangle^\ell$ setwise.*

*Proof*. Let $M \preceq \mathbb{C}$, where $\mathbb{C}$ is the monster model of $T$ and $\bar{a} \in \mathbb{C}$, such that $\mathrm{tp}(\bar{a}/M)$ is the generic type of $\varphi(\bar{x}, \bar{b}, \bar{c})$. Let $f$ be an automorphism of $\mathbb{C}$ that fixes $\bar{a}$ pointwise and $M$ setwise. $f$ can be naturally extended to $\Lambda^2\mathbb{C}$. We consider as above

$$\begin{aligned}
\Phi_i &= \sum_{\ell<j<n} r^i_{\ell_j}(a_\ell \wedge a_j) + \sum_{j<n}(b^i_j \wedge a_j) + \psi_i \\
f(\Phi_i) &= \sum_{\ell<j<n} r^i_{\ell_j}(a_\ell \wedge a_j) + \sum_{j<n}(f(b^i_j) \wedge a_j) + f(\psi_i)
\end{aligned}$$

in $\Lambda^2\mathbb{C}$ over $\Lambda^2(M)$. Since $M$ is strong in $\mathbb{C}$ $f$ must fix the subspace $\langle \Phi_0 \ldots \Phi_{n-1}\rangle^\ell$ over $\Lambda^2 M$. That means $\langle \Phi_0 - \psi_0, \ldots, \Phi_{n-1} - \psi_{n-1}\rangle^\ell = f\langle \Phi_0 - \psi_0, \ldots, \Phi_{n-1} -$

$\psi_{n-1}\rangle^\ell$. Hence $f(\langle b^i_j : i,j < n\rangle^\ell) = \langle b^i_j : i,j < n\rangle^\ell$. Then for every $i$ $\quad f(\Phi_i) = \sum\limits_{j<n} s^i_j \Phi_j$ modulo $\Lambda^2 M$ and therefore

$$f(\Phi_i) - \sum_{j<n} s^i_j \Phi_j = f(\psi_i) - \sum_{j<n} s^i_j \psi_j \in N(M).$$

If $\hat{\psi}_i$ is the image of $\psi_i$ in $M$, then $f(\hat{\psi}_i) = \sum\limits_{j<n} s^i_j \hat{\psi}_j$ in $M$. Hence $\langle \bar{c}\rangle^\ell = f(\langle \bar{c}\rangle^\ell)$.

□

**Definition** Let $\mathcal{X}$ be the formulas we get from $\mathcal{X}^{\text{home}}$ if we work with the canonical parameters of the generic types of the formulas in $\mathcal{X}^{\text{home}}$.

Lemma 8.7 says that we can work with the quantifier free formulas in $\mathcal{X}^{\text{home}}$ if the we want to check P(III) – P(VII).

By the definition of $\mathcal{X}$, Lemma 8.7, and $\delta$-computations P(III) is clear. The first part of P(IV) is Theorem 8.5. The rest follows from the description of minimal strong extensions $A_{i+1}/A_i$ where $\delta(A_{i+1}/A_i) = 0$.

To obtain P(V)let $\bar{a}$ be a solution of some $\varphi(\bar{x}, \bar{b}) \in \mathcal{X}^{\text{home}}$ and $\bar{b} \subseteq B \subseteq R(\mathbb{C})$. We say $\bar{a}$ is $\downarrow^w$-generic over $B$, if $\delta(\bar{a}/B) = 0$ and $\bar{a}$ is linearly independent over $B$. Note that different $\downarrow^w$-generics over $B$ are isomorphic over $B$. The first part of P(V) is clear. To show the second part consider $U \leq M$ and solutions $\bar{e}_0, \bar{e}_1, \ldots$ with $\bar{e}_i \not\subseteq \langle U, B, \bar{e}_0, \bar{e}_1, \ldots, \bar{e}_{i-1}\rangle^\ell$. Then we compute

$$0 \leq \delta(B\bar{e}_0 \ldots \bar{e}_i/U) = \delta(B/U) + \sum_{j \leq i} \delta(j)$$

where $\delta(j) = \delta(\bar{e}_j/UB\bar{e}_0 \ldots \bar{e}_{j-1})$. Since $\delta(j) \leq 0$ there are at most $\text{l.dim}(B/U)$ many $j$ with $\delta(j) < 0$. $\delta(j) = 0$ implies that $\bar{e}_j$ is $\downarrow^w$-generic over $\langle U, B\bar{e}_0 \ldots, \bar{e}_{j-1}\rangle^\ell$ since $\bar{e}_j \not\subseteq \langle U, B, \bar{e}_0, \ldots, \bar{e}_{j-1}\rangle^\ell$.

It remains to show P(VI). Let $C \supseteq B \subseteq A$ be strong subspaces of $R(M)$ linearly independent over $B$ and both minimal strong extension of $B$ given by generic solution of formulas in $\mathcal{X}^{\text{home}}$. Then $A + C$ is a free amalgam over $B$. Let $\bar{e} \in A + C$ be a solution of $\varphi(\bar{x}, \bar{b}, \bar{d})$ where $\bar{b}, \bar{d} \in A + C$ and $\bar{e}$ is $\downarrow^w$-generic over $A + E$ and $C + E$ where $\langle E\rangle \supseteq \bar{b}\bar{d}$. Since $A + C$ is the free amalgam over $B$ $\quad N(A+C) = N(A) \bigoplus\limits_{N(B)} N(C)$. Therefore the $\Phi_i$ in the definition of $\varphi_1(\bar{e}, \bar{b}, \bar{\psi})$ have the form

$$\sum_{j<n} e_j \wedge b^i_j + \psi_i \quad \text{ where } \quad b^i_j \in B$$

and $e_j = a_j + c_j$ where $a_0, \dots, a_{n-1}$ is linearly independent over $C + E$ and $c_0, \dots, c_{n-1}$ is linearly independent over $A + E$. Hence

$$\Phi_i = \sum_j (a_j \wedge b^i_j) + \psi^a_i + \sum_j (c_j \wedge b^i_j) + \psi^c_i$$

where $\sum_j (a_j \wedge b^i_j) + \psi^a_i \in N(A)$ and $\sum_j (c_j \wedge b^i_j) + \psi^c_i \in N(C)$. We see that $\varphi(\bar{x}, \bar{b}, \bar{\psi})$ defines a torsor set where the underlying group set is defined by $\sum_{j<n}(x_j \wedge b^i_j)$ and has the parameter in $B$ as desired.

P(VII) is true, since for each $H$ with $\langle R_1(H) \rangle = H$ and $\operatorname{acl}(H) \cap R_1(H) = R_1(H)$ we can use the following function:

$$\eta(a, b) = \begin{cases} a, & \text{if } \langle a \rangle^\ell = \langle b \rangle^\ell, \\ [a, b], & \text{otherwise.} \end{cases}$$

Then $H = \{\eta(a, b) : a, b \in H_1\}$.

We have shown

**Theorem 8.8** *$T$ is a theory that satisfies the conditions* P(I) – P(VII).

**Corollary 8.9** *$T$ provides us uncountably categorical theories $\Gamma(T^\mu)$ of Morley rank 2 where $R_1(x)$ is a strongly minimal set. By interpretation we get the corresponding theories of nilpotent groups of class 2 and exponent $p > 2$.*

# 9 Red fields and fusion over a vectorspace

In this chapter it is shown, how the results in [BMPZ3] and [BMPZ4] can be proved using the results of this paper. B. Poizat [Po2] has constructed algebraically closed fields of characteristic $p > 0$ with a (red) predicate $R(x)$ for a subgroup of the additive group of the field, such that its theory $T_{R,p}$ is $\omega$-stable of infinite Morley rank. To give a short description of $T_{R,p}$ consider fields in $\mathrm{ACF}_p$ $(p > 0)$ as $\mathbb{F}_p$-vectorspaces with extra structure. We add a predicate $R(x)$ for a subspace and define for every finite subspace $H$ of such a structure $M$

$$\delta(H) = 2\operatorname{tr}\deg(H) - \mathrm{l.dim}\, R(H).$$

This definition is due to B. Poizat [Po2]. $\operatorname{tr}\deg(H)$ is the transcendence degree of $H$, and $R(H) = R(M) \cap H$. Furthermore $\operatorname{tr}\deg(H/K)$ is the transcendence degree of $H + K$ over $K$ and

$$\delta(H/K) = 2\operatorname{tr}\deg(H/K) - \mathrm{l.dim}(H/K).$$

Now we use the $\delta$-function as in Chapter 8. Note that we do **not** restrict the subspaces $H$, $K$ to $R(M)$. We define strongness as there. We get

**Lemma 9.1** $\delta(H + K) \leq \delta(H) + \delta(K) - \delta(H \cap K)$.

**Definition** Let $\mathbb{K}$ be the class of all $M$ in $\mathrm{ACF}_p$ with extra predicate for a subspace such that $\langle 0\rangle^\ell \leq M$.

We work in structures $M$ in $\mathbb{K}$. Then we define $d$ and $\mathrm{cl}_d$ as in Chapter 8 and again we have

**Lemma 9.2** i) *The intersection of strong subspaces is strong.*

ii) $\mathrm{cl}_d$ *defines a pregeometry on subspaces of* $R(M)$ *with dimension function* $d$.

iii) *Strongness is transitive.*

iv) *If* $V \leq U$, *then* $X \cap V \leq X \cap U$ *for every subspace* $X$. *(*$U$, $V$ *not restricted to* $R(M)$*.)*

By i) we can define $\mathrm{CSS}(H)$ as the intersection of all $K$ that are strong in $M$ and contain $H$. Then $\mathrm{CSS}(H) \subseteq \mathrm{acl}^M(H)$.

**Theorem 9.3 ([Po2])** *The class* $\mathbb{K}$ *has the amalgamation property with respect to strong embeddings and the asymmetric amalgamation.*

By standard methods we obtain

**Corollary 9.4** *There is a countable structure* $M_{R,p}$ *in* $\mathbb{K}$ *that satisfies the following condition:*

(rich) *If* $K \leq H \subseteq M \in \mathbb{K}$ *and there is a strong partial* $L_{\text{field}}$*-elementary embedding* $f$ *of* $K$ *into* $M_{R,p}$ *then there exists an extension* $\bar{f}$ *of* $f$, *that extends* $f$ *and is a partial* $L_{\text{field}}$*-elementary strong embedding of* $H$ *in* $M_{R,p}$.

B. Poizat has defined that a structure in $\mathbb{K}$ is rich if it satisfies the condition (rich). He showed

**Theorem 9.5** *If $M$ and $N$ are rich $\mathbb{K}$-structures $\langle \bar{a} \rangle^{\ell} \leq M$, $\langle \bar{b} \rangle^{\ell} \leq N$, and $\mathrm{tp}^{M}_{\mathrm{field}}(\bar{a}) = \mathrm{tp}^{N}_{\mathrm{field}}(\bar{b})$ then $(M, \bar{a}) \equiv_{L_{\infty,\omega}} (N, \bar{b})$.*

Hence we have a complete theory $T_{R,p}$ of rich $\mathbb{K}$-structures. B. Poizat gave an axiomatization of $T$ and showed

**Theorem 9.6** *Rich $\mathbb{K}$-structures are the $\omega$-saturated models of $T_{R,p}$.*

Let $K \leq H \subseteq M \in \mathbb{K}$, such that $H$ is minimal over $K$. That means that there is no $K \subsetneq J \subsetneq H$ with $J \leq H$. Then we have the following cases:

1. $H = \langle K, a \rangle^{\ell}$, $R(H) = R(K)$
   a) If $a$ is transcendental over $K$, then we say $H$ is white transcendental over $K$.
   b) If $a$ is algebraic over $K$, then we say $H$ is algebraic over $K$.
2. $H = \langle K, \bar{a} \rangle^{\ell}$ and $\bar{a}$ is a basis of $R(H)$ over $R(K)$.
   a) $\mathrm{l.dim}(\bar{a}) \geq 2$ and $\delta(H/K) = 0$. Then we call $H$ prealgebraic over $K$.
   b) $\mathrm{l.dim}(\bar{a}) = 1$ and a transcendental over $K$. Then we call $H$ red transcendental over $K$.

**Theorem 9.7** *$T_{R,p}$ fulfils* P(I) – P(VII).

*Proof*. P(I) is clear. Every white generic is a product of two red generics. Hence every element of $M$ is the sum of two products of red elements.

P(II) Let $A, B \subseteq R(M)$ and $a \in R(M)$. By the definitions $A \leq M$ and $a \in \mathrm{cl}_d(A)$ is part of $\mathrm{tp}(A)$ and $\mathrm{tp}(a, A)$, respectively. Furthermore $A \subseteq \mathrm{CSS}(A) \leq M$ and $\mathrm{CSS}(A) \subseteq \mathrm{acl}(A) \cap R(M)$. Algebraic extensions $A$ of strong spaces $B$ are algebraic with respect to $\mathrm{ACF}_p$ and white over $B$. Hence they do not exist in $R(M)$. The rest of P(II) follows from Theorem 9.5.

P(III) Codes are already used by B. Poizat. We choose the codes from [BMPZ3]. They describe the minimal prealgebraic extensions.

P(IV) The first part is given by Theorem 9.5 and 9.6. The $\delta$-analysis of finite strong extension of strong subspaces implies P(IV). Only prealgebraic extensions occur, since we consider subspaces of $R(M)$.

For P(V) and P(VI) the relation $\mathop{\smile\!\!\!\!\!\mid}^{w}$ is non-forking in $\mathrm{ACF}_p$ and therefore $\mathop{\smile\!\!\!\!\!\mid}^{w}$-generic means $\mathop{\smile\!\!\!\!\!\mid}^{\mathrm{ACF}_p}$-generic.

P(V) follows by a computation as in Chapter 8. $\bar{e}_i \not\subseteq \langle UBe_0, \ldots, e_{i-1}\rangle^{\ell}$ and $\delta(\bar{e}_i/\langle U, B, \bar{e}_0, \ldots, \bar{e}_{i-1}\rangle^{\ell}) = 0$ implies that $e_i$ is $\mathop{\smile\!\!\!\!\!\mid}^{w}$-independent from $\bar{e}_0, \ldots, \bar{e}_{i-1}$ over $\langle U, B\rangle^{\ell}$.

P(VI) will be proved as in [BMPZ3]. Note that non-forking in that proof is non-forking for $\mathrm{ACF}_p$.

P(VII) As mentioned above the formula $\theta(x_1, x_2, y)$ is $x_1 \cdot x_2 = y$.

The elimination of quantifiers and the elimination of imaginaries in $\mathrm{ACF}_p$ provides quantifier-free formulas in P(IV). $\square$

**Corollary 9.8** *The red fields from* [BMPZ3] *can be obtained using the approach of this paper.*

This is not a surprise, since the frame developed in this paper uses strongly the ideas of [BMPZ3].

Now we turn to the fusion. The fusion over a vectorspace without the collapse is described in [HH]. We start with two strongly minimal theories $T_1$ and $T_2$ with the DMP. Let $L_1$ and $L_2$ be the corresponding languages and $L_1 \cap L_2 = L_0$ the language of a vectorspace over a finite field. We assume quantifier elimination for both theories. W.l.o.g. we assume that $L_i$ contains only relational symbols besides $L_0$. Our language is $L = L_1 \cup L_2$. Note that $R(x)$ is the predicate $x = x$! Furthermore $\langle\ \rangle = \langle\ \rangle^{\ell}$.

We consider models $U$ of $T_1^{\forall} \cup T_2^{\forall}$. Then $U$ is a vectorspace and a substructure of the monster models $\mathbb{C}_1 \models T_1$ and $\mathbb{C}_2 \models T_2$. For finite models of $T_1^{\forall} \cup T_2^{\forall}$ we write again $A, B, \ldots$ We define

$$\delta(A) = \mathrm{tr}_1(A) + \mathrm{tr}_2(A) - \mathrm{l.dim}(A).$$

Let $\mathbb{K} = \{U : U \models T_1^{\forall} \cup T_2^{\forall} \text{ and } \delta(A) \geq 0 \text{ for all } A \subseteq U\}$. It is easy to check that

$$\delta(A + B) \leq \delta(A) + \delta(B) - \delta(A \cap B).$$

As above we can use $\delta$ to define strongness "$\leq$", the pregeometry $a \in \mathrm{cl}_d(A)$, the dimension function $d$ and $\mathrm{CSS}(A)$.

We have the following minimal strong extensions $A \leq B$.

i) $\delta(A/B) = 0$, $A = \langle B, a\rangle^\ell$ for some $a \in A \setminus B$ algebraic over $B$ either in $T_1$ or $T_2$. We call it algebraic minimal extension.

ii) $\delta(A/B) = 0$. $A$ is neither algebraic in $T_1$ nor in $T_2$. We call it minimal prealgebraic extension.

iii) $\delta(A/B) = 1$ and $A = \langle B, a\rangle^\ell$ where $a$ is neither algebraic over $B$ in $T_1$ nor in $T_2$. We call it minimal transcendental extension.

Again $\mathbb{K}$ has the amalgamation property with respect to strong embeddings and the asymmetric amalgamation. We get a Fraïssé-Hrushovski-Limit $M_{Fu}$. We define that a structure $M$ in $\mathbb{K}$ is rich, if for all $B \leq A$ in $\mathbb{K}$ and $B \leq M$ we find an embedding of $A$ in $M$ over $B$. Since rich structures in $\mathbb{K}$ are infinite and closed under $\mathrm{acl}^{L_1}$ and $\mathrm{acl}^{L_2}$ they are models of $T_1 \cup T_2$. Then we obtain that rich structures are $(L_1 \cup L_2)_{\infty,\omega}$-equivalent. If $\langle \bar{a}\rangle^\ell \leq M$, where $M$ is rich, then $\mathrm{tp}^{L_1}(\bar{a}) \cup \mathrm{tp}^{L_2}(\bar{a})$ determines the type of $\bar{a}$ in $M$. Let $T_{Fu}$ be the theory of the rich $\mathbb{K}$-structures. $T_{Fu}$ can be axiomatized by

i) $M \in \mathbb{K}$.

ii) $M \models T_1 \cup T_2$.

iii) $M$ is rich with respect to $B \leq A$ minimal algebraic or prealgebraic. This can be expressed using the asymmetric amalgamation.

**Theorem 9.9** *Let $M$ be a model of $T_{Fu}$.*

i) *$M$ is rich if and only if it is $\omega$-saturated.*

ii) *For every finite subspace $A$ of $M$ we have*

$$\mathrm{l.dim}(A) \leq \mathrm{tr}_1(A) + \mathrm{tr}_2(A).$$

iii) *If $A \leq M$, then $\mathrm{tp}^{L_1}(A) \cup \mathrm{tp}^{L_2}(A)$ implies $\mathrm{tp}^M(A)$.*

iv) *Let $M \subseteq N$ be models of $T_{Fu}$. Then $M \preceq N$ if $M$ is an elementary substructure of $N$ in the sense of $T_1$ and $T_2$.*

For $T_{Fu}$ we introduce $\mathop{\smile\!\!\!\!\!|}^{w}$ as independent in the sense of $T_1$ and $T_2$. Furthermore we choose the prealgebraic codes from [BMPZ4] as the set of formula $\mathcal{X}$. Then we get the following

**Theorem 9.10** *$T_{F_u}$ satisfies* P(I) – P(VII).

Note P(VII) follows from $M = R(M)$ for $M \vDash T_{Fu}$. As in Corollary 7.2 we get a $L^{\mu}$-theory $T^{\mu}_{Fu}$ with an extra predicate $R^{\mu} = P^{\mu}$. If $M \vDash T^{\mu}_{Fu}$ then $R^{\mu}(M)$ is closed under $\mathrm{acl}^{L_1}$ and $\mathrm{acl}^{L_2}$. Hence $R^{\mu}(M)$ is again a model of $T_1 \cup T_2$. We can go back to the original languages of $T_1$ and $T_2$ and get

**Theorem 9.11** *$\Gamma(T^{\mu}_{Fu})$ is a strongly minimal theory that contains $T_1 \cup T_2$. Furthermore* i) – iv) *from Theorem* 9.9 *are also true for this theory.*

# References


[Bau1] A. Baudisch: Classification and interpretation. *J. Symbolic Logic* 54 (1989), 138–159.

[Bau2] A. Baudisch: A new uncountably categorical group. *Trans. Amer. Math. Soc.* 348 (1996), 3889–3940.

[BH] J. Baldwin and K. Holland: Constructing $\omega$-stable structures: rank 2 fields. *J. Symbolic Logic* 65 (2000), 371–391.

[BMPZ1] A. Baudisch, A. Martin-Pizarro and M. Ziegler: Hrushovski's fusion. In: *Festschrift für Ulrich Felgner zum 65. Geburtstag*, eds. F. Haug, B. Löwe and T. Schatz. Studies in Logic, Vol 4, College Publications, London 2007, 15–31.

[BMPZ2] A. Baudisch, A. Martin-Pizarro and M. Ziegler: On fields and colors. *Algebra i Logika* 45(2) (2006); http://arxiv.org/math.LO/0605412.

[BMPZ3] A. Baudisch, A. Martin-Pizarro and M. Ziegler: Red fields. *J. Symbolic Logic* 72 (2007), 207–225.

[BMPZ4] A. Baudisch, A. Martin-Pizarro and M. Ziegler: Fusion over a vector space. *J. Math. Logic* 6 (2006), 141–162.

[BHMPW] A. Baudisch, M. Hils, A. Martin-Pizarro and F. Wagner: Die böse Farbe. J. de l'Institut de Mathèmatiques de Jussieu, à paraître.

[JG] J.B. Goode: Hrushovski's geometries. Proceedings of the 7th Easter Conference on Model Theory 1989, Seminarberichte der Humboldt-Universität zu Berlin, Nr. 104, 106–117.

[HH] A. Hasson and M. Hils: Fusion over sublanguages. *J. Symbolic Logic* 71(2) (2006), 361–398.

[Hr1] E. Hrushovski: A new strongly minimal set. *Ann. Pure Appl. Logic* 62 (1993), 147–166.

[Hr2] E. Hrushovski: Strongly minimal expansions of algebraically closed fields. *Israel J. Math.* 79 (1992), 129–151.

[Po0] B. Poizat: Groupes stables. Une tentative de conciliation entre la géométrie algébrique et la logique mathématique. Nur al-Mantiq wal-Marifah, Bruno Poizat, Lyon, 1987.

[Po1] B. Poizat: Le carré de l'égalité. *J. Symbolic Logic* 64 (1999) 3, 1338-1355.

[Po2] B. Poizat: L'égalité au cube. *J. Symbolic Logic* 66 (2001) 4, 1647–1676.

[Z] M. Ziegler: A note on generic types. Unpublished, 2006, http://arxiv.org/math.lo/0608433.