Mekler's construction preserves CM-triviality

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Abstract

For every structure M of finite signature A. Mekler [11] has constructed a group G such that for every κ the maximal number of n-types over an elementary equivalent model of cardinality κ is the same for M and G. These groups are nilpotent of class 2 and of exponent p, where p is a fixed prime greater than 2. We consider stable structures M only and show that M is CM-trivial if and only if G is CM-trivial. Furthermore we obtain that the free group $F_2(p,\omega)$ in the variety of 2-nilpotent groups of exponent $p > 2$ with ω free generators has a CM-trivial ω-stable theory.

1 Introduction

Let S be a complete theory of finite signature. Let $\lambda_S(\kappa)$ be the maximal size $|S_1(M)|$ of the Stone space $S_1(M)$ for $M \models S$ with $\kappa = |M|$. For every such S Alan Mekler [11] constructed a complete group theory T_S such that S is interpretable in T_S and $\lambda_S(\kappa) = \lambda_{T_S}(\kappa)$ for every cardinality κ. It follows that S and T_S have the same stability spectrum. Furthermore it can be shown that S is simple if and only if T_S is simple.

The models of T_S are nilpotent groups of class 2 and exponent p (> 2). Already Ju.L. Eršov [7] used similar ideas to Mekler's to prove the undecidability of the variety of these groups. We call the models of such theories T_S Mekler groups. If S is superstable then T_S is superstable. Under the assumption that S is superstable it is shown in [3] that S has NDOP if T_S has NDOP. If Depth(S) exists, then

$$\mathrm{Depth}(S) \leq \mathrm{Depth}(T_S) \leq \mathrm{Depth}(S) + 1.$$

This is used to construct ω-stable groups with NDOP and Depth n for every $n < \omega$. In [4] it is used to construct superstable NDOP–NOTOP groups. Note that Mekler groups have infinite rank and are not $\aleph_0$-categorical. Since Mekler groups are not abelian by finite they are not one-based, as by [10] one-based stable groups are abelian by finite.

For stable theories one-basedness implies the non-interpretability of a field. For uncountably categorical theories B. Zil'ber [16] conjectured the converse. In [9] E. Hrushovski refuted this conjecture and introduced CM-triviality. We define this notion in the next section. It is weaker than one-basedness but it still implies the non-interpretability of a field. It is an open question whether Zil'ber's conjecture is true if we replace one-basedness by CM-triviality. In [6] an ω-stable, non-CM-trivial theory is constructed that does not allow the interpretation of a field. But this theory has infinite rank.
In this paper we show that S is stable and CM-trivial if and only if the corresponding theory T_S of Mekler-groups is stable and CM-trivial. Hence Mekler's construction can be used to produce examples of stable CM-trivial groups. A. Pillay proved in [12] that CM-trivial groups of finite Morley rank are nilpotent by finite. F. Wagner [15] got similar results replacing the condition to be of finite Morley rank by stability plus several other conditions. Note that the new $\aleph_1$-categorical group constructed in [5] is CM-trivial.

Let $F_c(p, \omega)$ be the free group in the variety of nilpotent groups of class c and of exponent p (prime $> c$) with ω free generators. In [1] and [2] it is shown that $\mathrm{Th}(F_c(p, \omega))$ is ω-stable and non-multidimensional with exactly c dimensions. Our proof of the main result of this paper gives us furthermore that $\mathrm{Th}(F_2(p, \omega))$ is CM-trivial. Hence it is not possible to interpret a field.

In Section 2 we define CM-triviality and consider interpretations without new information.

In Section 3 we consider the bilinear maps $\mathcal{F}(G)$ for $G \models T_S$ that are given by the commutator. They live in G^{eq}. Often it is more convenient to work in $\mathcal{F}(G)$.

In Section 4 we follow W. Hodges [8] to decribe Mekler's construction.

In Section 5 we prove that S is simple if and only if T_S is simple. This is common work with my student Alexander Pentzel.

In Section 6 we introduce the notion of a good subspace of $V_G = G/Z(G)$ for $G \models T_S$. We show that every element a of the commutator subgroup G' is interalgebraic with a finite good subspace of V_G.

In Section 7 we explain how each good subspace of $V_{\mathfrak{C}}$ corresponds to a good subspace that respects V_G for $G \preceq \mathfrak{C} \models T_S$.

In Section 8 we formulate the main result and prove it for the tame case. The lemmas we show are important for the general case.

In Section 9 we discuss the case of the mixed elements.

In Section 10 the results of Section 8 and 9 are used to prove the main result.

In Section 11 CM-triviality of $F_2(p, \omega)$ is proved.

2 CM-triviality and interpretation

Let T be any complete stable theory. We use $\mathfrak{C}$ to denote the monster model. Often we work in T^{eq} and $\mathfrak{C}^{\mathrm{eq}}$. In this paper the definable and the algebraic closure $\mathrm{dcl}(A)$ and

acl(A) respectively are considered in $\mathfrak{C}^{\mathrm{eq}}$. $\mathrm{Cb}(\mathrm{tp}(\overline{a}/A))$ is used to denote the canonical base of the type $\mathrm{tp}(\overline{a}/A)$. In fact it is the canonical base of the strong type of $\overline{a}/A$, that means of $\mathrm{tp}(\bar{a}/\mathrm{acl}(A))$.
CM-triviality was introduced by E. Hrushovski [9]. It is a property of T^{eq}.

Definition T is *CM-trivial* if whenever $c \in \mathfrak{C}^{\mathrm{eq}}$, $A \subseteq B$ are algebraically closed sets in $\mathfrak{C}^{\mathrm{eq}}$ with $\mathrm{acl}(cA) \cap B = A$, then $\mathrm{Cb}(\mathrm{tp}(c/A)) \subseteq \mathrm{acl}(\mathrm{Cb}(\mathrm{tp}(c/B)))$.

A. Pillay [12] proved that it is sufficient to consider only models $M \preceq N$ of T instead of $A \subseteq B$, and tuples $\overline{c}$ in $\mathfrak{C}$. As usual in C^{eq} we often do not distinguish between models M of T and $M^{\mathrm{eq}} = \mathrm{acl}^{\mathrm{eq}}(M)$. Note that it is also possible to assume that M and N are saturated.
Let S and T be theories, and L_S and L_T be the corresponding languages. At the moment we do not assume completeness as in the rest of the section. We use G and H to denote the models of T, and M and N to denote the models of S.

Definition An *interpretation* Γ of S in T is a uniform definition of an S-model $\Gamma(G)$ in every model G^{eq} of T^{eq} as a relativised reduct of a definitional expansion of a finite slice of G^{eq}.

As in W. Hodges book [8] an interpretation Γ of S in T induces a functor $\mathrm{Func}(\Gamma)$ from the category of models of T and elementary embeddings to the category of models of S and elementary embeddings.
We want to describe this notion in more detail. An unnested atomic formula of L_S is a formula of the form $x_0 = x_1$, $x_0 = c$, $R(x_0, \ldots, x_{m-1})$ or $f(x_0, \ldots, x_{m-2}) = x_{m-1}$, where c is a constant symbol of L_S, R is a relation symbol of L_S, and f is a function symbol of L_S. Γ can be given by an L_T-formula $\gamma(\overline{x})$ and L_T-formulas $\gamma(\varphi)(\overline{x}_0, \ldots, \overline{x}_{m-1})$ for every unnested atomic formula $\varphi(x_0, \ldots, x_{m-1})$ of L_S, where all $\overline{x}$, $\overline{x}_i$ are of the same length n. We write $\overline{x} \sim \overline{y}$ instead of $\gamma(x = y)(\overline{x}, \overline{y})$. Since we want that these formulas describe an L_S-structure $\Gamma(G)$ in every model G of T, the following elementary conditions must be satisfied by T:

i) $\sim$ is an equivalence relation.

ii) $\forall \overline{x}\, \overline{y}(\overline{x} \sim \overline{y} \wedge \gamma(\overline{x}) \rightarrow \gamma(\overline{y}))$

iii) $\forall \overline{x}_0 \ldots \overline{x}_{m-1} \overline{y}_0 \ldots \overline{y}_{m-1} \Big(\bigwedge_{i<m} \overline{x}_i \sim \overline{y}_i \wedge \gamma(\varphi)(\overline{x}_0, \ldots, \overline{x}_{m-1}) \rightarrow \gamma(\varphi)(\overline{y}_0, \ldots, \overline{y}_{m-1}) \Big)$
for all unnested atomic formulas $\varphi(x_0, \ldots, x_{m-1})$ of L_S.

Let the domain of $\Gamma(G)$ be the set of all $\sim$-classes a in G^{eq}, where $a = \overline{a}/\sim$ for some tuple $\overline{a}$ satisfying $\gamma(\overline{x})$. $\sim$ plays the role of equality. Then we can define the L_S-structure $\Gamma(G)$ on this set by

$$\Gamma(G) \models \varphi(a_0, \ldots, a_{m-1}) \quad \text{iff} \quad G \models \gamma(\varphi)(\overline{a}_0, \ldots, \overline{a}_{m-1}) \tag{2.1}$$

for all $\overline{a}_i$ in a_i and all unnested atomic formulas $\varphi(x_0, \ldots, x_{m-1})$ of L_S. Γ is an interpretation of S in T if $\Gamma(G)$ is a model of S. We can define L_T formulas $\gamma(\varphi)$ for all L_S formulas φ in such a way that (2.1) remains true.

Let $\Gamma(x)$ be the L_T^{eq}-formula that describes the domain of $\Gamma(G)$. Using (2.1) we can define for every $\varphi(x_0, \ldots, x_{m-1})$ of L_S a L_T^{eq}-formula $\Gamma(\varphi)(x_0, \ldots, x_{m-1})$ such that for all $a_0, \ldots, a_{m-1}$ in $\Gamma(G)$

$$(2.2) \qquad \Gamma(G) \models \varphi(a_0, \ldots, a_{m-1}) \quad \text{iff} \quad G^{\mathrm{eq}} \models \Gamma(\varphi)(a_0, \ldots, a_{m-1}).$$

Now we assume that S and T are complete.

Definition Γ is an interpretation without new information, if the following equivalent conditions are fulfilled:

i) For every model G of T there is an elementary extension H such that it is possible to extend every automorphism of $\Gamma(G)$ (with respect to L_S) to an automorphism of H (with respect to L_T).

ii) For every formula $\psi(x_0, \ldots, x_{m-1})$ of L_T^{eq} with

$$T^{\mathrm{eq}} \models \forall x_0 \ldots x_{m-1}(\psi(x_0, \ldots, x_{m-1}) \longrightarrow \bigwedge_{i<m} \Gamma(x_i))$$

there exists some formula $\varphi(x_0, \ldots, x_{m-1})$ of L_S such that

$$T^{\mathrm{eq}} \models \forall x_0 \ldots x_{m-1}(\psi(x_0, \ldots, x_{m-1}) \longleftrightarrow \Gamma(\varphi)(x_0, \ldots, x_{m-1})).$$

Interpretations Γ without new information are considered in [2] and [3]. For the rest of this section we assume that T is stable and therefore S is stable. Furthermore let Γ be an interpretation without new information. If T is stable, then for every model M of S there is some model G of T such that $M \cong \Gamma(G)$. Using the Open Mapping Theorem for stable theories it is shown [3]:

Lemma 2.1 (*T stable*).
Let G be a model of T. Let $\psi(x_0, \ldots, x_{m-1}, \overline{a})$ be a formula of $(L_T)^{\mathrm{eq}}$ with parameter $\overline{a}$ and

$$G^{\mathrm{eq}} \models \psi(x_0, \ldots, x_{m-1}, \overline{z}) \longrightarrow \bigwedge_{i<m} \Gamma(x_i).$$

Then there is some L_S formula $\varphi(x_0, \ldots, x_{m-1}, \overline{b})$ with $\overline{b}$ in $\Gamma(G)$ such that

$$G^{\mathrm{eq}} \models \psi(x_0, \ldots, x_{m-1}, \overline{a}) \longleftrightarrow \Gamma(\varphi)(x_0, \ldots, x_{m-1}, \overline{b}).$$

Assume $\overline{c}, A \subseteq \Gamma(\mathfrak{C})$. Then we define

$$\mathrm{tp}_\Gamma(\overline{c}/A) = \{\Gamma(\varphi)(\overline{x}, \overline{a}) : \varphi(\overline{x}, \overline{y}) \in L_S,\ \overline{a} \subseteq A,\ \Gamma(\mathfrak{C}) \models \varphi(\overline{c}, \overline{a})\}.$$

Here $\Gamma(\mathfrak{C})$ is used to denote the L_S-structure defined in $\mathfrak{C}^{\mathrm{eq}}$. It is also used to denote the subset of $\mathfrak{C}^{\mathrm{eq}}$ defined by the formula $\Gamma(x)$. Clearly, $\mathrm{tp}_\Gamma(\overline{c}/A)$ is a subset of $\mathrm{tp}(\overline{c}/A)$. But the definition of an interpretation without new information gives us

Lemma 2.2 $\mathrm{tp}_\Gamma(\overline{c}/A)$ *implies* $\mathrm{tp}(\overline{c}/A)$.

We can consider $\mathrm{tp}_\Gamma(\overline{c}/A)$ as a type for the theory S. In [3] it is shown

Lemma 2.3

i) *Assume* $\overline{c}$, A *and* B *are in* $\Gamma(\mathbb{C}) \subseteq \mathbb{C}^{\mathrm{eq}}$ *and* $A \subseteq B$. *In* T^{eq} *we have* $\mathrm{tp}(\overline{c}/B)\mathrm{dnf}/A$ *if and only if in* S $\mathrm{tp}_\Gamma(\overline{c}/B)\mathrm{dnf}/A$.

ii) *Assume* $\overline{c} \subseteq \Gamma(\mathbb{C})$ *and* $G \preceq H \preceq \mathbb{C}$. *Then*

$$\begin{array}{rl} \textit{in } T^{\mathrm{eq}} & \mathrm{tp}(\overline{c}/H)\mathrm{dnf}/G \ \textit{ if and only if} \\ \textit{in } S & \mathrm{tp}_\Gamma(\overline{c}/\Gamma(H))\mathrm{dnf}/\Gamma(G). \end{array}$$

Again consider $\overline{c}$ and A in $\Gamma(\mathbb{C}) \subseteq \mathbb{C}^{\mathrm{eq}}$. According to this situation there is a subset $\mathrm{Cb}_\Gamma(\mathrm{tp}_\Gamma(\overline{c}/A))$ of $\Gamma(\mathbb{C})^{\mathrm{eq}} \subset \mathbb{C}^{\mathrm{eq}}$ which is the canonical base of $\mathrm{tp}_\Gamma(\overline{c}/A)$ if we consider it as a type with respect to S. For $A \subseteq \Gamma(\mathbb{C})$ the algebraic closure with respect to S is the algebraic closure in $\Gamma(\mathbb{C})^{\mathrm{eq}}$ with respect to T: $\mathrm{acl}_S(A) = \mathrm{acl}_T(A) \cap \Gamma(\mathbb{C})^{\mathrm{eq}}$.

Lemma 2.4 *Let* S *and* T *be stable and assume that* Γ *is an interpretation without new information of* S *in* T.

i) *For* $\overline{c}, A \subseteq \Gamma(\mathbb{C}) \subseteq \mathbb{C}^{\mathrm{eq}}$

$$\mathrm{Cb}(\mathrm{tp}(\overline{c}/A)) = \mathrm{Cb}_\Gamma(\mathrm{tp}_\Gamma(\overline{c}/A)).$$

ii) *If* T *is* CM*-trivial, then* S *is* CM*-trivial.*

Proof. i) If we consider $\mathrm{tp}_\Gamma(\overline{c}/A)$ as a type with respect to S, then $\mathrm{stp}_\Gamma(\overline{c}/A)$ is well-defined over $\mathrm{acl}_S(A)$ and of course over $\mathrm{Cb}_\Gamma(\mathrm{tp}_\Gamma(\overline{c}/A))$. Now we consider the situation in T. Since Γ is an interpretation without new information the defining scheme for $\mathrm{stp}_\Gamma(\overline{c}/A)$ induces a defining scheme for $\mathrm{stp}(\overline{c}/A)$. Hence $\mathrm{Cb}(\mathrm{tp}(\overline{c}/A)) = \mathrm{Cb}_\Gamma(\mathrm{tp}_\Gamma(\overline{c}/A))$.
ii) Let $M \preceq N \preceq \Gamma(\mathbb{C})$ be models of S and $\overline{c}$ be in $\Gamma(\mathbb{C})$ such that $\mathrm{acl}_S(\overline{c}M) \cap N = M$. We consider M and N as subsets of $\mathbb{C}^{\mathrm{eq}}$ closed under acl_S. Inside $\Gamma(\mathbb{C})$ acl_S and acl_T are the same. Note that $\mathrm{acl}_T(\overline{c}M) \cap \Gamma(\mathbb{C}) = \mathrm{acl}_S(\overline{c}M)$. Hence

$$\mathrm{acl}_S(\overline{c}M) \cap N = \mathrm{acl}_T(\overline{c}M) \cap \Gamma(\mathbb{C}) \cap N = \mathrm{acl}_T(\overline{c}M) \cap N. \tag{2.3}$$

To use the CM-triviality of T we need

$$\mathrm{acl}_T(\overline{c}M) \cap \mathrm{acl}_T(N) = \mathrm{acl}_T(M). \tag{2.4}$$

Note that there is some $G \preceq \mathbb{C}$ such that $\Gamma(G) = N$. To show this we use the fact from [3] that there is some T-model G' with $\Gamma(G') \simeq N$. W.l.o.g. we can assume $G' \preceq \mathbb{C}$.

Since Γ is an interpretation without new information there is an L_T automorphism f of $\mathbb{C}$ with $f(\Gamma(G')) = N$. Then $f(G')$ is the desired G.

Let e be an element of $\mathrm{acl}_T(\overline{c}M) \cap \mathrm{acl}_T(N)$. We choose a $L_{T^{\mathrm{eq}}}$-formula $\theta(x, \overline{b})$ with parameters $\overline{b}$ from $\Gamma(\mathbb{C})$, such that

$$\mathbb{C} \models \theta(e, \overline{b})$$

and the number $|\theta(\mathbb{C}, \overline{b})|$ is finite and minimal, where $\theta(\mathbb{C}, \overline{b})$ is used to denote the set of elements that satisfy $\theta(x, \overline{b})$. Since $e \in G$ and $G \preceq \mathbb{C}$ we can find $\overline{b}$ in N. Let $\overline{b}_1$ be any tuple in $\Gamma(\mathbb{C})$ such that $\mathbb{C} \models \exists\, x(\theta(x, \overline{b}) \wedge \theta(x, \overline{b}_1))$ and $|\theta(\mathbb{C}, \overline{b}_1)| = |\theta(\mathbb{C}, \overline{b})|$. By assumption $\theta(\mathbb{C}, \overline{b}_1) = \theta(\mathbb{C}, \overline{b})$. Let $\vartheta(\overline{y})$ be a formula that says that $\overline{y}$ is in $\Gamma(\mathbb{C})$, $|\theta(\mathbb{C}, \overline{y})| = |\theta(\mathbb{C}, \overline{b})|$ and for every $\overline{y}_1$ with $|\theta(\mathbb{C}, y_1)| = |\theta(\mathbb{C}, b)|$ we have

$$\theta(\mathbb{C}, y_1) = \theta(\mathbb{C}, y) \quad \text{or} \quad \theta(\mathbb{C}, y_1) \cap \theta(\mathbb{C}, y) = \emptyset.$$

For $\overline{b}_1$ and $\overline{b}_2$ in $\Gamma(\mathbb{C})$ with $\models \vartheta(\overline{b}_1) \wedge \vartheta(\overline{b}_2)$ we define

$$\overline{b}_1 \simeq \overline{b}_2 \quad \text{iff} \quad \exists\, x(\theta(x, \overline{b}_1) \wedge \theta(x, \overline{b}_2)).$$

$\simeq$ is an equivalence relation. Since Γ is an interpetation without new information there is a L_S-formula $\chi(\overline{y}_1, \overline{y}_2)$ such that $\Gamma(\chi)(\overline{y}_1, \overline{y}_2)$ defines $\simeq$. Let $\tilde{\overline{b}}$ be the $\simeq$-class of $\overline{b}$. $\tilde{\overline{b}}$ is an element of N. But it is also an element of $\mathrm{acl}_T(e)$. By (2.3)

$$\tilde{\overline{b}} \in \mathrm{acl}_S(\overline{c}\, M) \cap N = M.$$

W.l.o.g. we can choose $\overline{b}$ in M. Hence $e \in \mathrm{acl}_T(M)$, as desired in (2.4). The CM-triviality of T implies

$$\mathrm{Cb}(\mathrm{tp}_T(\overline{c}/M)) \subseteq \mathrm{acl}_T(\mathrm{Cb}(\mathrm{tp}_T(\overline{c}/N)).$$

By i) we have

$$\mathrm{Cb}_\Gamma(\mathrm{tp}_\Gamma(\overline{c}/M)) \subseteq \mathrm{acl}_T(\mathrm{Cb}_\Gamma(\mathrm{tp}_\Gamma(\overline{c}/N))).$$

For S this means

$$\mathrm{Cb}(\mathrm{tp}_S(\overline{c}/M)) \subseteq \mathrm{acl}_S(\mathrm{Cb}(\mathrm{tp}_S(\overline{c}/N))).$$

The CM-triviality of S is proved. $\square$

3 Alternating bilinear maps

Often it is helpful for our purposes to consider alternating bilinear maps besides nilpotent groups of class 2 and exponent $p > 2$. We follow similar ideas as in [5] but we do not assume that the centre of such a group is the commutator subgroup.

We fix a prime p greater than 2. Let $\mathbb{G}_{2,p}$ be the category of all nilpotent groups G of class 2 and exponent p. We write $\mathbb{G}^P_{2,p}$ if we have an additional unary predicate $P(x)$ such that $P(G)$ is a subgroup between the commutator subgroup G' and the centre $Z(G)$. The morphisms of $\mathbb{G}^P_{2,p}$ are the monomorphisms with respect to the signature "$1, \cdot, P(x)$". Later we fix a theory U in $\mathbb{G}_{2,p}$ and consider the subcategory of $G^P_{2,p}$ of all substructures of models of U, where P is considered as a predicate for the centre of the model in which the substructure is considered. So the typical situation is the following: U is complete, $\mathbb{C}$ is a monster model of U, $P(\mathbb{C}) = Z(\mathbb{C})$ and we consider all small substructures of $\mathbb{C}$ with respect to the signature "$1, \cdot, P(x)$".

First some notation. We often write AB instead of $A \cup B$. In groups G we use $[a, b] = a^{-1}b^{-1}ab$. $\langle X \rangle$ denotes the subgroup generated by $X \subseteq G$. Then $G' = \langle \{[a, b] : a, b \in G\} \rangle$ is the commutator subgroup. $Z(G)$ denotes the centre of G. Since in this paper all considered groups G are in $\mathbb{G}_{2,p}$ we work with abelian subgroups (e.g. $Z(\mathbb{C})$) or abelian factor groups (e.g. $G/Z(G)$) as with vector spaces over the field $\mathbb{F}_p$ with p elements. We speak about linear independence of elements of such abelian groups.
Let $\mathbb{B}_p$ be the category of all alternating bilinear maps $\langle V, W, \beta \rangle$ where V and W are vector spaces over the field $\mathbb{F}_p$ with p elements and β is an alternating bilinear map of $V \times V$ into W. Similarly as in groups we call elements $\beta(x, y)$ in W commutators. The morphisms of $\mathbb{B}_p$ are the embeddings of this class of structures. Hence a morphism of $\langle V_1, W_1, \beta_1 \rangle$ into $\langle V_2, W_2, \beta_2 \rangle$ is a pair (f, g) of vector space embeddings

$$f : V_1 \to V_2 \quad \text{and} \quad g : W_1 \to W_2$$

such that the following diagram is commutative:

$$\begin{array}{ccc} V_1 \times V_1 & \xrightarrow{\beta_1} & W_1 \\ \big\downarrow f \times f & & \big\downarrow g \\ V_2 \times V_2 & \xrightarrow{\beta_2} & W_2 \end{array} \quad .$$

Note that for every $\mathbb{F}_p$-vector space V there is a free alternating bilinear map $\langle V, \Lambda^2 V, \wedge \rangle$. It is called *the exterior square of* V and it is defined by the following property:

If $\langle V, W, \beta \rangle$ is any alternating bilinear map over V, then there is a vector space homomorphism f_β such that:

$$
\begin{array}{ccc}
V \times V & \xrightarrow{\quad \wedge \quad} & \Lambda^2 V \\
 & \searrow^{\beta} & \downarrow f_\beta \\
 & & W
\end{array}
\quad .
$$

In a canonical way we can define a functor $\mathcal{F}$ of $\mathbb{G}^P_{2,p}$ into $\mathbb{B}_p$: For $G \in \mathbb{G}^P_{2,p}$ let $\mathcal{F}(G)$ be $\langle G/P(G), P(G), [x,y]\rangle \in \mathbb{B}_p$. Here $G/P(G)$ and $P(G)$ are considered as vector spaces over $\mathbb{F}_p$. Note that $[x_0,y_0] = [x_1,y_1]$ if $x_0x_1^{-1} \in P(G) \subseteq Z(G)$ and $y_0y_1^{-1} \in P(G)$. For a $G^P_{2,p}$-embedding f of G in H let $\mathcal{F}(f)$ be the pair $(\overline{f}, f \downharpoonleft P(G))$, where $\overline{f}$ is the embedding of $G/P(G)$ into $H/P(H)$ induced by f and $f \downharpoonleft P(G)$ is the restriction of f to $P(G)$.
Since f is a monomorphism with respect to the signature "$1, \cdot, P(x)$", we have

$$
\begin{array}{ccc}
G/P(G) \times G/P(G) & \xrightarrow{\quad [\,,\,] \quad} & P(G) \\
\Big\downarrow \overline{f} \times \overline{f} & & \Big\downarrow f \downharpoonleft P(G) \\
H/P(H) \times H/P(H) & \xrightarrow{\quad [\,,\,] \quad} & P(H)
\end{array}
\quad .
$$

Note that $\mathcal{F}$ is a functor. Often we write $\mathcal{F}(G) = \langle V_G, W_G, \beta_G\rangle$.

Lemma 3.1 *Let G and H be groups in $G^P_{2,p}$ and let (g,h) be an embedding of $\mathcal{F}(G)$ into $\mathcal{F}(H)$. Let $\{c_\alpha : \alpha < \nu\}$ be a subset of G such that $\{c_\alpha/P(G) : \alpha < \nu\}$ is a basis of V_G and let $\{a_\alpha : \alpha < \nu\}$ be a subset of H with $g(c_\alpha/P(G)) = a_\alpha/P(H)$. Then $f\Big(\prod\limits_\alpha c_\alpha^{r_\alpha} d\Big) = \prod\limits_\alpha a_\alpha^{r_\alpha} h(d)$ where $d \in P(G)$ and $r_\alpha = 0$ for all but finitely many α, defines an $G^P_{2,p}$-embedding of G into H with $\mathcal{F}(f) = (g,h)$.*

Proof. Every element of G can be written as $\prod\limits_\alpha c_\alpha^{r_\alpha} d$ where $d \in P(G)$ and $r_\alpha = 0$ for all but finitely many α. We use the uniqueness of such representations $\prod\limits_\alpha c_\alpha^{r_\alpha} d$ and $\prod\limits_\alpha a_\alpha^{r_\alpha} h(d)$ of elements of G and H respectively. Since h is an embedding of $P(G)$ into $P(H)$ it shows that f is a well-defined injection.
To produce the standard representation of a product on both sides the "same" commutators are used:

$$
\begin{aligned}
c_\gamma c_\alpha &= c_\alpha c_\gamma [c_\gamma, c_\alpha] \qquad \text{and} \\
a_\gamma a_\alpha &= a_\alpha a_\gamma [a_\gamma, a_\alpha] = a_\alpha a_\gamma [g(c_\gamma), g(c_\alpha)] = a_\alpha a_\gamma h([c_\gamma, c_\alpha]).
\end{aligned}
$$

This proves that f is a homomorphism. □

Corollary 3.2 *Assume $G_0 \subseteq G$ and $H_0 \subseteq H$ are in $\mathbb{G}^P_{2,p}$ and there are a $\mathbb{G}^P_{2,p}$-isomorphism f_0 of G_0 onto H_0 and a $\mathbb{B}_p$-isomorphism (g,h) of $\mathcal{F}(G)$ onto $\mathcal{F}(H)$ that extends $\mathcal{F}(f_0)$. Then there is a $\mathbb{G}^P_{2,p}$-isomorphism f of G onto H that extends f_0 and fulfils $\mathcal{F}(f) = (g,h)$.*

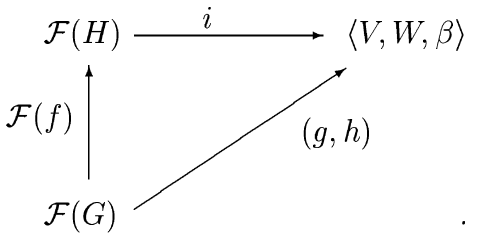
Proof. To apply Lemma 3.1 choose $\{c_\alpha : \alpha < \nu\}$ in G such that $\{c_\alpha/P(G) : \alpha < \mu\}$ is a basis for V_{G_0} for some $\mu \leq \nu$. Furthermore choose $a_\alpha = f_0(c_\alpha)$ for $\alpha < \mu$. Then Lemma 3.1 gives the desired isomorphism. □

We can use a similar proof to obtain the following consequence of Lemma 3.1 that will be used in the proof of the main theorem of the paper.

Corollary 3.3 *Let G be a substructure of $\mathbb{C}$ in $\mathbb{G}^P_{2,p}$ where $P(\mathbb{C}) = Z(\mathbb{C})$. Let A be a subset of $\mathbb{C}$ linearly independent modulo $\langle Z(\mathbb{C}) \cup G\rangle$. Let f_0 be an automorphism of G and let (g,h) be an automorphism of $\mathcal{F}(\mathbb{C})$ that extends $\mathcal{F}(f_0)$ with $g(a/Z(\mathbb{C})) = a/Z(\mathbb{C})$ for $a \in A$. Then there is an automorphism f of $\mathbb{C}$ that extends f_0 with $f(a) = a$ for $a \in A$ and $\mathcal{F}(f) = (g,h)$.*

Proof. To apply Lemma 3.1 we choose $\{c_\alpha : \alpha < \nu\}$ such that $\{c_\alpha/Z(\mathbb{C}) : \alpha < \nu\}$ is a basis of $V_{\mathbb{C}}$, $\{c_\alpha/Z(\mathbb{C}) : \alpha < \mu\}$ is a basis of V_G, and $A \subseteq \{c_\alpha : \mu \leq \alpha < \nu\}$. Then we choose $a_\alpha = f_0(c_\alpha)$ for $\alpha < \mu$ and $a_\alpha = c_\alpha$ for $c_\alpha \in A$. Lemma 3.1 provides the desired result. □

Lemma 3.4 *Let G be in $\mathbb{G}^P_{2,p}$. Assume that there is a $\mathbb{B}_p$-embedding (g,h) of $\mathcal{F}(G)$ into an alternating bilinear map $\langle V, W, \beta\rangle$. Then there are a group H with a predicate $P(H)$ in $\mathbb{G}^P_{2,p}$, a $\mathbb{B}_p$-isomorphism $i = (i_0, i_1) : \mathcal{F}(H) \cong \langle V, W, \beta\rangle$, and a $\mathbb{G}^P_{p,2}$-embedding f of G into H such that $i \circ \mathcal{F}(f) = (g,h)$:*



If (g,h) is surjective, then f is surjective, and H is uniquely determined up to isomorphisms respecting the embedding of G.

Proof. First we define H. There are ordinals $\nu \leq \mu$ and a basis $\{a_\alpha : \alpha < \mu\}$ of V such that $\{a_\alpha : \alpha < \nu\}$ is a basis of $g(G/P(G))$. The elements of H are pairs $\left(\sum_{\alpha<\mu} r_\alpha a_\alpha, b\right)$ where $b \in W$ and $r_\alpha = 0$ for all but finitely many α. The group multiplication is defined by

$$(3.1)\quad \left(\sum_\alpha r_\alpha a_\alpha, b_1\right)\left(\sum_\alpha s_\alpha a_\alpha, b_2\right) = \left(\sum_\alpha (r_\alpha + s_\alpha) a_\alpha, b_1 + b_2 + \sum_{\gamma<\alpha} r_\alpha s_\gamma \beta(a_\alpha, a_\gamma)\right)$$

Furthermore we define $P(H) = \{(0,b) : b \in W\}$.
It is easy to show that H is a group in $\mathbb{G}_{2,p}$ and that $H' \subseteq P(H) \subseteq Z(H)$. It follows that $\left(\sum_\alpha r_\alpha a_\alpha, b\right) = \prod_\alpha (a_\alpha, 0)^{r_\alpha}(0,b)$ and the image of $\{(a_\alpha, 0) : \alpha < \mu\}$ is a basis of $H/P(H)$. We have in H

$$(3.2)\qquad \left[\left(\sum_\alpha r_\alpha a_\alpha, b_1\right), \left(\sum_\alpha s_\alpha a_\alpha, b_2\right)\right] = \left(0, \sum_{\gamma<\alpha}(r_\alpha s_\gamma - s_\alpha r_\gamma)\beta(a_\alpha, a_\gamma)\right).$$

Hence $i = (i_0, i_1)$ with

$$i_0\left(\left(\sum_\alpha r_\alpha a_\alpha, b\right)/P(H)\right) = \sum_\alpha r_\alpha a_\alpha$$

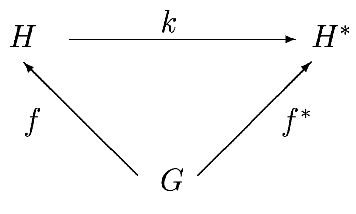
and

$$i_1((0,b)) = b$$

is the desired isomorphism of $\mathcal{F}(H)$ onto $\langle V, W, \beta\rangle$.
To define f we choose $\{c_\alpha : \alpha < \nu\}$ in G with $g(c_\alpha/P(G)) = a_\alpha$. Then $\{c_\alpha/P(G) : \alpha < \nu\}$ is a basis for $G/P(G)$. Hence every element of G has a unique representation of the form $\prod_\alpha c_\alpha^{r_\alpha} d$ where $d \in P(G)$. Then we define

$$f\left(\prod_\alpha c^{r_\alpha} d\right) = \left(\sum_\alpha r_\alpha a_\alpha, h(d)\right).$$

By the uniqueness of the representation of the elements of both sides and since h is a vector space embedding, f is a well-defined injection into H. Because of the definition of the multiplication in H by (3.1) it is an embedding. $i \circ \mathcal{F}(f) = (g,h)$ follows from the definitions. They also imply the surjectivity of f, if (g,h) is surjective.
Assume that H^*, f^*, and i^* also satisfy the conditions for H, f and i. Then $(i^*)^{-1}i$ defines an $\mathbb{B}_p$-isomorphism of $\mathcal{F}(H)$ onto $\mathcal{F}(H^*)$. By Lemma 3.1 we can lift $(i^*)^{-i}i$ to an $\mathbb{G}^P_{2,p}$-isomorphism k of H onto H^*. We can choose k in such a way that $kf = f^*$:



This implies the uniqueness of H over G. □

Lemma 3.4 shows that for every bilinear map M in $\mathbb{B}_p$ there is a group G in $\mathbb{G}^P_{2,p}$ with $\mathcal{F}(G) = M$. Furthermore it follows that all possible G are isomorphic.

Corollary 3.5 *The functor $\mathcal{F}$ gives a bijection between the isomorphism types of structures in $\mathbb{G}^P_{2,p}$ and $\mathbb{B}_p$.*

The functor $\mathcal{F}$ provides an interpretation of $\mathbb{B}_p$ in $\mathbb{G}^P_{2,p}$. If $G \equiv H$ are in $\mathbb{G}^P_{2,p}$, then $\mathcal{F}(G) \equiv \mathcal{F}(H)$.
Hence we can define for every complete theory T of structures in $\mathbb{G}^P_{2,p}$ a complete theory $\mathcal{F}(T)$ of the bilinear maps that lives in the models of T. Again Lemma 3.1 implies:

Corollary 3.6 *The interpretations of complete theories given by $\mathcal{F}$ are interpretations without new information.*

Especially we can apply this to $\mathbb{G}_{2,p}$ if we define $P(G) = Z(G)$. At the end of the section we show that there are stronger model-theoretic connections between $\mathbb{G}^P_{2,p}$ and $\mathbb{B}_p$.

Corollary 3.7 *Let G and H be structures in $\mathbb{G}^P_{2,p}$. Let $A \subseteq G$ and $f_0(A) = B \subseteq H$ be substructures, where f_0 is an isomorphism. Then $\mathrm{tp}_G(A) = \mathrm{tp}_H(B)$ if and only if $\mathrm{tp}_{\mathcal{F}(G)}(\mathcal{F}(A)) = \mathrm{tp}_{\mathcal{F}(H)}(\mathcal{F}(B))$.*

Proof. We add constant symbols $\{c_a : a \in A\}$ to the language of $\mathbb{G}^P_{2,p}$ and constant symbols $\{c_e : e \in \mathcal{F}(A)\}$ to the language of $\mathbb{B}_p$.
Let κ be a cardinal such that for a theory of size $\aleph_0 + |A|$ there exists a special model. Since $\mathcal{F}$ is an interpretation $\mathrm{tp}_G(A) = \mathrm{tp}_H(B)$ implies $\mathrm{tp}_{\mathcal{F}(G)}(\mathcal{F}(A)) = \mathrm{tp}_{\mathcal{F}(H)}(\mathcal{F}(B))$. Let $\langle G^*, A\rangle$ be an elementary extension of $\langle G, A\rangle$ that is a special model of cardinality κ. Let $\langle H^*, B\rangle$ be an elementary extension of $\langle H, B\rangle$ that is a special model of cardinality κ. Then $\langle \mathcal{F}(G^*), \mathcal{F}(A)\rangle$ is an elementary extension of $\langle \mathcal{F}(G), \mathcal{F}(A)\rangle$ and it is a special model of cardinality κ. Analogously $\langle \mathcal{F}(H^*), \mathcal{F}(B)\rangle$ is an elementary extension of $\langle \mathcal{F}(H), \mathcal{F}(B)\rangle$ and it is a special model of cardinality κ. They are all models of the same theory, since $\mathrm{tp}_{\mathcal{F}(G)}(\mathcal{F}(A)) = \mathrm{tp}_{\mathcal{F}(H)}(\mathcal{F}(B))$. Hence the special models $\langle \mathcal{F}(G^*), \mathcal{F}(A)\rangle$ and $\langle \mathcal{F}(H^*), \mathcal{F}(B)\rangle$ are isomorphic and the isomorphism extends $\mathcal{F}(f_0)$. We can lift the automorphism to a group automorphism f of G^* and H^* that extends f_0 by Corollary 3.2. Hence $\mathrm{tp}_G(A) = \mathrm{tp}_H(B)$, as desired. □

Corollary 3.8 *For G and H in $\mathbb{G}^P_{2,p}$ we have*

$$G \equiv H \quad \text{if and only if} \quad \mathcal{F}(G) \equiv \mathcal{F}(H).$$

If we consider $\mathbb{G}_{2,p}$ and interpret $P(x)$ as $Z(x)$, then $\mathcal{F}(G)$ for $G \in \mathbb{G}_{2,p}$ is well-defined. If T is a complete theory of groups in $\mathbb{G}_{2,p}$, then let $\mathcal{F}(T) = \mathrm{Th}(\mathcal{F}(G) : G \in \mathrm{Mod}(T))$. It is clear that stability or simplicity of T implies stability or simplicity of $\mathcal{F}(T)$ respectively. We prove the converse.

Corollary 3.9 *Let T be a complete theory of groups in $\mathbb{G}_{2,p}$. Then T is λ-stable if and only if $\mathcal{F}(T)$ is λ-stable.*

Proof. To show the non-trivial direction assume that there are λ^+ types over a model G of T with $|G| = \lambda$. W.l.o.g. we can assume that these types have the form $\mathrm{tp}(a/G)$ with $a \in \mathfrak{C}$, where $\mathfrak{C}$ is a large elementary extension of G. If we have λ^+-types $\mathrm{tp}(a/G)$ with $a \notin \langle G \cup Z(\mathfrak{C})\rangle$, then their "images" $\mathrm{tp}(aZ(\mathfrak{C})/\mathcal{F}(G))$ are pairwise different by Corollary 3.7. To prove this we assume that $\mathrm{tp}(a, G)$ and $\mathrm{tp}(c/G)$ are two different types of this kind. If $\mathrm{tp}(aZ(\mathfrak{C})/\mathcal{F}(G))$ and $\mathrm{tp}(cZ(\mathfrak{C})/\mathcal{F}(G))$ are equal, then we consider $\langle G \cup a\rangle$ and $\langle G \cup c\rangle$. By Lemma 3.1 we can extend the identity on G to an isomorphism f_0 of these sugroups with $f_0(a) = c$. Now we can apply Corollary 3.7 and obtain a contradiction. Otherwise we have λ^+ types $\mathrm{tp}(a/G)$ with $a \in \langle G \cup Z(\mathfrak{C})\rangle$. Then we have λ^+ types $\mathrm{tp}(a/G)$ with $a \in Z(\mathfrak{C})$. Similarly as above by Corollary 3.7 their "images" are different. □

Note λ is a strong limit cardinal, if λ is a limit cardinal and $2^\kappa < \lambda$ for all $\kappa < \lambda$. For strong limit cardinals λ we have $\lambda^{\mathrm{cf}(\lambda)} = 2^\lambda$ and $\lambda^{<\mathrm{cf}(\lambda)} = \lambda$. For every regular κ there are arbitrary large strong limit cardinals λ with $\mathrm{cf}(\lambda) = \kappa$. The following characterization of simplicity is essentially due to S. Shelah [14]:

Theorem 3.10 *Let κ be $|T|^+$ and $\lambda > 2^\kappa$ be a strong limit cardinal with $\mathrm{cf}(\lambda) = \kappa$. T is not simple if and only if there are 2^λ pairwise contradictory 1-types of power κ over a set A of cardinality λ.*

Corollary 3.11 *Let T be a complete theory of infinite groups in $\mathfrak{G}_{2,p}$. Then T is simple if and only if $\mathcal{F}(T)$ is simple.*

Proof. The tree property for $\mathcal{F}(T)$ implies the tree property for T. Hence simplicity of T implies simplicity of $\mathcal{F}(T)$. Now assume that T is not simple. We have the situation described in Theorem 3.10. W.l.o.g. we assume that A is a model and the types p_i are over elementary submodels A_i of A ($i < 2^\lambda$, $|A_i| = \kappa$). Then we can assume that either all p_i are realized outside of $\langle A \cup Z(\mathfrak{C})\rangle$ or all p_i are realized inside. In the second case we can assume that there is some $a \in A$ such that the realizations have the form ab_i where $b_i \in Z(\mathfrak{C})$, since $\mathrm{cf}(2^\lambda) > \lambda$. Hence w.l.o.g. $p_i = \mathrm{tp}(b_i/A_i)$ where either $b_i \notin \langle A \cup Z(\mathfrak{C})\rangle$ for all i, or $b_i \in Z(\mathfrak{C})$ for all i. In both cases Corollary 3.7 implies that the $\mathcal{F}(T)$-images of the types are also contradictory. Hence Theorem 3.10 implies that $\mathcal{F}(T)$ is not simple. □

4 Mekler's construction

Let p be a prime greater than 2. Let S^* be any theory of finite similarity type. A.H. Mekler [11] has given a uniform construction of groups $G(M)$ for every model M of S^*, a theory T^* of all groups $G(M)$ for M in S^*, and an interpretation Γ of S^* in T^* such that

i) T^* is a theory of nilpotent groups of class 2 and of exponent p.

ii) If $G \models T^*$, then there is an $M \models S^*$ such that $G(M) \equiv G$.

iii) For S^*-models M and N we have $M \equiv N$ if and only if $G(M) \equiv G(N)$.

iv) $\Gamma(G(M)) \cong M$.

v) $\mathrm{Th}(M)$ is λ-stable iff $\mathrm{Th}(G(M))$ is λ-stable.

The aim of this paper is to prove that

vi) $\mathrm{Th}(M)$ is stable and CM-trivial if and only if $\mathrm{Th}(G(M))$ is stable and CM-trivial.

Before we show:

vii) $\mathrm{Th}(M)$ is simple if and only if $\mathrm{Th}(G(M))$ is simple.

A.H. Mekler has done his construction for the theory S^* of nice graphs. This restriction is possible since this theory is universal for biinterpretation (in the notation of W. Hodges [8]).
In this section we want to describe Mekler's construction. We follow the detailed presentation in W. Hodges [8]. The notation is slightly changed. We use T^* instead of T_{ng}. For 4.1 until 4.7 you find detailed proofs in [8]. Furthermore there is an explicit axiomatization of T^*.
A nice graph is a structure with only one binary symmetric and irreflexive relation $R(x, y)$ such that

a) $\exists x_0 x_1 (x_0 \neq x_1)$.

b) $\forall x_0 x_1 \, \exists y (x_0 \neq x_1 \longrightarrow y \neq x_0 \wedge y \neq x_1 \wedge R(x_0, y) \wedge \neg R(x_1, y))$.

c) $\forall x_0 x_1 x_2 \Big(\bigwedge_{i<j<3} x_i \neq x_j \longrightarrow \neg \Big(\bigwedge_{i<j<3} R(x_i, x_j) \Big) \Big)$.

d) $\forall x_0 x_1 x_2 x_3 \Big(\bigwedge_{i<j<4} x_i \neq x_j \longrightarrow \neg \Big(\bigwedge_{i<3} R(x_i, x_{i+1}) \wedge R(x_3, x_0) \Big) \Big)$.

c) and d) say that there are no triangles and squares. Let S^* be the theory of nice graphs. We use $M, N, \ldots$ to denote graphs and often models of S^* and $G, H, \ldots$ to denote nilpotent groups of class 2 and exponent p. We assume that the alternating bilinear map $\mathcal{F}(G) = \langle V_G, W_G, \beta_G \rangle$ is living in G^{eq} ($P(G) = Z(G)$).
Now let M be any graph. Then let $F(M)$ be the free nilpotent group of class 2 and exponent p that is freely generated by the domain of M. Let $Z(M)$ be the centre of $F(M)$. We can consider $Z(M)$ and $F(M)/Z(M)$ as vector spaces over the field $\mathbb{F}_p$ with p elements. $F(M)/Z(M)$ has a basis $\{a/Z(M) : a \in M\}$. We assume that M is ordered by a relation $<$ that is not in the language. Then $\{[a, b] : a, b \in M, a < b\}$ is a basis of $Z(M)$, where $[a, b] = a^{-1} b^{-1} ab$. Let $H(M)$ be the subgroup $\langle \{[a, b] : a, b \in M, M \models R(a, b)\} \rangle$. Then A.H. Mekler defined $G(M) = F(M)/H(M)$.

If the meaning is clear, then we write Z instead of $Z(G(M))$ or $Z(G)$. First we study the elements of $G(M)$. Note that $\{[a,b] : a,b \in M, a < b$ and $M \models \neg R(a,b)\}$ is a basis of the centre of $G(M)$. Furthermore $(G(M))' = Z(G(M))$. We define, for any elements g, h in any group G

$$\begin{aligned} g &\sim h, \text{ if } C(g) = C(h), \\ g &\approx h, \text{ if there is some } r \ (0 < r < p) \text{ such that } g = h^r \cdot c \text{ where } c \in Z. \\ g &\equiv_Z h, \text{ if } g/Z = h/Z. \end{aligned}$$

These are three $\emptyset$-definable equivalence relations and we have $g \equiv_Z h \Rightarrow g \approx h \Rightarrow g \sim h$.
g/Z is the $\equiv_Z$-class of g, $g^{\approx}$ is the $\approx$-class of g, and $g^{\sim}$ is the $\sim$-class of g. Each $g^{\sim}$ is a union of $\approx$-classes. Hence we can define:

Definition Let g be an element of any nilpotent group G of class 2 and exponent p. Assume $g \notin Z$.

- g is of type q, if q is the number of different $\approx$-classes in $g^{\sim}$.
- g is isolated if $G \models [g,h] = 1$ implies $h \approx g$ or $h \in Z$.
- g is of type q^{ι}, if g is of type q and isolated.
- g is of type q^{ν} if g is of type q and not isolated.

If q is finite, then all notions above are first order definable. Now we consider nice graphs M. In the notation of W. Hodges [8] A.H. Mekler [11] proved:

Theorem 4.1 *Let M be a nice graph.*

i) *Every non-central element in $G(M)$ is of type 1, $p-1$, or p.*

ii) *An element of $G(M)$ is $\approx$-equivalent to a (unique) vertex of M if and only if it is of type 1^{ν}.*

iii) *For every element g in $G(M)$ of type p, there is an element b of type 1^{ν} such that $G(M) \models [g,b] = 1$, and $b^{\approx}$ is uniquely determined.*

There is a formula $\gamma(x)$ that says "x is of type 1^{ν}". Hence we can recover M from $G(M)$. There is an injective map of the elements of M onto the $\approx$-classes of the non-central elements of $G(M)$ of type 1^{ν}. We speak about vertexes $g^{\approx}$ and $h^{\approx}$ in $G(M)$ (or better $G(M)^{\mathrm{eq}}$). If X is a set of such elements of typ 1^{ν} that are in different $\approx$-classes, then X is linearly independent modulo Z. Two such vertexes $g^{\approx}$ and $h^{\approx}$ are joined in M if and only if g and h commute. Hence

Corollary 4.2 *There is an interpetation Γ of the theory S^* of nice graphs in the theory of nilpotent groups of class 2 and exponent p such that $\Gamma(G(M)) \cong M$.*

Let T^* be $\mathrm{Th}(\{G(M) : M \models S^*\})$. W. Hodges [8] has given 10 axiom schemes that axiomatize T^*. Below we give some of the properties of the models of T^*.
Let G be a model of T^*.

- Every non-central element in G is of type 1, $p-1$, or p (Theorem 4.1i).
- $\Gamma(G)$ is a nice graph.
- Elements of type p or $p-1$ in G are not isolated. Hence we distinguish only types 1^ν, 1^ι, p and $p-1$.
- Elements of type $p-1$ are the product of two $\sim$-inequivalent elements of type 1^ν.
- If g is an element of type p, then by Theorem 4.1 there is some b of type 1^ν such that g and b commute.

In the last paragraph $b^\sim = b^\approx$ is uniquely determined this properties. We call the element b a handle for g. We also speak about the handle $b^\sim = b^\approx$ of g. Note that in this case $C(g) = \{g^\alpha b^\beta c : c \in Z, 0 \leq \alpha, \beta < p\}$. Then $b^\approx \in \mathrm{dcl}(g)$ and $b/Z \in \mathrm{acl}(g)$.

Definition Let M be a nice graph. A graph $M^+ \supseteq M$ is a cover of M if for every vertex g in $M^+ \setminus M$ either there is a vertex b in M such that b is the unique vertex in M^+ which is joined to g and b is joined to infinitely many vertexes in M, or g is not joined to any other vertex.
A cover M^+ of M is a λ-cover if for every vertex $b \in M$ the number of vertexes g in $M^+ \setminus M$ joined to b is λ, if b is joined to infinitely many vertexes in M, and zero otherwise, and the number of new vertexes in $M^+ \setminus M$, which are not joined to any other vertex, is λ if M is infinite and 0 otherwise.

Note that a cover of a nice graph is not nice in general. Now we introduce the notion of a transversal following W. Hodges in [8].

Definition Let G be a model of T^*.

- An 1^ν-*transversal* X_{1^ν} of G is a set consisting of one representative of each $\sim$-class of elements of type 1^ν.
- An element of G is *proper* if it is not a product of elements of type 1^ν.
- A p-transversal X_p of G is a set of representatives of $\sim$-classes of proper elements of type p, which is maximal with the property that if Y is a finite subset of X_p and all elements of Y have the same handle, then Y is independent modulo the subgroup generated by the elements of type 1^ν and $Z(G)$.
- An 1^ι-transversal is a set of representatives of $\sim$-classes of proper elements of type 1^ι, which is maximal independent modulo the subgroup generated by the elements of type 1^ν and p and $Z(G)$.

- A subset X of G is a transversal if $X = X_{1^\nu} \cup X_p \cup X_{1^\iota}$ where X_{1^ν} is an 1^ν-transversal, X_p is a p-transversal, and X_{1^ι} is an 1^ι-transversal.
- $X_{p,e}$ is used to denote the subset of the elements with handle e of X_p.

$\gamma(x)$ was the formula that says that x of type 1^ν. Hence $\langle\gamma(G)\rangle$ is the subgroup generated by the elements of type 1^ν. Then $\langle\gamma(G)\rangle \supseteq Z(G)$. If X_{1^ν} is an 1^ν-transversal, then

$$\langle\gamma(G)\rangle/Z(G) = \bigoplus_{a\in X_{1^\nu}} (\langle a\rangle Z(G)/Z(G)).$$

Lemma 4.3 *Assume $G \models T^*$ and $X = X_{1^\nu} \cup X_p \cup X_{1^\iota}$ is a transversal of G. We can consider X as a graph M^+ with $R(a,b)$ if and only if $G \models [a,b] = 1$. Then M^+ is a cover of $M^+ \restriction X_{1^\nu}$ and $M^+ \restriction X_{1^\nu}$ can be identified with $\Gamma(G)$.*

Note that transversals exist in models of T^*. As shown by A.H. Mekler transversals can be used to describe the structure of models of T^*:

Theorem 4.4 *Let G be a model of T^*. Let X be a transversal of G. Let us consider X as a cover M^+ of $\Gamma(G)$ as in Lemma 4.3. Then there is an elementary abelian p-group H such that*

$$G = \langle X\rangle \oplus H \quad \text{and} \quad \langle X\rangle \cong G(M^+).$$

Corollary 4.5

i) *If in Theorem 4.4 G is an infinite special model of T^* (e.g. G is saturated), then M^+ is a $|G|$-cover and $|H| = |G|$.*

ii) *If G_1 and G_2 are special models of T^* of the same infinite cardinality, then every isomorphism from $\Gamma(G_1)$ to $\Gamma(G_2)$ lifts to an isomorphism from G_1 onto G_2.*

Corollary 4.6 *If M and N are elementarily equivalent nice graphs, then $G(M) \equiv G(N)$.*

Let $G \models T^*$ and X be a transversal of G. Assume $G = \langle X\rangle \oplus H$ as in Theorem 4.4. Let $<$ be a well-ordering of X. We may assume $X_{1^\nu} < X_p < X_{1^\iota}$. Let $X = \{x_\alpha : \alpha < \kappa\}$ be an enumeration of X with respect to $<$. If g is an element of G, then we call

$$g = \prod_{\alpha<\kappa} x_\alpha^{r_\alpha} \prod_{\alpha<\beta<\kappa} (x_\alpha, x_\beta)^{s_{\alpha\beta}} \cdot h$$

where $h \in H$ and $0 \le r_\alpha$, $s_{\alpha\beta} < p$, a representation of g with respect to X, $<$, and H.

Corollary 4.7 *In the situation described above every element $g \in G$ has a unique representation with respect to X, $<$, and H.*

For the rest of the paper we consider completions S of S^* with an infinite model. By Corollary 4.6 the theory $T_S = \mathrm{Th}(\{G(M) : M \models S\})$ is complete. Using Theorem 4.4 A.H. Mekler showed:

Corollary 4.8 *For all infinite cardinals λ we have that S is λ-stable if and only if T_S is λ-stable.*

If necessary we consider transversals X in models G of T_S as covers over its 1^ν-part X_{1^ν}. Similarly as above we have:

Lemma 4.9 *Let $G \models T_S$.*

i) *Two transversals of G are isomorphic as graphs. The isomorphism respects the 1^ν-, p-, and 1^ι-parts.*

ii) *If there is a graph isomorphism of a transversal X of $G \models T_S$ onto a transversal Y of G, then it lifts to an automorphism of G.*

Corollary 4.10 Γ *is an interpretation without new information.*

Corollary 4.11 *If T_S is stable and CM-trivial, then S is stable and CM-trivial.*

Proof. Use Lemma 2.4 and 4.10. □

To prove the converse of Corollary 4.11 we need some further results about Mekler-groups.

Definition A subset Y of a transversal X is called a part of X if for every element $g \in X_p \cap Y$ there is a handle of g in Y.

Note that a part of a transversal can be defined similarly as a transversal:
Let G be a model of T^*. Y is a part of a transversal if $Y = Y_{1^\nu} \cup Y_p \cup Y_{1^\iota}$ where
Y_{1^ν} is a subset of elements of type 1^ν where two different elements are in different $\sim$-classes,
Y_p is a set of elements of type p that is linearly independent modulo the subgroup generated by all elements of type 1^ν and $Z(G)$, and for every $y \in Y_p$ there is a handle in Y_{1^ν}, and
Y_{1^ι} is a set of elements of type 1^ι that is linearly independent modulo the subgroup generated by all elements of type 1^ν and p and $Z(G)$.

Lemma 4.12 *In a model G of T_S let h be a bijection between two parts of transversals Y and $h(Y)$ such that h respects the 1^ν, p-, and 1^ι-parts, the handles, and $\mathrm{tp}_\Gamma(Y_{1^\nu}^{\approx}) = \mathrm{tp}_\Gamma(h(Y_{1^\nu}^{\approx}))$. Then $\mathrm{tp}(Y) = \mathrm{tp}(h(Y))$.*

Proof. Let G^* be an elementary extension of G that is a special model of $\mathrm{Th}(G)$. Also in G^* the sets Y and $h(Y)$ are parts of a transversal and $\mathrm{tp}_\Gamma(Y_{1^\nu}^{\approx}) = \mathrm{tp}_\Gamma(h(Y_{1^\nu}^{\approx}))$.
Let $X_{1^\nu} \supseteq Y_{1^\nu}$ be an 1^ν-transversal of G^*. Since $\mathrm{tp}_\Gamma(Y_{1^\nu}^{\approx}) = \mathrm{tp}_\Gamma(h(Y_{1^\nu}^{\approx}))$ we can extend h to a graph-automorphism g of $\Gamma(G^*)$. As in Corollary 4.5ii) we can lift g to a group-automorphism f of G^*. Using Theorem 4.4 it is easy to do this in such a way that f extends h. □

The next lemma is shown in [3].

Lemma 4.13 *Let $G \subseteq H$ be models of T_S. Then $G \preceq H$ if and only if $\Gamma(G) \preceq \Gamma(H)$ and every transversal of G can be extended to a transversal of H.*

For the rest of the paper we use $\mathfrak{C}$ to denote a monster model of T_S.

Definition

i) Assume $G \preceq \mathfrak{C} \models T_S$. A transversal X of $\mathfrak{C}$ respects G if $X = X^0 \cup X^1$, where $X^0 = X \cap G$ and X^0 is a transversal of G.

ii) X is a part of a transversal of $\mathfrak{C}$ that respects G, if X can be extended to a transversal of $\mathfrak{C}$ that respects G.

Lemma 4.14 *Let T_S be stable. Assume $G \preceq \mathfrak{C} \models T_S$ and Y is a finite part of a transversal of $\mathfrak{C}$ that respects G. Then*

$$\mathrm{Cb}(\mathrm{tp}(Y/G)) = (Y \cap G) \cup \mathrm{Cb}_\Gamma(\mathrm{tp}_\Gamma(Y_{1^\nu}^{\approx}/\Gamma(G)).$$

Proof. W.l.o.g. we can assume that G is saturated. To show the nontrivial direction consider some automorphism f of $\mathfrak{C}$ that fixes G setwise and $(Y \cap G) \cup \mathrm{Cb}_\Gamma(\mathrm{tp}_\Gamma(Y_{1^\nu}^{\approx}/\Gamma(G)))$ pointwise. Then $\mathrm{tp}_\Gamma(Y_{1^\nu}^{\approx}/\Gamma(G)) = \mathrm{tp}_\Gamma(f(Y_{1^\nu}^{\approx})/\Gamma(G))$. Let X be a part of a transversal of $\mathfrak{C}$ such that $Y \subseteq X$, $X \cap G$ is a transversal of G, and $X = (X \cap G) \cup Y$. Let h be a map of X into $\mathfrak{C}$ such that $h \restriction (X \cap G) = \mathrm{id}$ and $h(a) = f(a)$ for $a \in Y$. By Lemma 4.12 $\mathrm{tp}(X) = \mathrm{tp}(h(X))$. Hence $\mathrm{tp}(Y/G) = \mathrm{tp}(h(Y)/G) = \mathrm{tp}(f(Y)/G)$, as desired. □

Since we consider only nice graphs we have the following fact:

Lemma 4.15 *If $G \preceq \mathfrak{C} \models T_S$, then the following is impossible: a, b, and c are elements of type 1^ν that are pairwise not $\approx$-equivalent, $a, b \in G$, $c \notin G$, and $[a,c] = [b,c] = 0$.*

Proof. Assume we have the above situation. Since $G \preceq \mathfrak{C}$ there is some c' in G such that c' is not $\approx$-equivalent to a and b and $[c', a] = 0$, $[c', b] = 0$. We have a square in $\Gamma(\mathfrak{C})$, which is forbidden. □

Hence up to $\approx$-equivalence there is at most one $a \in G$ with $\beta(c, a) = 0$ for c of type 1^ν not in G. We call $a^{\approx}$ the root of c in G.

Corollary 4.16 *If $G \preceq \mathfrak{C} \models T_S$ and c is an element of type 1^ν not in G, then c has at most one root in G.*

5 Simplicity

As before S is a complete theory of nice graphs, and T_S is the corresponding Mekler-theory. Of course simplicity of T_S implies simplicity of S. We show the converse. This is joint work with my student Alexander Pentzel.

Theorem 5.1 *S is simple if and only if T_S is simple.*

Proof. To show the non-trivial direction we assume that T_S is not simple. We use Theorem 3.10. Let λ and κ be cardinals chosen as in Theorem 3.10. Let $\Gamma(\mathfrak{C})$ be a monster-model of S living in $\mathfrak{C} \models T_S$. We have to find a subset $A \subseteq \Gamma(\mathfrak{C})$ of cardinality λ and 2^λ pairwise contradictory 1-types of power κ over A. By assumption we have this situation in $\mathfrak{C} \models T_S$. There are a model $G \preceq \mathfrak{C}$ with $|G| = \lambda$ and 2^λ pairwise contradictory types $p_i \in S_1(\langle D^i \rangle)$ with $|D^i| \leq \kappa$, $D^i \subseteq G$, and $i < 2^\lambda$. Let X be a transversal of G. By Lemma 4.13 we find a transversal of the form XY of $\mathfrak{C}$. By Theorem 4.4 there are elementary abelian p-groups $\langle I \rangle \subseteq Z(G)$ and $\langle J \rangle \subseteq Z(\mathfrak{C})$ such that $G = \langle X \rangle \oplus \langle I \rangle$ and $\mathfrak{C} = \langle XY \rangle \oplus \langle I \rangle \oplus \langle J \rangle$, where I and J are bases of the abelian p-groups. W.l.o.g. we can assume that $D^i \subseteq X \cup I$ and $\langle D^i \rangle \preceq G$ for all $i < 2^\lambda$. Then we have that the D^i are closed under handles: If d is an element of type p in D^i, then the handle of d is in D^i. Now we consider realizations of the p_i. We can assume that

$$p_i = \mathrm{tp}(t_i(X^i Y^i I^i J^i)/\langle D^i \rangle)$$

where $X^i \subseteq X$, $Y^i \subseteq Y$, $I^i \subseteq I$, and $J^i \subseteq J$ are finite, $X^i Y^i$ is closed under handles, and the t_i are terms. W.l.o.g. $X^i I^i \subseteq D^i$.
Let $Y^i_{1^\nu}$ be the 1^ν-part of Y^i, $Y^i_{p,e}$ be the p-part of Y^i with handle $e \in X^i_{1^\nu} Y^i_{1^\nu}$, and $Y^i_{1^\iota}$ be the 1^ι-part of Y^i.
Since $\mathrm{cf}(2^\lambda) > \lambda$ we can assume w.l.o.g. that there are a term t and finite subsets $X^* \subseteq X$ and $I^* \subseteq I$ such that $t_i = t$, $X_i = X^*$, and $I^i = I^*$ for all $i < 2^\lambda$. By the same argument we can furthermore assume that all J^i have the same length and there are bijections h_{ij} of Y^i onto Y^j such that

$$h_{ij}(Y^i_{1^\nu}) = Y^j_{1^\nu}, \quad h_{ij}(Y^i_{p,e}) = Y^j_{p,h_{ij}(e)}, \quad \text{and } h_{ij}(Y^i_{1^\iota}) = Y^j_{1^\iota}.$$

Finally we can w.l.o.g. assume that there is some $J^* \subseteq J$ with $J^i = J^*$ for all $i < 2^\lambda$.

Now we show:

Claim If $\mathrm{tp}_\Gamma(Y^{i\approx}_{1^\nu}/\Gamma(\langle D^i \rangle)) \cup \mathrm{tp}_\Gamma(Y^{j\approx}_{1^\nu}/\Gamma(\langle D^j \rangle))$ is consistent, then $p_i \cup p_j$ is consistent.

The claim implies that the S-types $q_i = \mathrm{tp}_\Gamma(Y^{i\approx}_{1^\nu}/\Gamma(\langle D^i \rangle))$ are pairwise contradictory. Hence we have the desired 2^λ pairwise contradictory types q_i over $\Gamma(G)$ where $|\Gamma(G)| = \lambda$ and $|q_i| = \kappa_\iota$.

Proof of the Claim. Since $\Gamma(\mathbb{C})$ is a monster model of the theory S there is a set E_{1^ν} of elements of type 1^ν in $\mathbb{C}$ such that $E_{1^\nu}^{\approx}$ is a common realization of $\mathrm{tp}_\Gamma(Y_{1^\nu}^{i\approx}/\Gamma(\langle D^i\rangle))$ and $\mathrm{tp}_\Gamma(Y_{1^\nu}^{j\approx}/\Gamma(\langle D^j\rangle))$.
Choosing the right elements in the $\approx$-classes we can assume that $E_{1^\nu} \subseteq X_{1^\nu}Y_{1^\nu}$. We have $E_{1^\nu} \cap D_{1^\nu}^i = E_{1^\nu} \cap D_{1^\nu}^j = \emptyset$. Now we choose $E_p \subseteq Y_p$, $E_{1^\iota} \subseteq Y_{1^\iota}$, and $E = E_{1^\nu}E_pE_{1^\iota}$ such that there is a bijection h^i of Y^iD^i onto ED^i that is the identity on D^i, h_i respects the 1^ν-, p-, and 1^ι-parts and the handles, and

$$\mathrm{tp}_\Gamma(Y_{1^\nu}^{i\approx}D_{1^\nu}^{i\approx}) = \mathrm{tp}_\Gamma(h^i(Y_{1^\nu}^{i\approx}D_{1^\nu}^{i\approx})) = \mathrm{tp}_\Gamma(E_{1^\nu}^{i\approx}D_{1^\nu}^{i\approx}).$$

Then we have also such a bijection h^j between Y^jD^j and ED^j. By Lemma 4.12 we have

$$\mathrm{tp}(ED^i) = \mathrm{tp}(Y^iD^i), \text{ and } \mathrm{tp}(ED^j) = \mathrm{tp}(Y^jD^j),$$

and therefore

$$\mathrm{tp}(E\langle D^i\rangle J^*) = \mathrm{tp}(Y^i\langle D^i\rangle J^*), \text{ and } \mathrm{tp}(E\langle D^j\rangle J^*) = \mathrm{tp}(Y^j\langle D^j\rangle J^*).$$

Hence EJ^* is a common realization of $\mathrm{tp}(Y^iJ^*/\langle D^i\rangle)$ and $\mathrm{tp}(Y^jJ^*/\langle D^j\rangle)$. But then $p_i = \mathrm{tp}(t(X^*Y^iI^*J^*)/\langle D^i\rangle)$ and $p_j = \mathrm{tp}(t(X^*Y^jI^*J^*)/\langle D^j\rangle)$ have the common realization $t(X^*EI^*J^*)$. Note that by construction $X^*I^* \subseteq D^\ell$ for all ℓ. □

6 Lifting elements of the commutator subgroup

Let S be a complete theory of nice graphs and let T_S be the corresponding theory of Mekler-groups G. Often it is convenient to work in $\mathcal{F}(G) = \langle V_G, W_G, \beta_G\rangle$ (see Section 3). We assume that $\mathcal{F}(G)$ lives in G^{eq}. If we work in $\mathcal{F}(G)$ then we use additive notation for $V_G = G/Z(G)$ and $W_G = Z(G)$. If $G = \mathbb{C}$ is the monster model of T_S, then we leave out the index $G = \mathbb{C}$.

Definition A subset X of V_G is (a part of) a transversal if a set of representatives in G is (a part of) a transversal.

Note that this is equivalent to say that all sets of representatives in G are (parts of) transversals.
In V_G we also use other notions for G like $\sim$-equivalence, $\approx$-equivalence, type $1^\nu, \ldots$

Definition If $a, b, c \in V$ are of type 1^ν, $a \not\approx b$, $c \not\approx a$, $c \not\approx b$, $\beta(a,c) = 0$, and $\beta(b,c) = 0$, then $c^\approx$ is called the connection between a and b.

If such a connection $c^\approx$ of a and b exists, then it is uniquely determined by the niceness of $\Gamma(\mathbb{C})$.

Definition A subspace $U \subseteq V$ is called good, if it has a basis X such that X is a part of a transversal and for the handles in X all connections are represented in X.

Let V_{1^ν} be the subspace of V generated by all elements of type 1^ν. Then $V_{1^\nu} = \bigoplus_{a \in X_{1^\nu}} \langle a \rangle$ for every 1^ν-transversal X_{1^ν} of V.
Let $V_{1^\nu,p}$ be the subspace of V generated by all elements of type 1^ν or p.

Lemma 6.1 *Let v be an element of $V_{1^\nu,p}$. Then there is a finite part $X \cup Y \cup \{u_e : e \in Y\}$ of a transversal such that*

i) *$X \cup Y$ is a part of an 1^ν-transversal and $X \cap Y = \emptyset$.*

ii) *$\{u_e : e \in Y\}$ is a part of a p-transversal and e is the handle of u_e.*

iii) *If $d \in X$ and $e \in Y$, then $\beta(d, e) \neq 0$.*

iv) *$v = \sum_{d \in X} d + \sum_{e \in Y} u_e$.*

Furthermore let $C(Y)$ be a set of representatives of the connections between the elements of Y. Then $U(v) = \langle X \cup Y \cup C(Y) \cup \{u_e : e \in Y\} \rangle$ is definable over v and it is the smallest good subspace that contains v.

Proof. Let Q be any transversal of V. Then v is in

$$V_{1^\nu,p} = \bigoplus_{a \in Q_{1^\nu}} \langle a \rangle \oplus \bigoplus_{e \in Q_{1^\nu}} \langle Q_{p,e} \rangle.$$

If we replace some elements of Q_{1^ν} by $\approx$-equivalent elements in a suitable manner, then there are finite subsets X^* and Y of Q_{1^ν} such that

$$v = \sum_{a \in X^*} a + \sum_{e \in Y} w_e$$

where $w_e \neq 0$, and $w_e \in \langle Q_{p,e} \rangle$. Let X be $\{a \in X^* : \beta(a, e) \neq 0 \text{ for all } e \in Y\}$. Let f be an automorphism of $\langle V, W, \beta \rangle$ that fixes v. Then f permutes $Q_{1^\nu}^{\approx}$ and

$$f(Q_{p,e}) \subseteq \bigoplus_{\substack{a \in Q_{1^\nu} \\ \beta(a, f(e)) = 0}} \langle a \rangle \oplus \langle Q_{p,f(e)} \rangle$$

is a p-transversal for the handle $f(e)$. The inclusion follows since $\beta(b, f(e)) = 0$ for every $b \in f(Q_{p,e})$ and the right side is the subspace of all d with $\beta(d, f(e)) = 0$. Since $v = f(v)$ has a unique representation as a linear combination over Q, we get $f(X^{\approx}) = X^{\approx}$ and $f(Y^{\approx}) = Y^{\approx}$ and X and Y are contained in every good subspace of V that contains v. It follows $f(\langle C(Y) \rangle) = \langle C(Y) \rangle$ and this subspace is also contained in every good subspace that contains v. Now we add every summand a for $a \in X^* \setminus X$ to w_a if $a \in \langle Y \rangle$ or to some w_e with $\beta(e, a) = 0$ otherwise. The result we call u_e. W.l.o.g. we assume that $u_e = w_e$ and $X = X^*$.
Now we have

$$v = \sum_{d \in X} d + \sum_{e \in Y} u_e$$

where $X \cap Y = \emptyset$, $X \cup Y \cup \{u_e : e \in Y\}$ is part of a transversal Q, and $u_e \in Q_{p,e}$. Then $v = f(v)$ implies as above

$$\sum_{d \in X} d + \sum_{e \in Y} u_e = \sum_{d \in X} f(d) + \sum_{e \in Y} f(u_e).$$

We have the stronger conclusion that f permutes X and the set $\{u_e : e \in Y\}$ modulo $\langle C(Y) \rangle$. Hence

$$v \in \bigoplus_{a \in X \cup Y \cup CY} \langle a \rangle \oplus \bigoplus_{e \in Y} \langle u_e \rangle = U(v)$$

and for automorphisms f with $f(v) = v$ we have $f(U(v)) = U(v)$, as desired. □

Corollary 6.2 *In Lemma* 6.1 *the representation* iv) $v = \sum_{d \in X} d + \sum_{e \in Y} u_e$ *is unique modulo* $\langle C(Y) \rangle$ *by the properties* i) – iii).

Corollary 6.3 *For every $v \in V$ there exists a smallest good subspace $U(v)$ that contains v. It is finite and definable over v.*

Proof. If $v \in U_{1^\nu,p}$, then the result is proved in Lemma 6.1. If $v \notin V_{1^\nu,p}$, then v is an element of type 1^ι. Therefore $U(v) = \langle v \rangle$. □

Corollary 6.4 *If V^0 is a subspace of V, then there is a smallest good subspace $U(V^0)$ that contains V^0. If V^0 is finite, then $U(V^0)$ is finite and definable over V^0.*

Proof. By Lemma 6.1 for every $v \in V^0 \cap V_{1^\nu,p}$ there are $X(v) \cup Y(v) \cup CY(v) = X^*_{1^\nu}(v)$ and $\{u(v)_e : e \in Y(v)\} = X^*_p(v)$ such that $v \in \langle X^*_{1^\nu}(v) X^*_p(v) \rangle$ and $\langle X^*_{1^\nu}(v) X^*_p(v) \rangle$ is finite, good, definable over v, and contained in each good subspace that contains v. There is a part $X^1_{1^\nu}$ of a 1^ν-transversal such that $\langle X^1_{1^\nu} \rangle = \langle \bigcup_{v \in V^0 \cap V_{1^\nu,p}} X^*_{1^\nu}(v) \rangle$. Let $X_{1^\iota} \subseteq V^0$ be a basis of V^0 modulo $V_{1^\nu,p}$. For every $e \in X^1_{1^\nu}$ we choose a maximal subset $X_{p,e}$ of $\{u(v)_e : v \in V^0 \cap V_{1^\nu,p}\}$ linearly independent modulo V_{1^ν}. Then it is possible to represent every $u(v)_e$ as a linear combination $u^*(v)_e$ over $X_{p,e}$ modulo V_{1^ν}. We obtain X_{1^ν}, if we add all necessary elements of V_{1^ν} to $X^1_{1^\nu}$ to represent $u(v)_e - u^*(v)_e$ for all $v \in V^0 \cap V_{1^\nu,p}$. Now $X_{1^\nu} \cup \bigcup_{e \in X_{1^\nu}} X_{p,e} \cup X_{1^\iota} = X$ is a finite part of a transversal, $V^0 \subseteq \langle X \rangle$, and X is contained in every good subspace U with $V^0 \subseteq U$. □

If we analyse the proof of Corollary 6.4, then we obtain:

Lemma 6.5 *For a subspace $V^0 \subseteq V$*

$$U(V^0) = \left\langle \bigcup_{v \in V^0} U(v) \cup R(V^0) \right\rangle$$

where $R(V^0)$ is a set of non-$\approx$-equivalent elements of type 1^ν that occur in the representation of elements of $\left\langle \bigcup_{v \in V^0} U(v) \right\rangle \cap V_{1^\nu}$.

Now we easily show:

Lemma 6.6 *The intersection of two good subspaces is good.*

Proof. Let U_0 and U_1 be two good subspaces of V. Then $U_0 \cap U_1$ contains all $U(v)$ for $v \in U_0 \cap U_1$ and the set $R(U_0 \cap U_1)$ described in Lemma 6.5. Hence $U(U_0 \cap U_1) = U_0 \cap U_1$. $\square$

Let X be a transversal for V. Assume that X is ordered by an additional relation $<$. By Corollary 4.7

$$\{\beta(a,b) : a, b \in X,\ a < b,\ \beta(a,b) \neq 0\}$$

is a basis for the vector space $\langle \beta(V,V) \rangle$. That means that elements of $\beta(V,V)$ have a unique representation as a linear combination over this basis. For $a \in \langle \beta(V,V) \rangle$ (or $a \in G'$) define $U(a)$ as the intersection of all good subspaces V_0 of V with $a \in \langle \beta(V_0, V_0) \rangle$. Since by Corollary 6.4 every finite subspace is contained in a finite good subspace, $U(a)$ is finite; it is good by Lemma 6.6. We want to show that $a \in \beta(U(a), U(a))$ and that $U(a) \subseteq \operatorname{acl}(a)$. For the last assertion it is sufficient to show that $U(a)$ is definable over a.
As for groups we use the following notation:

Definition. Let U be a good subspace of $V = V_{\mathfrak{C}}$ and let X be a part of a transversal. X respects U if X can be extended to a transversal Y of $V_{\mathfrak{C}}$ such that $Y \cap U$ is a transversal for U. A good subspace respects U if it is generated by a part of a transversal that respects U.

Let X be a part of a transversal and let U be a good subspace of V. Then X_{1^ν} represents all elements of type 1^ν up to $\approx$-equivalence and it is linearly independent. If $u \in U$ is a linear combination $u = \sum r_i a_i$ with $a_i \in X_{1^\nu}$, then $a_i \in U$ since U is good (Lemma 6.1). Hence we have $X_{1^\nu} = X^U_{1^\nu} \cup X^+_{1^\nu}$ where $X^U_{1^\nu} \subseteq U$ and $X^+_{1^\nu}$ is linearly independent over U.
Note that X respects U if and only if

$$\begin{aligned} X_{p,e} &= X^U_{p,e} \cup X^+_{p,e} \quad \text{for every } e \in X^U_{1^\nu} \text{ and} \\ X_{1^\iota} &= X^U_{1^\iota} \cup X^+_{1^\iota} \end{aligned}$$

where $X^U_{p,e} \subseteq U$, $X^U_{1^\iota} \subseteq U$,

$$\begin{aligned} &X^+_{p,e} \text{ is linearly independent modulo } \langle V_{1^\nu} \cup U \rangle \text{ and} \\ &X^+_{1^\iota} \text{ is linearly independent modulo } \langle V_{1^\nu,p} \cup U \rangle. \end{aligned}$$

Lemma 6.7 *If $U_1 \subseteq U_2 \subseteq V$ are good subspaces, then every part of a transversal X that generates U_1 can be extended to a part of a transversal XY that generates U_2.*

Proof. Since U_2 is good there is some part Y^0 of a transversal that generates U_2. W.l.o.g. $Y^0_{1^\nu} = X_{1^\nu} \cup Y_{1^\nu}$ for some set Y_{1^ν} of elements of type 1^ν. For every $e \in X_{1^\nu} \cup Y_{1^\nu}$ we choose a maximal subset $Y_{p,e}$ of $Y^0_{p,e}$ that is linearly independent modulo $\langle V_{1^\nu} \cup U_1 \rangle$. If $y \in Y^0_{p,e} \setminus Y_{p,e}$, then

$$y \in \langle V_{1^\nu} \cup U_1 \cup Y_{p,e} \rangle.$$

Then there is a linear combination y' over $X_{p,e} \cup Y_{p,e}$ such that

$$y = y' \text{ modulo } V_{1^\nu} \cap U_2.$$

Hence $y \in \langle X_{1^\nu} Y_{1^\nu} X_{p,e} Y_{p,e} \rangle$. We have shown that

$$X_{1^\nu} \cup Y_{1^\nu} \cup \bigcup_{e \in X_{1^\nu} Y_{1^\nu}} (X_{p,e} \cup Y_{p,e})$$

is a part of a transversal that generates $U_2 \cap V_{1^\nu,p}$. Then we find a maximal $Y_{1^\iota} \subseteq Y^0_{1^\iota}$ that is linearly independent modulo $\langle V_{1^\nu,p} \cup U_1 \rangle$. Similarly as above $Y^0_{1^\iota} \subseteq \langle (U_2 \cap V_{1^\nu,p}) \cup X_{1^\iota} \cup Y_{1^\iota} \rangle$. □

Lemma 6.8 *If U_0 and U_1 are good subspaces of V with $a \in \beta(U_0, U_0)$ and $a \in \beta(U_1, U_1)$, then $a \in \beta(U_0 \cap U_1, U_0 \cap U_1)$.*

Proof. $U_0 \cap U_1$ is good by Lemma 6.6. Let X be a generating part of a transversal for $U_0 \cap U_1$. By Lemma 6.7 there are Y^0 and Y^1 such that XY^0 and XY^1 are parts of transversals that generate U_0 and U_1 respectively. W.l.o.g. we can assume, that there is no part XY^* of a transversal in U_1 such that $|Y^*_{1^\iota}| < |Y^1_{1^\iota}|$ and $a \in \beta(\langle XY^* \rangle, \langle XY^* \rangle)$. First we show $Y^1_{1^\iota} \subseteq U_0$ modulo $V_{1^\nu,p}$. Otherwise we can choose $Y^1_{1^\iota}$ in such a way that $Y^1_{1^\iota} = Y^2_{1^\iota} \cup Y^3_{1^\iota}$ where $Y^2_{1^\iota}$ is linearly independent modulo $\langle U_0 \cup V_{1^\nu,p} \rangle$ and $Y^3_{1^\iota} \subseteq \langle U_0 \cup V_{1^\nu,p} \rangle$. Let $\overline{X}$ be a transversal that contains $XY^0Y^2_{1^\iota}$. By assumption (minimality of $|Y^1_{1^\iota}|$) the representation of a over XY^1 contains each $y \in Y^2_{1^\iota}$ in some commutator. But then we have the same for the unique representation of a over $\overline{X}$. This contradicts $a \in \beta(U_0, U_0)$. Hence $Y^2_{1^\iota} = \emptyset$ and $Y^1_{1^\iota} \subseteq U_0$ modulo $V_{1^\nu,p}$. For each $y^1 \in Y^1_{1^\iota}$ there is some $y^0 \in U_0$ such that $y^1 - y^0 \in V_{1^\nu,p}$ and hence

$$y^1 - y^0 \in V_{1^\nu,p} \cap \langle U_0 \cup U_1 \rangle.$$

If we add suitable $u_0 \in V_{1^\nu,p} \cap U_0$ and $u_1 \in V_{1^\nu,p} \cap U_1$ to y^0 and y^1 respectively, then we have w.l.o.g. $Y^1_{1^\iota} \subseteq U_0$ and therefore $Y^1_{1^\iota} \subseteq U_0 \cap U_1$. Hence $Y^1_{1^\iota} = \emptyset$. Similarly we can assume w.l.o.g. that $Y^0_{1^\iota} = \emptyset$.
For all transversals $\overline{X} \supseteq X$ the set $I^{\approx}$ of handles $e^{\approx}$ such that $\overline{X}_{p,e}$ is used in representation of a over $\overline{X}$ as a linear combination of commutators is the same. Hence we can assume that $I \subseteq X$ and $C(I) \subseteq X$. Now $Y^1 = \bigcup_{e \in I} Y^1_{p,e} \cup Y^1_{1^\nu}$. We fix some $e \in I$ and assume that we have chosen a part XY^* of a transversal in U_1 such that $|Y^*_{p,e}|$ is minimal and $a \in \beta(\langle XY^* \rangle, \langle XY^* \rangle)$. W.l.o.g. $Y^1 = Y^*$. As above for $Y^1_{1^\iota}$ and $Y^0_{1^\iota}$ we have $Y^1_{p,e} = Y^0_{p,e} = \emptyset$. Then $Y^0 = Y^0_{1^\nu}$, $Y^1 = Y^1_{1^\nu}$, and hence XY^0Y^1 is a part

of a transversal. Then all elements of any transversal $\overline{X} \supseteq XY^0Y^1$ that are used in commutators of the representation of a over $\overline{X}$ are in X. Since I and CI are also in X we get $Y^0 = Y^1 = \emptyset$, as desired. $\square$

We have shown that $a \in \beta(U(a), U(a))$. Finally we prove:

Theorem 6.9 *For every $a \in \beta(V, V)$ ($a \in \mathbb{C}'$ respectively) there is a smallest good subspace $U(a) \subseteq V$ such that $a \in \beta(U(a), U(a))$. $U(a)$ is finite and definable over a.*

Proof. It is clear that $a \in \beta(V^0, V^0)$ for some finite good subspace. By Lemma 6.6 we can define $U(a)$ as the intersection of all good subspaces U that contain a in $\beta(U, U)$. By Lemma 6.8 $a \in \beta(U(a), U(a))$. $U(a)$ is finite, hence definable, and $\{a\}$-automorphism invariant, hence definable over a. $\square$

Let $\mathrm{Aut}_{\{G\}(d/V_G)}(\mathbb{C})$ be the subgroup of all automorphisms of $\mathbb{C}$ that fix d/V_G and fix G setwise.

Lemma 6.10 *Let $G \preceq \mathbb{C} \models T_S$. Let $d = \sum_{1 \leq i \leq n} c_i + \sum_{1 \leq i \leq m} d_i + \sum_{1 \leq i \leq \ell} w_i + x$ be an element of $\langle V_G \cup V_{1^\nu, p} \rangle$ where $c_1, \ldots, c_n, e_1, \ldots, e_m$ are elements of type 1^ν not in V_G in different $\approx$-classes, each d_i is an proper element of type p with handle e_i, for every i either $\beta(c_i, e_j) \neq 0$ for all j or c_i is a connection of two e_j, any $w_i \notin \langle V_G \cup V_{1^\nu} \rangle$ and it is a proper element of type p with handle $f_i \in V_G$, different f_i are in different $\approx$-classes, $\beta(c_i, f_j) \neq 0$ for all i, j, and $x \in V_G$. Then there is a subgroup $\mathbb{K}$ of $\mathrm{Aut}_{\{G\}(d/V_G)}(\mathbb{C})$ of finite index such that every $f \in \mathbb{K}$ fixes $\{c_1, \ldots, c_n, e_1, \ldots, e_m, d_1, \ldots, d_m, f_1, \ldots, f_\ell\}$ pointwise and $f(w_i) = w_i$ modulo V_G.*

Proof. Let C be the set of connections of the handles e_i and f_j. By Lemma 6.1

$$\{c_1, \ldots, c_n, e_1, \ldots, e_m, d_1, \ldots, d_m, f_1, \ldots, f_\ell, w_1, \ldots, w_\ell\} \cup C$$

generates $U(d - x)$ the smallest good subspace that contains $d - x$.
Hence $\langle V_G \cup U(d - x) \rangle$ is the smallest good subspace that contains V_G and d. It is fixed setwise by every $f \in \mathrm{Aut}_{\{G\}(d/V_G)}(\mathbb{C})$. Such an automorphism f determines an automorphism of the finite space $\langle V_G \cup U(d-x) \rangle / V_G$. Hence there is a subgroup $\mathbb{K}$ of $\mathrm{Aut}_{\{G\}(d/V_G)}(\mathbb{C})$ of finite index such that $f \in \mathbb{K}$ fixes $\{c_1, \ldots, c_n, e_1, \ldots, e_m, d_1, \ldots, d_m, w_1, \ldots, w_\ell\}$ pointwise modulo V_G and therefore $f_1, \ldots, f_\ell$ pointwise. By Lemma 4.15 $|\{u \in V_G : \beta(u, e_i) = 0\}| \leq p$. Hence we can decrease $\mathbb{K}$ in such a way that it is still of finite index but every $f \in \mathbb{K}$ fixes $\{c_1, \ldots, c_n, e_1, \ldots, e_m, d_1, \ldots, d_m\}$ pointwise. To show this first assume $c = c_i$ or $c = e_j$. Then $f(c) = c + v$ where $v \in V_G$ and $f \in \mathrm{Aut}_{\{G\}(d/V_G)}(\mathbb{C})$ implies $f(c) = c$ since $c + v$ with $v \neq 0$ is not of type 1^ν. Now consider $f(d_i) = d_i + v$ where $v \in V_G$ and $f \in \mathrm{Aut}_{\{G\}(d/V_G)}(\mathbb{C})$. Then $\beta(v, e_i) = 0$. Since $e_i \notin V_G$ there is no proper element of type p with handle e_i in V_G. By Lemma 4.15 $\beta(v, e_i) = 0$ has at most one nontrivial solution up to $\approx$-equivalence. $\square$

7 Good subspaces that respect V_G

Let $\mathbb{C}$ be a monster model of T_S and $G \preceq \mathbb{C}$. Let D be a finite good subspace of $V = V_{\mathbb{C}}$. Our aim is to have a good subspace D^* that contains D, is definable over D, and respects V_G. This is impossible in general. We will develop a substitute. First we give some notations and definitions.

Let M be a model of any elementary theory T and D be a subset of M. Then $\mathrm{Aut}_{(D)}(M) = \{f \in \mathrm{Aut}(M) : f(d) = d \text{ for } d \in D\}$ is the pointwise stabilizer of D and $\mathrm{Aut}_{\{D\}}(M) = \{f \in \mathrm{Aut}(M) : f(D) = D\}$ is the setwise stabilizer of D.

Definition Two subgroups of some group are commensurable, if their intersection has finite index in both of them.

Fact 7.1 *Let $\mathbb{C}$ be a monster model of T, $M \preceq \mathbb{C}$ saturated, $A \subseteq \mathbb{C}$, and $E \subseteq M^{\mathrm{eq}}$. If $\mathrm{Aut}_{(E)}(M)$ and $\mathbb{J} = \{f \in \mathrm{Aut}(M) : \mathrm{tp}(A/M) = f(\mathrm{tp}(A/M))\}$ are commensurable, then $\mathrm{acl}(E) = \mathrm{acl}(\mathrm{Cb}(\mathrm{tp}(A/M)))$.*

Note that

$$\mathbb{J} = \{f \upharpoonright M : f \in \mathrm{Aut}_{\{M\}(A)}(\mathbb{C})\}.$$

Furthermore:

Fact 7.2 *If $A \subseteq B \subseteq \mathrm{acl}(A)$, then $\mathrm{Cb}(\mathrm{tp}(A/M)) \subseteq \mathrm{Cb}(\mathrm{tp}(B/M)) \subseteq \mathrm{acl}(\mathrm{Cb}(\mathrm{tp}(A/M)))$. If $A \subseteq B \subseteq \mathrm{dcl}(A)$, then $\mathrm{Cb}(\mathrm{tp}(A/M)) = \mathrm{Cb}(\mathrm{tp}(B/M))$.*

Fact 7.3 *If in the situation of Fact 7.1 $A = (A \cap M) \cup B$, then $\mathrm{Cb}(\mathrm{tp}(A/M)) = \mathrm{dcl}((A \cap M) \cup \mathrm{Cb}(\mathrm{tp}(B/M)))$.*

Now we come back to T_S and $G \preceq \mathbb{C} \models T_S$. As before we often use $\mathcal{F}(G) = \langle V_G, W_G, \beta_G \rangle \preceq \mathcal{F}(\mathbb{C}) = \langle V, W, \beta \rangle$ in $\mathbb{C}^{\mathrm{eq}}$.

Definition $\Box = (X^\Box, Y^\Box, D^\Box)$ is a special triple with respect to V_G of subsets of V, if

i) $D^\Box$ is linearly independent modulo V_G.

ii) $X^\Box Y^\Box$ is a part of a transversal that respects V_G, $X^\Box \subseteq V_G$, $Y^\Box$ is linearly independent modulo V_G,

$$X^\Box = X^\Box_{1^\nu} \cup X^\Box_{1^\iota}, \quad Y^\Box = Y^\Box_p = \bigcup_{e \in X^\Box_{1^\nu}} Y^\Box_{p,e},$$

and all $Y^\Box_{p,e}$ for $e \in X^\Box_{1^\nu}$ are non-empty.

iii) There are a bijection $d \to x_d$ of $D^\Box$ onto $X^\Box_{1\iota}$ and an injection $d \to y_d$ of $D^\Box$ into $\langle Y^\Box \rangle$ such that

$$d = x_d + y_d \quad \text{for every} \quad d \in D^\Box.$$

iv) $\langle V_G Y^\Box \rangle$ is the smallest good subspace that contains V_G and $D^\Box$.

If X is a subset of V, then we use $\langle X \rangle$ to denote the subspace of V that is generated by X as above. $\langle X \rangle_{\mathcal{F}}$ denotes the restriction of the considered alternating bilinear map to the subspace $\langle X \rangle$. Given a special triple $\Box = (X^\Box, Y^\Box, D^\Box)$ as above we introduce an equivalence relation $\theta_\Box(X^0, X^1)$:

Definition Let $\overline{x}^0$ and $\overline{x}^1$ be subsequences of V of the length of $D^\Box$, namely $\overline{x}^i = (x^i_d : d \in D^\Box)$. Define $\theta_\Box(\overline{x}^0, \overline{x}^1)$ to hold if there is a vector space homomorphism f of $\langle Y^\Box \rangle$ into V that can be extended to an $\mathbb{B}_p$-homomorphism $\overline{f}$ of $\langle X^\Box_{1\nu} Y^\Box \rangle_{\mathcal{F}}$ into $\mathcal{F}(\mathbb{C})$ with $\overline{f}(x) = x$ for $x \in X^\Box_{1\nu}$ and

$$x^0_d = x^1_d + f(y_d).$$

Note:

- In the definition above we have $\beta(e, f(y)) = 0$ for $e \in X^\Box_{1\nu}$ and $y \in Y^\Box_{p,e}$.
- We obtain $\langle X^\Box_{1\nu} Y^\Box \rangle_{\mathcal{F}}$ from the free alternating bilinear map over the vector space $\langle X^\Box_{1\nu} Y^\Box \rangle$, if we factorize it by all $\beta(a_0, a_1) = 0$ for $a_0, a_1 \in X^\Box_{1\nu}$ that come from $\langle X^\Box_{1\nu} \rangle_{\mathcal{F}}$ and by $\beta(y, e) = 0$ for $e \in X^\Box_{1\nu}$ and $y \in Y^\Box_{p,e}$.
  Hence a vector space homomorphism f of $\langle Y^\Box \rangle$ in V can be extended to an $\mathbb{B}_p$-homomorphism $\overline{f}$ of $\langle X^\Box_{1\nu} Y^\Box \rangle_{\mathcal{F}}$ into $\mathcal{F}(\mathbb{C})$ with $\overline{f}(x) = x$ for $x \in X^\Box_{1\nu}$ if and only if $\beta(e, f(y)) = 0$ for all $y \in Y^\Box_{p,e}$.
- $\theta_\Box(\overline{x}^0, \overline{x}^1)$ defines an equivalence relation.
- $\theta_\Box$ is definable over $X^\Box_{1\nu}$.
- In a situation where $X^\Box_{1\nu}$ is fixed we can consider the classes of $\theta_\Box$ as elements of $\mathbb{C}^{\mathrm{eq}}$.
- We use $\underline{\Box}(X^\Box_{1\iota})$ to denote the $\theta_\Box$-class of $X^\Box_{1\iota}$. Often we only write $\underline{\Box}$.

Theorem 7.4 *Let D be a finite good subspace of V. Then there are a finite good subspace $\langle XX^\Box YY^\Box \rangle \supseteq D$, where $XX^\Box YY^\Box$ is a part of a transversal that respects G with $\langle XX^\Box YY^\Box \rangle \cap V_G = \langle XX^\Box \rangle$, and a subset $D^\Box$ of $\langle X^\Box Y^\Box \rangle$ such that*

i) *$\langle V_G Y Y^\Box \rangle$ is the smallest good subspace that contains V_G and D. $D \subseteq \langle XYD^\Box \rangle$.*

ii) *$\Box = (X^\Box, Y^\Box, D^\Box)$ is a special triple and*

$$X^\Box \cap X = X^\Box_{1\nu}, \quad Y^\Box \cap Y = \emptyset.$$

iii) $\mathrm{Aut}_{\{G\}(D)}(\mathfrak{C})$ *and* $\mathrm{Aut}_{\{G\}(XYD^\square)}(\mathfrak{C})$ *are commensurable.*

iv) *If* H *is a saturated model of* T_S *with* $G \preceq H \preceq \mathfrak{C}$ *and* $\mathrm{acl}(D \cup G) \cap H = G$*, then* $\mathrm{Aut}_{\{H\}(D)}(\mathfrak{C})$ *and* $\mathrm{Aut}_{\{H\}(XYD^\square)}(\mathfrak{C})$ *are commensurable.*

We call $(X, X^\square, Y, Y^\square, D^\square)$ with the properties in Theorem 7.4 a platform for D.

Proof. By Lemma 6.6 $D \cap V_G$ is good. By Lemma 6.7 there is a part of a transversal I for D such that $I \cap V_G$ is a part of a transversal for $D \cap V_G$. It is possible that I does not respect V_G. Starting from I we construct X, Y, $X^\square$, $Y^\square$ and $D^\square$. In each step we say which constructed elements belong to which set.
Assume $I = I^0 \cup I^1$ where $I^0 = I \cap V_G$. Then I^0 will be part of X and $I^1_{1^\nu}$ will be part of Y_{1^ν}. Note that $e \in I^1_{1^\nu}$ implies that $I_{p,e} \subseteq I^1$. Then $I_{p,e}$ is linearly independent modulo $\langle V_G \cup V_{1^\nu}\rangle$. $I_{p,e}$ will be part of Y. Now we consider $I^1_{p,e}$ for $e \in I^0_{1^\nu}$. W.l.o.g. $I^1_{p,e} = I^2_{p,e} \cup I^3_{p,e}$ where $I^2_{p,e} \subseteq \langle V_G \cup V_{1^\nu}\rangle$ and $I^3_{p,e}$ is linearly independent over $\langle V_G \cup V_{1^\nu}\rangle$. $I^3_{p,e}$ will be $Y_{p,e}$. Up to now all constructed elements of X and Y are elements of D. Therefore they are fixed by all automorphisms that fix D pointwise. If $w \in I^2_{p,e}$, then

$$w = \sum_{1 \le i \le n} c_i + x \tag{7.1}$$

where $c_1, \ldots, c_n$ are elements of type 1^ν not in V_G in different $\approx$-classes with $\beta(e, c_i) = 0$ and $x \in V_G$ is a proper element of type p with handle e in V_G. The $\approx$-classes of the c_i and e are uniquely determined by w (Lemma 6.10).
Let $X^2_{p,e} \subseteq V_G$ be the set of the x's that correspond to the w's $\in I^2_{p,e}$ in (7.1). Then $X_{p,e} = I^0_{p,e} \cup X^2_{p,e}$ is linearly independent modulo V_{1^ν}. It will be the p-transversal for the handle e in $X \subseteq V_G$. Furthermore all c_i will be elements of $\langle Y_{1^\nu}\rangle$. In the situation (7.1) we have by Lemma 6.10

$$\mathrm{Aut}_{\{G\}(w)}(\mathfrak{C}) \quad \text{and} \quad \mathrm{Aut}_{\{G\}(\{x, c_1, \ldots, c_n\})}(\mathfrak{C}) \quad \text{are commensurable.} \tag{7.2}$$

By Lemma 6.1 $\{c_1, \ldots, c_n\} \subseteq \mathrm{acl}(D \cup V_G)$. If H is as in iv), then $\mathrm{acl}(D \cup V_G) \cap H \subseteq G$ implies that $\{c_1, \ldots, c_n\}$ is linearly independent modulo V_H. We obtain by Lemma 6.10

$$\mathrm{Aut}_{\{H\}(w)}(\mathfrak{C}) \quad \text{and} \quad \mathrm{Aut}_{\{H\}(\{x, c_1, \ldots, c_n\})}(\mathfrak{C}) \quad \text{are commensurable.} \tag{7.2^H}$$

Finally we consider $I^1_{1^\iota}$. Again we split w.l.o.g. $I^1_{1^\iota} = I^2_{1^\iota} \cup I^3_{1^\iota} \cup I^4_{1^\iota}$ where $I^2_{1^\iota} \subseteq \left\langle V_G \cup V_{1^\nu} \cup \bigcup_{e \notin V_G} Y^0_{p,e}\right\rangle$, $I^3_{1^\iota} \subseteq \langle V_G \cup V_{1^\nu,p}\rangle$ but it is linearly independent modulo $\left\langle V_G \cup V_{1^\nu} \cup \bigcup_{e \notin V_G} Y^0_{p,e}\right\rangle$, and $I^4_{1^\iota}$ is linearly independent modulo $\langle V_G \cup V_{1^\nu,p}\rangle$. Again w.l.o.g. we can assume that $I^4_{1^\iota} \subseteq Y_{1^\iota}$. Similarly as in (7.1) we have for $w \in I^2_{1^\iota}$

$$w = \sum_{1 \le i \le n} c_i + \sum_{1 \le i \le m} d_i + x \tag{7.3}$$

where the c_i's are elements of type 1^ν not in V_G in different $\approx$-classes, the d_i's are proper elements of type p not in V_G with different handles e_i not in V_G, $\beta(c_i, e_j) \neq 0$ for all i and j, and $x \in V_G$ is an element of type 1^ι. By Lemma 6.10 we have for (7.3):

$$(7.4) \qquad \mathrm{Aut}_{\{G\}(w)}(\mathbb{C}) \text{ and } \mathrm{Aut}_{\{G\}(c_1,\ldots,c_n,e_1,\ldots,e_m,d_1,\ldots,d_m,x)}(\mathbb{C}) \text{ are commensurable.}$$

Let $X^2_{1^\iota}$ be the set of the x''s in (7.3) for the w's in $I^2_{1^\iota}$. Then $X_{1^\iota} = I^0_{1^\iota} \cup X^2_{1^\iota}$ is linearly independent modulo $V_{1^\nu,p}$. X_{1^ι} is the 1^ι-part of X.
We will put $c_1, \ldots, c_n, e_1, \ldots, e_m$, and $d_1, \ldots, d_m$ into $\langle Y \rangle$. Note (7.4) implies $\{c_1, \ldots, c_n, e_1, \ldots, e_m, d_1, \ldots, d_m\} \subseteq \mathrm{acl}(V_G \cup D)$. By $\mathrm{acl}(V_G \cup D) \cap H \subseteq G$ we have that this set is linearly independent modulo V_H. Since the handle e_i of d_i is not in V_H we have $d_i \notin \langle V_H \cup V_{1^\nu} \rangle$. Otherwise we would get a proper element of type p in V_H with handle $e_i \notin V_H$. Again we can apply Lemma 6.10:

$$(7.4^H) \qquad \mathrm{Aut}_{\{H\}(w)}(\mathbb{C}) \text{ and } \mathrm{Aut}_{\{H\}(c_1,\ldots,c_n,e_1,\ldots,e_m,d_1,\ldots,d_m,x)}(\mathbb{C}) \text{ are commensurable.}$$

The set of elements we have collected for X up to now we call X^0. We will have $X^0_p = X_p$ and $X^0_{1_\iota} = X_{1^\iota}$. Using all elements we have choosen for $\langle Y \rangle$ we form Y^0 such that X^0Y^0 is a part of a transversal that respects V_G with $V_G \cap \langle X^0Y^0 \rangle = \langle X^0 \rangle$. The elements of X^0Y^0 are either elements of D or they occur in (7.1) or (7.3). By (7.2) and (7.4) we get therefore a subgroup $\mathbb{K}^0$ of $\mathrm{Aut}_{\{G\}(D)}(\mathbb{C})$ of finite index such that each $f \in \mathbb{K}^0$ fixes X^0Y^0 pointwise. Analogously by (7.2^H) and (7.4^H) we have $\mathbb{K}^{0H}$ in $\mathrm{Aut}_{\{H\}(D)}(\mathbb{C})$ of finite index such that $f \in \mathbb{K}^{0H}$ fixes X^0Y^0 pointwise.
Finally we consider $w \in I^3_{1^\iota}$. We have

$$(7.5) \qquad w = \sum_{1 \leq i \leq n} c_i + \sum_{1 \leq i \leq m} d_i + \sum_{1 \leq i \leq \ell} w_i + x$$

where $c_1, \ldots, c_n, e_1, \ldots, e_m$ are elements of type 1^ν not in V_G in different $\approx$-classes, each d_i is an proper element of type p with handle e_i, $\beta(c_i, e_j) \neq 0$ for all i, j, the element $w_i \notin \langle V_G \cup V_{1^\nu} \rangle$ is proper of type p with handle $f_i \in V_G$, different f_i are in different $\approx$-classes, $\beta(c_i, f_j) \neq 0$ for i, j, and $x \in V_G$.
Then x is a proper element of type 1^ι. By Lemma 6.10 there is a subgroup $\mathbb{K}^w$ of $\mathrm{Aut}_{\{G\}(w)}(\mathbb{C})$ of finite index such that every $f \in \mathbb{K}^w$ fixes $\{c_1, \ldots, c_n, e_1, \ldots, e_m, d_1, \ldots, d_m, f_1, \ldots, f_\ell\}$ pointwise and $f(w_i) = w_i$ modulo V_G.
Let d be $x + \sum_{1 \leq i \leq \ell} w_i$. We have $f(d) = d$ for $f \in \mathbb{K}^w$. So

$$(7.6) \qquad \mathrm{Aut}_{\{G\}(w)}(\mathbb{C}) \text{ and } \mathrm{Aut}_{\{G\}(d,c_1,\ldots,c_n,d_1,\ldots,d_m,e_1,\ldots,e_m,f_1,\ldots,f_\ell)}(\mathbb{C}) \text{ are commensurable.}$$

By Lemma 6.1 $\{c_1, \ldots, c_n, e_1, \ldots, e_m, d_1, \ldots, d_m, w_1, \ldots, w_\ell\}$ are in $\mathrm{acl}(V_G \cup D)$. As $\mathrm{acl}(DG) \cap H = G$ we know that these elements are linearly independent modulo V_H. We have almost the situation (7.5) for w with respect to V_H. The only difference is that it is possible that some w_i are in $\langle V_H V_{1^\nu} \rangle$. In this case $w_i = w_i^* + x_i^*$ where $x_i^* \in V_H \cap V_{p,f_i}$ and w_i^* is a sum of elements v of type 1^ν that are not in V_H and satisfy

$\beta(v, f_i) = 0$. By Lemma 4.15 we have $\beta(v, f_j) \neq 0$ for $i \neq j$. If $\beta(v, e_j) = 0$, then v is a connection between f_i and e_j. Let $w_i^* = w_i$ if $w_i \notin \langle V_H \cup V_{1^\nu}\rangle$. Otherwise we work with w_i^* and add x_i^* to x. Let x^* be sum of x and these x_i^*. Then we have

$$(7.5^H) \qquad w = \sum_{1\le i\le n} c_i + \sum_{1\le i\le m} d_i + \sum_{1\le i\le \ell} w_i^* + x^*,$$

where $c_1, \dots, c_n, e_1, \dots, e_m, d_1, \dots, d_m, f_1, \dots, f_\ell$ behave with respect to V_H as in (7.5) with respect to V_G. Furthermore $x^* \in V_H$; if $w_i = w_i^*$ then w_i^* is a proper element of type p with handle f_i and $w_i^* \notin V_H$, and if $w_i \neq w_i^*$ then w_i^* is a sum of non $\approx$-equivalent elements v of type 1^ν not in V_H with $\beta(v, f_i) = 0$, $\beta(v, f_j) \neq 0$ for $i \neq j$, and $\beta(v, e_j) \neq 0$ for all j or v is a connection of two handles e_j, f_i. In the last case f_i is uniquely determined by v (Lemma 4.15). In this case we define $w_i^{**} = w_i^* - v$ and $d_j^{**} = d_j + v$. It can hapen that we add several v to d_j. We call the final result again d_j^{**}. We can write (7.5^H) as

$$w = \sum_{1\le i\le n} c_i + \sum_{1\le i\le m} d_i^{**} + \sum_{1\le i\le \ell} w_i^{**} + x^*.$$

Now we apply Lemma 6.10 and obtain $I\!K^{wH} \subseteq \mathrm{Aut}_{\{H\}(w)}(\mathbb{C})$ of finite index such that every $f \in I\!K^{wH}$ fixes all c_i, all e_i, all d_i^{**}, all v in w_i^{**} if $w_i^* \neq w_i$, and for $w_i = w_i^*$ the handle f_i and w_i modulo V_H. Note for $w_i \neq w_i^*$ that by Lemma 4.15 $f_i^\approx$ is the only $\approx$-class in V_H with $\beta(v, f_i) = 0$. Hence w.l.o.g. we can assume that $f(f_i) = f_i$ also in this case. Finally w.l.o.g. $f(v) = v$ for a connection v of f_i and some e_j. Hence $f(d_i) = d_i$ and $f(d) = d$ for $d = x + \sum\limits_{1\le i\le \ell} w_i = x^* + \sum :_{1\le i\le \ell} w_i^*$. We have shown:

(7.6^H) $\mathrm{Aut}_{\{H\}(w)}(\mathbb{C})$ and $\mathrm{Aut}_{\{H\}(d,c_1,\dots,c_n,e_1,\dots,e_m,d_1,\dots,d_m,f_1,\dots,f_\ell)}(\mathbb{C})$ are commensurable.

Let $I\!K$ be the intersection of $I\!K^0$ and all $I\!K^w$ for $w \in I^3_{1^\iota}$ and $I\!K^H$ the intersection of $I\!K^{0H}$ and all $I\!K^{wH}$.

Let $X^\square_{1^\nu} \subseteq V_G$ be a part of a 1^ν-transversal that consists of representatives of all $\approx$-classes of the f_i in (7.5) for $w \in I^3_{1^\iota}$.

Now we define X as $X^0 \cup X^\square_{1^\nu}$, where we assume that X and X^0 contain the same element a of type 1^ν, if they both represent the $\approx$-class of a. We enlarge Y^0 to Y^1 in such a way that all $c_1, \dots, c_n, e_1, \dots, e_m$, and $d_1, \dots, d_m$ from (7.5) for $w \in I^3_{1^\iota}$ generate Y^1 over Y^0 and XY^1 is a part of a transversal with $\langle XY^1\rangle \cap V_G = \langle X\rangle$. Y^1 is fixed pointwise by $I\!K$ and $I\!K^H$.

Let f_i be an element of $X^\square_{1^\nu}$. For each w in $I^3_{1^\iota}$ either there is no element of type p with handle f_i in (7.5) or there is some w_i that we call $w_i(w)$. Let $Y^\square_{p,f_i}$ be a subset of $\{w_i(w) : w \in I^3_{1^\iota}\}$ such that $Y_{p,f_i}Y^\square_{p,f_i}$ is a basis of $\langle\{w_i(w) : w \in I^3_{1^\iota}\}\rangle \cup Y_{p,f_i}$ modulo $\langle V_G \cup V_{1^\nu}\rangle$. Define $Y^\square = \bigcup\limits_{e\in X^\square_{1^\nu}} Y^\square_{p,e}$. The elements of $Y^\square$ are fixed modulo V_G by the automorphisms of $I\!K$ and fixed modulo V_H by the automorphisms of $I\!K^H$, since this was true for the w_i.

We consider again (7.5). By definition $d = x + \sum_{1 \le i \le \ell} w_i$. Hence $d = x + v + y + y_d$ where $y_d \in \langle Y^\square \rangle$, $y \in \langle Y^1 \rangle$, and $v \in V_{1^\nu}$ modulo V_G. We have $v = \sum_{1 \le i \le \ell} w_i - y - y_d$. Then y is fixed by every $f \in I\!\!K$ and $g \in I\!\!K^H$. We have $f(v) = v$ modulo V_G for $f \in I\!\!K$ and $g(v) = v$ modulo V_H for $g \in I\!\!K^H$ since this was true for the w_i and Y^1. Then the elements of type 1^ν in v, which are not in V_G, and therefore not in V_H, are w.l.o.g. fixed by the automorphisms of $I\!\!K$ and $I\!\!K^H$ (Lemma 6.10). We enlarge Y^1 to Y such that the $\approx$-classes of these elements are represented in Y_{1^ν}. Now we substract all summands of d that are from Y and call the result again d. We have $d = x_d + y_d$ with $x_d \in V_G$ is of type 1^ι and $y_d \in \langle Y^\square \rangle$. $y_d = 0$ is impossible since this would imply $w \in I^2_{1^\iota}$.
Let $D^\square$ be the set of these elements d we obtain from the w's in $I^3_{1^\iota}$ this way via (7.5). We define $X^\square_{1^\iota} = \{x_d : d \in D^\square\}$ and $X^\square = X^\square_{1^\nu} \cup X^\square_{1^\iota}$. Assume w.l.o.g. that X_{1^ν} and Y_{1^ν} contain all connections of the handles. By construction it is easily seen that $(X, X^\square, Y, Y^\square, D^\square)$ fulfils the conditions of Theorem 7.4:

$$D \subseteq \langle X X^\square Y Y^\square \rangle, \quad D \subseteq \langle X Y D^\square \rangle,$$

$XX^\square YY^\square$ is a part of a transversal that respects V_G, $\langle XX^\square YY^\square \rangle \cap V_G = \langle XX^\square \rangle$.
Every element y in $YY^\square$ is either an element of D or it was the result of an application of Lemma 6.10 in (7.1), (7.3) and (7.5). But if we consider $w - x$ in (7.1), (7.3), and (7.5) then we can apply Lemma 6.1 and obtain the element y as an element of $U(w-x)$. Hence $\langle V_G Y Y^\square \rangle$ is the smallest good subspace that contains V_G and D. This proves i).
ii) is clear by construction. We obtain that $\langle V_G Y^\square \rangle$ is the smallest good subspace that contains V_G and $D^\square$ by a similar argument as above for i).
Finally we show iii) and iv). Since $D \subseteq \langle XYD^\square \rangle$ we have that $\mathrm{Aut}_{\{G\}(XYD^\square)}(\mathbb{C})$ is a subgroup of $\mathrm{Aut}_{\{G\}(D)}(\mathbb{C})$ and $\mathrm{Aut}_{\{H\}(XYD^\square)}(\mathbb{C})$ is a subgroup of $\mathrm{Aut}_{\{H\}(D)}(\mathbb{C})$. Conversely $I\!\!K$ is a subgroup of finite index of $\mathrm{Aut}_{\{G\}(D)}(\mathbb{C})$. Analogously $I\!\!K^H$ is a subgroup of finite index in $\mathrm{Aut}_{\{H\}(D)}(\mathbb{C})$. In fact (7.2), (7.4), (7.6) and (7.2^H), (7.4^H), (7.6^H) imply $I\!\!K \subseteq \mathrm{Aut}_{\{G\}(XYD^\square)}(\mathbb{C})$ and $K^H \subseteq \mathrm{Aut}_{\{H\}(XYD^\square)}(\mathbb{C})$. □

Corollary 7.5 *Let D be a finite good subspace of V. Let $(X, X^\square, Y, Y^\square, D^\square)$ be a platform for D. Then the following is true:*

v) $YY^\square \subseteq \mathrm{acl}(XX^\square D)$.

vi) *There is a subgroup $I\!\!K \subseteq \mathrm{Aut}_{\{G\}(D)}(\mathbb{C})$ of finite index fixing $YY^\square$ modulo V_G pointwise.*

vii) $\mathrm{Aut}_{\{G\}(XY\{\square\})}(\mathbb{C}) \cap \mathrm{Aut}_{\{G\}(D)}(\mathbb{C})$ *has finite index in* $\mathrm{Aut}_{\{G\}(D)}(\mathbb{C})$.

viii) *For every $f \in \mathrm{Aut}_{\{G\}(XY\{\square\})}(\mathbb{C})$ there is some $g \in \mathrm{Aut}_{\{G\}(D)}(\mathbb{C})$ such that $f \upharpoonright G = g \upharpoonright G$.*

Proof. v) Since $D \subseteq \langle XX^{\square}YY^{\square}\rangle$ we have $d = x(d) + y(d)$ where $x(d) \in \langle XX^{\square}\rangle$ and $y(d) \in \langle YY^{\square}\rangle$. By Corollary 6.4 $U(\{y(d) : d \in D\}) \subseteq \mathrm{acl}(XX^{\square}D)$. If C is the set of handles and connections of $YY^{\square}$ in $XX^{\square}$, then $\langle CYY^{\square}\rangle$ is a good subspace that respects V_G and contains $U(\{y(d) : d \in D\})$. Since $\langle CYY^{\square}\rangle \cap V_G = \langle C\rangle$ and $U(\{y(d) : d \in D\}) \subseteq \langle CYY^{\square}\rangle$ we have $U(\{y(d) : d \in D\}) \cap V_G \subseteq \langle C\rangle$. If some $y \in YY^{\square}$ is not in $U(\{y(d) : d \in D\})$, then $\langle V_G \cup U(\{y(d) : d \in D\})\rangle$ is a proper good subspace of $\langle V_G YY^{\square}\rangle$ that contains D. This contradicts Theorem 7.4i).

vi) By Theorem 7.4i) we have for every $f \in \mathrm{Aut}_{\{G\}(D)}(\mathbb{C})$ that $f(\langle V_G YY^{\square}\rangle) = \langle V_G YY^{\square}\rangle$. Hence there is some subgroup $\mathbb{K}$ of $\mathrm{Aut}_{\{G\}(D)}(\mathbb{C})$ of finite index fixing $YY^{\square}$ modulo V_G pointwise.

vii) Theorem 7.4iii) implies this since $\mathrm{Aut}_{\{G\}(XYD^{\square})}(\mathbb{C}) \subseteq \mathrm{Aut}_{\{G\}(XY\cup\{\underline{\square}\})}(\mathbb{C})$. To show this let h be an element of $\mathrm{Aut}_{\{G\}(XYD^{\square})}(\mathbb{C})$. For $d \in D^{\square}$ we have $d = x_d + y_d$ with $x_d \in X^{\square}_{1\iota}$ and $y_d \in \langle Y^{\square}\rangle$, as $(X^{\square}, Y^{\square}, D^{\square})$ is special. So $d = h(d)$ implies $h(y_d) - y_d = x_d - h(x_d) \in V_G$. Furthermore we have $h(x) = x$ for $x \in X^{\square}_{1\nu} \subseteq X$. Hence for $y \in Y^{\square}$ the elements y and $h(y)$ are elements of type p with the same handle. The map $j(x) = x$ for $x \in X^{\square}_{1\nu}$ and $j(y) = y - h(y)$ for $y \in Y^{\square}$ induces an $\mathbb{B}_p$-homomorphism $\overline{j}$ of $\langle X^{\square}_{1\nu}Y^{\square}\rangle_{\mathcal{F}}$ into $\mathcal{F}(G)$. Hence $h(x_d) = x_d + \overline{j}(y_d)$ and therefore $(x_d : d \in D^{\square})$ and $(h(x_d) : d \in D^{\square})$ are in the same $\theta_{\square}$-class: $h(\underline{\square}) = \underline{\square}$.

To show viii) let f be an automorphism in $\mathrm{Aut}_{\{G\}(XY\underline{\square})}(\mathbb{C})$. We have to find an extension g of $f \restriction G$ with $f(d) = d$ for $d \in D^{\square}$. Then the assertion follows from $D \subseteq \langle XYD^{\square}\rangle$. Again we use $d = x_d + y_d$ with $x_d \in X^{\square}_{1\iota}$ and $y_d \in \langle Y^{\square}\rangle$. Since $(x_d : d \in D^{\square})$ and $(f(x_d) : d \in D^{\square})$ are in the same $\theta_{\square}$-class $x_d - f(x_d) \in V_G$ is an homomorphic image $\overline{j}(y_d)$, where $\overline{j}$ is an $\mathbb{B}_p$-homomorphism of $\langle X^{\square}_{1\nu}Y^{\square}\rangle_{\mathcal{F}}$ into $\mathcal{F}(G)$ with $\overline{j}(x) = x$ for $x \in X^{\square}_{1\nu}$. We define the desired g for a transversal $X^0YY^{\square}Y^*$ of $\mathbb{C}$. Let X^0 be any transversal for V_G and $X^0YY^{\square}$ be a part of a transversal of the smallest good subspace that contains $V_G \cup D$. On X^0YY^* g is defined like f and for $y \in Y^{\square}$ we define $g(y) = y + \overline{j}(y)$. By Lemma 4.9ii) we define an automorphism of $\mathbb{C}$ in this way. For this it is important that y and $g(y)$ are proper elements of type p with the same handle. Then $g(d) = g(x_d) + g(y_d) = f(x_d) + y_d + \overline{j}(y_d) = f(x_d) + y_d + x_d - f(x_d) = x_d + y_d = d$ for $d \in D^{\square}$ as desired. □

Corollary 7.6 *Let G be a saturated model of T_S, $G \preceq \mathbb{C}$, and D be a good subspace of V. Then*

$$\mathrm{acl}(\mathrm{Cb}(\mathrm{tp}(D/G))) = \mathrm{acl}(\mathrm{Cb}(\mathrm{tp}(XYD^{\square}/G))) = \mathrm{acl}(\mathrm{Cb}(\mathrm{tp}(XY\{\underline{\square}\}/G))) =$$
$$= \mathrm{acl}(\mathrm{Cb}(\mathrm{tp}(Y/G)) \cup X \cup \underline{\square}) = \mathrm{acl}(\mathrm{Cb}_{\Gamma}(\mathrm{tp}_{\Gamma}(Y^{\sim}_{1\nu}/\Gamma(G)) \cup X \cup \underline{\square})).$$

Proof. By Fact 7.1 and 7.2 and Theorem 7.4iii) we have the first equality. The second equality follows from Corollary 7.5vii) and viii). The next equality follows from Fact 7.3 and the last from Lemma 4.14. □

Finally we show the main assertion for good subspaces of V.

Corollary 7.7 *Assume T_S is stable and CM-trivial. Let $G \preceq H \preceq \mathbb{C} \models T_S$ be saturated models of T_S and D be a finite good subspace of V such that* $\mathrm{acl}(D \cup G) \cap H = G$. *Then* $\mathrm{Cb}(\mathrm{tp}(D/G)) \subseteq \mathrm{acl}(\mathrm{Cb}(\mathrm{tp}(D/H)))$.

Proof. Let $(X, X^\square, Y, Y^\square, D^\square, \underline{\square})$ be the platform for D with respect to G given by Theorem 7.4. By Corollary 7.6 we have to show $\mathrm{Cb}_\Gamma(\mathrm{tp}_\Gamma(Y_{1^\nu}^{\sim}/\Gamma(G)) \cup X \cup \underline{\square} \subseteq \mathrm{acl}(\mathrm{Cb}(\mathrm{tp}(D/H)))$.
By the last assertion in Theorem 7.4 there is a subgroup $\mathbb{K}$ of finite index in $\mathrm{Aut}_{\{H\}(D)}(\mathbb{C})$ such that all $f \in \mathbb{K}$ fix X, Y_{1^ν}, and $D^\square$ pointwise. As in the proof of Corollary 7.5vii) it follows $f(\underline{\square}) = \underline{\square}$. $\square$

8 The tame case

The aim of this paper is to show:

Theorem 8.1 *Assume that $G \preceq H \preceq \mathbb{C}$ are saturated models of T_S and A is a finite subgroup of the monster model $\mathbb{C}$ such that*

$$\mathrm{acl}(A \cup G) \cap H = G.$$

Then $\mathrm{Cb}(\mathrm{tp}(A/G)) \subseteq \mathrm{acl}(\mathrm{Cb}(\mathrm{tp}(A/H)))$.

Definition In the situation of Theorem 8.1 we say that A is tame if

$$A \cap (G \cdot \mathbb{C}') = (A \cap G)(A \cap \mathbb{C}').$$

We show Theorem 8.1 for the tame case. The lemmas we prove are also useful for the general case. By Theorem 6.9 we get

Lemma 8.2 *There is a finite good subspace $D_0 \subseteq V$ such that*

$$A \cap \mathbb{C}' \subseteq \mathrm{dcl}(D_0) \quad \textit{and} \quad D_0 \subseteq \mathrm{acl}(A \cap \mathbb{C}').$$

Now we work in the general situation that $G \preceq \mathbb{C}$ are any nilpotent groups of class 2 and exponent p.

Lemma 8.3 *Assume $G \preceq \mathbb{C}$ are in $\mathbb{G}_{2,p}$, G is saturated, and* $\mathrm{Th}(\mathbb{C})$ *is stable. Furthermore assume $A \cap (G \cdot Z(\mathbb{C})) = A \cap (G \cdot \mathbb{C}')$, $c_1, \ldots, c_n \in G$ ($c_i = 1$ is possible), and $d_1, \ldots, d_n \in Z(\mathbb{C})$ linearly independent modulo $G \cdot \mathbb{C}'$. Then*
$\mathrm{Cb}(\mathrm{tp}(A \cup \{c_1 d_1, \ldots, c_n d_n\}/G)) = \mathrm{dcl}(\mathrm{Cb}(\mathrm{tp}(A/G)) \cup \{c_1/Z(\mathbb{C}), \ldots, c_n/Z(\mathbb{C})\})$.

Proof. Every $f \in \mathrm{Aut}_{\{G\}(A\cup\{c_1d_1,\dots,c_nd_n\})}(\mathfrak{C})$ fixes $\mathrm{Cb}(\mathrm{tp}(A/G))$ and $\{c_1/Z(\mathfrak{C}),\dots,c_n/Z(\mathfrak{C})\}$ pointwise. Hence

$$\mathrm{Cb}(\mathrm{tp}(A/G)) \cup \{c_1/Z(\mathfrak{C}),\dots,c_n/Z(\mathfrak{C})\} \subseteq \mathrm{Cb}(\mathrm{tp}(A\cup\{c_1d_1,\dots,c_nd_n\}/G)).$$

To prove the converse let g be an automorphism of G that fixes each $c_i/Z(G)$ and $\mathrm{Cb}(\mathrm{tp}(A/G))$. It is sufficient to find an extension $f^* \in \mathrm{Aut}_{\{G\}}(\mathfrak{C})$ of g that fixes $A\cup\{c_1d_1,\dots,c_nd_n\}$ pointwise. Assume X is a basis for G modulo $Z(G)$. There is a basis XY for $\mathfrak{C}$ modulo $Z(\mathfrak{C})$. Then $G = \langle X\rangle \oplus K^G$ and $\mathfrak{C} = \langle XY\rangle \oplus K^G \oplus K$ where K^G and K are elementary abelian p-groups. Since $A\cap(G\cdot Z(\mathfrak{C})) = A\cap(G\cdot\mathfrak{C}')$ we have $A \subseteq \langle XY\rangle \oplus K^G$. Since g fixes $\mathrm{Cb}(\mathrm{tp}(A/G))$ there is an $f \in \mathrm{Aut}_{\{G\}(A)}(\mathfrak{C})$ that extends g such that $f(a) = a$ for $a\in A$. We have

$$f(\mathfrak{C}'\oplus K^G) = \mathfrak{C}'\oplus K^G.$$

Now $d_i = a_ib_i$ where $a_i \in \mathfrak{C}'\oplus K^G$ and $b_i \in K$. By assumption $b_1,\dots,b_n \in K$ are linearly independent. c_ia_i and $f(c_ia_i)$ are in $\langle XY\rangle\oplus K^G$. Now we define $f^* \in \mathrm{Aut}_{\{G\}}(\mathfrak{C})$ in such a way that $f^*(a) = f(a)$ for $a \in \langle XY\rangle \oplus K^G$ and

$$f^*(b_i) = f(c_ia_i)^{-1}(c_ia_i)b_i = f(c_i)^{-1}\,c_if(a_i)^{-1}a_ib_i.$$

Note $f(c_i)^{-1}c_i \in G\cap Z(\mathfrak{C}) = Z(G)$ since g fixes $c_i/Z(\mathfrak{C})$ and $f(a_i)^{-1}a_i \in \mathfrak{C}'\cdot Z(G)$ by definition. f^* extends g and fixes A pointwise since $A\subseteq\langle XY\rangle\oplus K^G$. Finally $f^*(c_id_i) = f^*(c_ia_ib_i) = f^*(c_ia_i)f^*(b_i) = f(c_ia_i)f^*(b_i) = f(c_ia_i)f(c_ia_i)^{-1}(c_ia_i)b_i = c_ia_ib_i = c_id_i$, as desired. □

Lemma 8.4 *Let $G \preceq \mathfrak{C}$ be in $\mathfrak{G}_{2,p}$. For every finite subgroup A of $\mathfrak{C}$ there is a subgroup $B \subseteq A$ such that $B\cap G = B\cap G'$, $B\cap(G\cdot Z(\mathfrak{C})) = B\cap(G\cdot\mathfrak{C}')$, and*

$$A = \langle B\cup(A\cap G)\cup\{c_1d_1,\dots,c_nd_n\}\rangle$$

where $c_1,\dots,c_n \in G$ and $d_1,\dots,d_n \in Z(\mathfrak{C})$ are linearly independent modulo $G\cdot\mathfrak{C}'$. Moreover, A is tame if and only if B is tame.

Proof. We consider the subgroups $\mathfrak{C}' \subseteq G\cdot\mathfrak{C}' \subseteq G\cdot Z(\mathfrak{C})$ of $\mathfrak{C}$. Since $G\preceq\mathfrak{C}$ we have $\mathfrak{C}'\cap G = G'$. Let X_1 be a basis of A modulo $G\cdot Z(\mathfrak{C})$. Let X_2 be a basis of $A\cap G\cdot Z(\mathfrak{C})$ modulo $G\cdot\mathfrak{C}'$. Let X_3 be a basis of $A\cap(G\cdot\mathfrak{C}')$ modulo $(A\cap G)(A\cap\mathfrak{C}')$. Let B be $\langle X_1\cup X_3\cup(A\cap\mathfrak{C}')\rangle$. Let $u_0,\dots,u_{r-1}$ be the elements of X_1, $v_0,\dots,v_{s-1}$ be the elements of X_3, and $w_0,\dots,w_{t-1}$ be a basis of $A\cap\mathfrak{C}'$. Every element of B can be written as $\prod_{i<r} u_i^{\alpha_i}\prod_{i<s} v_i^{\beta_i}\prod_{i<t} w_i^{\gamma_i}$, since every product of elements of $X_1\cup X_3\cup\{w_0,\dots,w_{t-1}\}$ can be brought into this form using some additional elements of $(A\cap\mathfrak{C}')$. It follows $B\cap G = A\cap\mathfrak{C}'\cap G = A\cap G'$. Hence $B\cap G = B\cap G'$. Furthermore by construction $\langle B\cup(A\cap G)\cup X_2\rangle = A$ and $B\cap(G\cdot Z(\mathfrak{C})) = B\cap(G\cdot\mathfrak{C}')$. A is tame iff $X_3=\emptyset$ iff B is tame. □

Corollary 8.5 *Let $G \preceq \mathfrak{C}$ be groups in $\mathfrak{G}_{2,p}$, G is saturated, and* $\mathrm{Th}(\mathfrak{C})$ *is stable. In the situation of Lemma* 8.4 *we have*

$$\mathrm{Cb}(\mathrm{tp}(A/G)) = \mathrm{dcl}[(A \cap G) \cup \mathrm{Cb}(\mathrm{tp}(B/G)) \cup \{c_1/Z(\mathfrak{C}), \ldots, c_n/Z(\mathfrak{C})\}].$$

Proof. This follows from Fact 7.3 and Lemma 8.3. □

Lemma 8.6 *Assume $G \preceq H \preceq \mathfrak{C}$ are in $\mathfrak{G}_{2,p}$ and B is a subgroup of $\mathfrak{C}$ with $B \cap G = B \cap G'$, $B \cap (G \cdot Z(\mathfrak{C})) = B \cap (G \cdot \mathfrak{C}')$, and* $\mathrm{acl}(B \cup G) \cap H = G$. *Then $B \cap H = B \cap G'$ and $B \cap (H \cdot Z(\mathfrak{C})) = B \cap (G \cdot \mathfrak{C}')$.*

Proof. By assumption we have $B \cap H = B \cap G = B \cap G'$. If $c \cdot d \in B \cap (H \cdot Z(\mathfrak{C}))$ with $c \in H$ and $d \in Z(\mathfrak{C})$, then $c/Z(\mathfrak{C}) \in H \cap \mathrm{acl}(B) \subseteq G$. Hence $c = c_0 d_0$ with $c_0 \in G$ and $d_0 \in Z(\mathfrak{C})$. We have $cd = c_0 d_0 d \in (G \cdot Z(\mathfrak{C})) \cap B$ and therefore $cd \in (G \cdot \mathfrak{C}') \cap B$. □

Corollary 8.7 *Let $G \preceq H \preceq \mathfrak{C}$ be in $\mathfrak{G}_{2,p}$, G and H are saturated, and* $\mathrm{Th}(\mathfrak{C})$ *is stable. If we want to show Theorem* 8.1 *for finite* (*tame*) *subgroups, then it is sufficient to show it for finite* (*tame*) *subgroups A with $A \cap G = A \cap G'$ and $A \cap (G \cdot Z(\mathfrak{C})) = A \cap (G \cdot \mathfrak{C}')$.*

Proof. Assume an arbitrary (tame) A is given. We choose B according to Lemma 8.4. We have $\mathrm{acl}(B \cup G) \cap H \subseteq \mathrm{acl}(A \cup G) \cap H = G$ since $B \subseteq A$. (If A is tame, then B is tame.) Since Theorem 8.1 is true for B by assumption we obtain

$$\mathrm{Cb}(\mathrm{tp}(B/G) \subseteq \mathrm{acl}(\mathrm{Cb}(\mathrm{tp}(B/H))).$$

By Corollary 8.5

$$\mathrm{Cb}(\mathrm{tp}(A/G)) = \mathrm{dcl}[(A \cap G) \cup \mathrm{Cb}(\mathrm{tp}(B/G)) \cup \{c_1/Z(\mathfrak{C}), \ldots, c_n/Z(\mathfrak{C})\}.$$

Since $(A \cap G) \cup \{c_1/Z(\mathfrak{C}), \ldots, c_n/Z(\mathfrak{C})\} \subseteq \mathrm{Cb}(\mathrm{tp}(A/H))$, the assertion follows. □

Now we return to Mekler groups.

Theorem 8.8 *Let $G \preceq H \preceq \mathfrak{C} \models T_S$ be saturated. Assume A is a finite subgroup of $\mathfrak{C}$, A is tame with respect to G, $A \cap G = A \cap G'$, $A \cap (G \cdot Z(\mathfrak{C})) = A \cap (G \cdot \mathfrak{C}')$, and* $\mathrm{acl}(A \cup G) \cap H = G$. *Then there is some finite good subspace D of V such that $D \subseteq$* $\mathrm{acl}(A)$, $\mathrm{acl}(\mathrm{Cb}(\mathrm{tp}(A/G))) = \mathrm{acl}(\mathrm{Cb}(\mathrm{tp}(D/G)))$, *and* $\mathrm{acl}(\mathrm{Cb}(\mathrm{tp}(A/H))) \subseteq \mathrm{acl}(\mathrm{Cb}(\mathrm{tp}(D/H)))$. *We have*

$$\mathrm{Cb}(\mathrm{tp}(A/G)) \subseteq \mathrm{acl}(\mathrm{Cb}(\mathrm{tp}(A/H))).$$

Proof. By assumption $A \cap (G \cdot Z(\mathfrak{C})) = A \cap (G \cdot \mathfrak{C}') = (A \cap G)(A \cap \mathfrak{C}') = (A \cap G')(A \cap \mathfrak{C}') = A \cap \mathfrak{C}'$. Hence $A = \langle (A \cap \mathfrak{C}') \cup E \rangle$ where E is a basis of A modulo $A \cap G \cdot Z(\mathfrak{C})$. Let $\overline{E}$ be the image of E in $V = \mathfrak{C}/Z(\mathfrak{C})$. By Theorem 6.9 for every $a \in (A \cap \mathfrak{C}')$ there is a good subspace $U(a) \subseteq V$ such that $a \in \beta(U(a), U(a))$ and $U(a) \subseteq \mathrm{acl}(a)$. By Corollary 6.4 there is a smallest good subspace $D \subseteq V$ such that

$\overline{E} \cup \bigcup\{U(a) : a \in A \cap \mathbb{C}'\} \subseteq D$ and $D \subseteq \mathrm{acl}(\overline{E} \cup \bigcup\{U(a) : a \in A \cap \mathbb{C}'\})$. D is finite. Hence $D \subseteq \mathrm{acl}(A)$ and therefore

$$\begin{aligned} \mathrm{Cb}(\mathrm{tp}(D/G)) &\subseteq \mathrm{acl}(\mathrm{Cb}(\mathrm{tp}(A/G))) \quad \text{and} \\ \mathrm{Cb}(\mathrm{tp}(D/H)) &\subseteq \mathrm{acl}(\mathrm{Cb}(\mathrm{tp}(A/H))). \end{aligned}$$

To show

$$\mathrm{acl}(\mathrm{Cb}(\mathrm{tp}(D/G))) = \mathrm{acl}(\mathrm{Cb}(\mathrm{tp}(A/G)))$$

let g be any automorphism of $\mathbb{C}$ that fixes D pointwise and G setwise. Let $\mathcal{F}(g)$ be the $\mathbb{B}_p$-automorphism of $\langle V, W, \beta\rangle$ induced by g. Then $\mathcal{F}(g)$ fixes D and therefore $A \cap \mathbb{C}' \subseteq \beta(D, D)$ and $\overline{E} \subseteq D$ pointwise. By Corollary 3.3 there is an automorphism $f \in \mathrm{Aut}_{\{G\}}(\mathbb{C})$ such that $g \upharpoonright G = f \upharpoonright G$, $\mathcal{F}(f) = \mathcal{F}(g)$ and $f(e) = e$ for $e \in E$. $\mathcal{F}(f) = \mathcal{F}(g)$ implies $f(a) = a$ for $a \in A \cap \mathbb{C}'$. Hence $f \in \mathrm{Aut}_{\{G\}(A)}(\mathbb{C})$ with $f \upharpoonright G = g \upharpoonright G$, as desired.
We have $\mathrm{acl}(D \cup G) \cap H = G$. Hence by Corollary 7.7 $\mathrm{Cb}(\mathrm{tp}(D/G)) \subseteq \mathrm{acl}(\mathrm{Cb}(\mathrm{tp}(D/H)))$. This implies the last assertion of Theorem 8.8. □

Corollary 8.9 *Theorem* 8.1 *is true for finite tame subgroups A.*

Proof. Apply Corollary 8.7 and Theorem 8.8. □

9 Mixed elements

In Theorem 8.8 we have excluded elements a in A with $a \notin G$, $a \notin \mathbb{C}'$ but $a \in G \cdot \mathbb{C}'$. We call these elements mixed elements with respect to G.
As before we consider saturated T_S-models $G \preceq H \preceq \mathbb{C}$, where $\mathbb{C}$ is a monster model. Let A be a finite subgroup of $\mathbb{C}$ such that $\mathrm{acl}(A \cup G) \cap H = G$.
We only work with mixed elements under the assumption that

(9.1) $\quad A \cap (G \cdot Z(\mathbb{C})) = A \cap (G \cdot \mathbb{C}')$.

If $a \in A$ is a mixed element with respect to H, then $a/Z(\mathbb{C}) \in H \cap \mathrm{acl}(A)$. Hence $a/Z(\mathbb{C}) \in G$, $a \in G \cdot Z(\mathbb{C})$ and therefore by (9.1) $a \in G \cdot \mathbb{C}'$. We have shown that a is a mixed element with respect to G. If a is a mixed element with respect to G, then a is a mixed element with respect to H. From now we will speak about mixed elements. Our aim is to show the following result.

Theorem 9.1 *Let a be a mixed element of A. Then there are elements $a^G \in G$, $a' \in \mathbb{C}'$ and a part of a transversal XU that respects G with $\langle XU\rangle \cap V_G = \langle X\rangle$ such that*

$$a = a^G \cdot \prod_{\substack{x \in X \\ u \in U}} [x, u]^{s_{xu}} a'$$

and the following is true:
There are a subgroup $I\!K^G$ *of* $\mathrm{Aut}_{\{G\}(a)}(\mathfrak{C})$ *of finite index and a subgroup* $I\!K^H$ *of* $\mathrm{Aut}_{\{H\}(a)}(\mathfrak{C})$ *of finite index such that for every* $f \in I\!K^G$ *and every* $g \in I\!K^H$ *we have* $f(a') = a'$, $g(a') = a'$, $f(x) = x$, $g(x) = x$ *for* $x \in X$, *and* $f(u) = u$ *modulo* V_G *and* $g(u) = u$ *modulo* V_H *for* $u \in U$.
The elements of U *are not of type* 1^ν. *If they are of type* p, *then their handle is in* X.
U *is linearly independent modulo* V_H.

By Theorem 9.1 we can assume w.l.o.g. that $a' \in A$ for a mixed element $a \in A$. That means we can restrict us to mixed elements of the form $a^G \prod\limits_{\substack{x\in X\\ u\in U}} [x,u]^{s_{xu}}$.

We will study mixed elements to prove Theorem 9.1. Assume $a = cd = c_1d_1$ where $c, c_1 \in G$ and $d, d_1 \in \mathfrak{C}'$. Then $dd_1^{-1} = c^{-1}c_1 \in G \cap \mathfrak{C}' = G'$. Here we use $G \preceq \mathfrak{C}$. Hence d is uniquely determined modulo G'. By Theorem 6.9 there is a finite good subspace $D \subseteq V$ with $d \in \beta(D,D)$ and $D \subseteq \mathrm{acl}(d)$. D is the smallest good subspace such that $d \in \beta(D,D)$. It follows that $\mathrm{Aut}_{\{d\}}(\mathfrak{C})$ and $\mathrm{Aut}_{(D)}(\mathfrak{C})$ are commensurable. By Theorem 7.4 there is a platform $(X, X^\square, Y, Y^\square, D^\square)$ for D with respect to G. Then $\langle V_G Y Y^\square\rangle$ is the smallest good subspace that contains V_G and D. If $f \in \mathrm{Aut}_{\{G\}(a)}(\mathfrak{C})$, then $a = cd = f(c)\cdot f(d)$. Then $f(D)$ is the smallest good subspace with $f(d) \in \beta(f(D), f(D))$ and $f(D) \subseteq \mathrm{acl}(f(d))$. Since $d = f(d)$ modulo G' it follows $f(d) \in \beta(\langle V_G D\rangle, \langle V_G D\rangle)$. Hence we have $f(D) \subseteq \langle V_G Y Y^\square\rangle$. Since $f(\langle V_G Y Y^\square\rangle)$ is the smallest good subspace that contains V_G and $f(D)$ we have $f(\langle V_G Y Y^\square\rangle) \subseteq \langle V_G Y Y^\square\rangle$. By similar arguments $D \subseteq f(\langle V_G Y Y^\square\rangle)$ and $f(\langle V_G Y Y^\square\rangle) = \langle V_G Y Y^\square\rangle$. It follows the existence of a subgroup $I\!K$ of $\mathrm{Aut}_{\{G\}(a)}(\mathfrak{C})$ of finite index such that

(a) $f(y) = y$ modulo V_G for all $y \in YY^\square$ and all $f \in I\!K$.

Let $(X^H, X^{H\square}, Y^H, Y^{H\square}, D^{H\square})$ be a platform for $\langle XYD^\square\rangle$ with respect to H. The smallest good subspace that contains V_G and D is $\langle V_G Y Y^\square\rangle$ and it contains $\langle XYD^\square\rangle$. Hence the smallest good subspace that contains V_H and D contains $\langle XYD^\square\rangle$. Since $D \subseteq \langle XYD^\square\rangle$ the subspace $\langle V_H Y^H Y^{H\square}\rangle$ is the smallest good subspace that contains V_H and D. By Theorem 7.4 for every saturated H^* with $H \preceq H^* \preceq C$ and $\mathrm{acl}(D \cup H) \cap H^* = H$ $\mathrm{Aut}_{\{H^*\}(D)}(\mathfrak{C})$ and $\mathrm{Aut}_{\{H^*\}(XYD^\square)}(\mathfrak{C})$ are commensurable. Note $\mathrm{acl}(D\cup H) \cap H^* = H$ implies $\mathrm{acl}(D \cup G) \cap H^* \subseteq H$ and therefore $\mathrm{acl}(D \cup G) \cap H^* = \mathrm{acl}(D \cup G) \cap H = G$. Since $(X^H, X^{H\square}, Y^H, Y^{H\square}, D^{H\square})$ is a platform for $\langle XYD^\square\rangle$ with respect to H we get that $\mathrm{Aut}_{\{H^*\}(D)}(\mathfrak{C})$ and $\mathrm{Aut}_{\{H^*\}(X^H Y^H D^{H\square})}(\mathfrak{C})$ are commensurable. Hence $(X^H, X^{H\square}, Y^H, Y^{H\square}, D^{H\square})$ is a platform for D with respect to H.
Let $f \in \mathrm{Aut}_{\{H\}(a)}(\mathfrak{C})$. Then $f(d) = d$ modulo H'. Hence

$$\langle V_H \cup D\rangle = \langle V_H \cup f(D)\rangle = f(\langle V_H \cup D\rangle) \text{ as above.}$$

As shown above $\langle V_H \cup Y^H \cup Y^{H\square}\rangle$ is the smallest good subspace that contains V_H and D. Similarly, $f(\langle V_H \cup Y^H \cup Y^{H\square}\rangle)$ is the smallest good subspace that contains V_H and $f(D)$. Since $\langle V_H \cup D\rangle = \langle V_H \cup f(D)\rangle$

$$f(\langle V_H \cup Y^H \cup Y^{H\square}\rangle) = \langle V_H \cup Y^H \cup Y^{H\square}\rangle.$$

Note that

$$YY^{\square} \subseteq \langle V_G Y D^{\square} \rangle \subseteq \langle V_H Y D^{\square} \rangle = \langle V_H Y^H Y^{H\square} \rangle.$$

Hence there exists a subgroup $\mathbb{K}^H$ of $\mathrm{Aut}_{\{H\}(a)}(\mathbb{C})$ of finite index such that

(a)^H $\quad f(y) = y$ modulo V_H for every $y \in YY^{\square}$ and $f \in \mathbb{K}^H$.

Using (a), Lemma 6.10, and Corollary 4.16 we obtain a subgroup $\mathbb{K}$ of $\mathrm{Aut}_{\{G\}(a)}(\mathbb{C})$ of finite index such that (a) is true and for every $f \in \mathbb{K}$:

(b) $\quad f(y) = y$ for every $y \in Y_{1^\nu}$ and every $y \in Y_{p,e}$ with $e \in Y_{1^\nu}$,
$f(x) = x$ for every handle $x \in X$ of some $y \in Y_{p,x} \cup Y^{\square}_{p,x}$,
and every root x of some $y \in Y_{1^\nu}$.

By Corollary 7.5

$$YY^{\square} \subseteq \mathrm{acl}(XX^{\square}D) \subseteq \mathrm{acl}(GD).$$

Hence $YY^{\square}$ is linearly independent modulo V_H. Then we can show:

(b)^H $\quad$ There is a subgroup $\mathbb{K}^H$ of $\mathrm{Aut}_{\{H\}(a)}(\mathbb{C})$ of finite index such that (a)^H is true and for every $g \in \mathbb{K}^H$ we have

$$g(y) = y \quad \text{for} \;\; y \in Y_{1^\nu} \cup \bigcup_{e \in Y_{1^\nu}} Y_{p,e}$$

and

$$g(x) = x \quad \text{for every handle } x \text{ of some } y \in Y_{p,x} \cup Y^{\square}_{p,x}$$
$$\text{and every root } x \text{ of some } y \in Y_{1^\nu}.$$

(b)^H follows from (a)^H by Lemma 6.10 and Corollary 4.16 in most cases. The critical case concerns the handle x of some $y \in Y_{p,x}Y^{\square}_{p,x}$ with $y \in \langle V_H V_{1^\nu} \rangle$. Then $y = \sum_{1 \le i \le r} v_i$ modulo V_H where the v_i are $\approx$-inequivalent elements of type 1^ν not in V_H. The handle e of y is an element of V_H. We have $\beta(e, v_i) = 0$. By Lemma 6.10 we can choose $\mathbb{K}^H$ such that $g(v_i) = v_i$, whence $g(e) = e$ for $g \in \mathbb{K}^H$ by Corollary 4.16.

We call $(X, X^{\square}, Y, Y^{\square}, D^{\square})$ a support for d with respect to G if

(i) $XX^{\square}YY^{\square}$ is a part of a transversal that respects V_G,

(ii) $\langle XX^{\square}YY^{\square} \rangle \cap V_G = \langle XX^{\square} \rangle$, $X \cap X^{\square} = X^{\square}_{1^\nu}$, $Y \cap Y^{\square} = \emptyset$,

(iii) $D^{\square}$ is linearly independent modulo $\langle V_G Y \rangle$, $X^{\square} = X^{\square}_{1^\nu} \cup X^{\square}_{1^\iota}$, $Y^{\square} = Y^{\square}_p = \bigcup_{e \in X^{\square}_{1^\nu}} Y^{\square}_{p,e}$, for $e \in X^{\square}_{1^\nu}$ the set $Y^{\square}_{p,e} \neq \emptyset$, and every $d \in D^{\square}$ has the form $d = x_d + y_d$, where $x_d \in X^{\square}_{1^\iota}$ and $y_d \in \langle Y^{\square} \rangle$.

(iv) d is a linear combination of commutators $\beta(u, v)$ with $u, v \in XYD^\square$.

(v) $YY^\square \subseteq \mathrm{acl}(aG) = \mathrm{acl}(dG)$.

Note that a support is not necessarily a platform.

We are interested in all situations $a = c \cdot d$, $(X, X^\square, Y, Y^\square, D^\square)$, $\mathbb{K}$, $\mathbb{K}^H$ where $(X, X^\square, Y, Y^\square, D^\square)$ is a support of d and (a), (a)^H, (b), (b)^H are true. We consider $XYD^\square$ as an ordered sequence, where $<$ denotes this ordering. After moving some summands $s\beta(x_1, x_2)$ with $x_1, x_2 \in X$ to c, we can replace (iv) by the following:

$$(9.2) \qquad d = \sum_{\substack{x \in X \\ y \in YD^\square}} s_{xy}\beta(x, y) + \sum_{\substack{y_1 < y_2 \\ y_i \in YD^\square}} s_{y_1 y_2}\beta(y_1, y_2).$$

Our aim is to show that we can choose d, the support $(X, X^\square, Y, Y^\square, D^\square)$, $\mathbb{K}$, $\mathbb{K}^H$ in such way that (a), (a)^H, (b), (b)^H, (9.2) are true and furthermore

(c) $\quad f(x) = x$ for all $x \in X$ and $f \in \mathbb{K}$

and

(c)^H $\quad g(x) = x$ for all $x \in X$ and $g \in K^H$.

We need further refinements to get (c) and (c)^H. After deleting some elements of X we can assume:

(9.3) For every $x \in X$ we have $s_{xy}\beta(x, y) \neq 0$ for some $y \in YD^\square$ in (9.2)
or x is a handle of some element of type p in $XYY^\square$
or x is a connection between two handles.
If $\beta(x, y) = 0$ for x, then $s_{xy} = 0$.

Finally we consider only situations such that

$$(9.4) \qquad X^e = \{x \in X : s_{xy}\beta(x, y) \neq 0 \text{ for some } y \in YD^\square \text{ and } \beta(x, y) \neq 0 \text{ for all } y \in YY^\square\} \text{ is of minimal size.}$$

Let us consider again

$$(9.2) \qquad d = \sum_{\substack{x \in X \\ y \in YD^\square}} s_{xy}\beta(x, y) + \sum_{\substack{y_1 < y_2 \\ y_i \in YD^\square}} s_{y_1 y_2}\beta(y_1, y_2).$$

If we use $D^\square \subseteq \langle X^\square_{1\iota} Y^\square \rangle$, then we obtain

$$(9.2)^{Y^\square} \qquad d = \sum_{\substack{x \in XX^\square \\ y \in YY^\square}} r_{xy}\beta(x, y) + \sum_{\substack{y_1 < y_2 \\ y_i \in YY^\square}} r_{y_1 y_2}\beta(y_1, y_2) \text{ modulo } \langle \beta(V_G, V_G) \rangle.$$

Since $XX^\square YY^\square$ is a part of a transversal that respects V_G we have that $\{\beta(v_1, v_2) : \beta(v_1, v_2) \neq 0,\ v_1, v_2 \in XX^\square YY^\square,\ v_1 < v_2\}$ is linearly independent modulo $\beta(V_G, V_G)$. Furthermore $D^\square \subseteq \langle Y^\square X^\square_{1^\iota}\rangle$. Hence

$$\sum_{y \in YY^\square} r_{xy}\beta(x, y) = \sum_{y \in YD^\square} s_{xy}\beta(x, y) \neq 0 \text{ modulo } \langle\beta(V_G, V_G)\rangle$$

for $x \in X^e$. We get that

$$X^e = \{x \in X : \sum_{y \in YY^\square} r_{xy}\beta(x, y) \neq 0 \text{ modulo } \langle\beta(V_G, V_G)\rangle \text{ and } \beta(x, y) \neq 0 \text{ for all } y \in YY^\square\}.$$

By property v) of a support we have $YY^\square \subseteq \mathrm{acl}(aG)$ and therefore

$$\{\beta(x, y) : y \in YY^\square,\ x \in X\} \subseteq \mathrm{acl}(aG).$$

Since $\mathrm{acl}(aG) \cap H = G$ we have that $\{\beta(x, y), y \in YY^\square\}$ is linearly independent modulo $\beta(V_H, V_H)$ for $x \in X^e$. Hence

$$X^e = \{x \in X : \sum_{y \in YY^\square} r_{xy}\beta(x, y) \neq 0 \text{ modulo } \langle\beta(V_H, V_H)\rangle \text{ and } \beta(x, y) \neq 0 \text{ for all } y \in YY^\square\}.$$

We can reformulate (9.4) in the following way: $(X, X^\square, Y, Y^\square, D^\square)$ is chosen in such a way that with respect to $(9.2)^{Y^\square}$ we have that

$$(9.4)^+ \qquad \begin{aligned} X^e &= \{x \in X : \sum_{y \in YY^\square} r_{xy}\beta(x, y) \neq 0 \text{ modulo } \langle\beta(V_G, V_G)\rangle \text{ and } \beta(x, y) \neq 0 \text{ for all } y \in YY^\square\} \\ &= \{x \in X : \sum_{y \in YY^\square} r_{xy}\beta(x, y) \neq 0 \text{ modulo } \langle\beta(V_H, V_H)\rangle \text{ and } \beta(x, y) \neq 0 \text{ for all } y \in YY^\square\} \text{ is of minimal size.} \end{aligned}$$

Under all the assumptions above we give

Proof of (c):
Let Q be a part of a 1^ν-transversal that is up to $\approx$-equivalence the set of roots of Y_{1^ν}. We assume that elements of Q that represent $\approx$-classes in $XX^\square$ are elements of $XX^\square$. See Corollary 4.16. W.l.o.g. we can assume:

$(c)^{\mathrm{weak}}$ Let x be an element of XQ. If $f(x) \in \langle XX^\square Q\rangle$ for all $f \in \mathbb{K}$, then $f(x) = x$ for all $f \in \mathbb{K}$.

By (b) and the definition of Q for all $f \in \mathbb{K}$ we have $f(x) = x$, if $x \in Q$, or x is a handle of an element in Y_p or $Y_p^\square$. Note that $\beta(x, y) = 0$ for some $y \in YY^\square$ implies that x is a root or x is a handle. Hence it is sufficient to show that $f(x) = x$

for $f \in I\!K$ and $x \in X^e$. Then we can choose $I\!K$ such that $I\!K$ fixes also handles of elements of X_p and connections. By (9.3) it follows (c). Our main argument will be that $f(d) = d$ modulo $\beta(V_G, V_G)$ for $f \in I\!K$. By ($\mathrm{c}^{\mathrm{weak}}$) it is sufficient to show that $f(x) \in \langle XX^\square Q\rangle$.
Hence we assume that:

(∗) There are some $f \in I\!K$ and $x \in X^e$ such that

$$f(x) \notin \langle XX^\square Q\rangle.$$

Under our assumptions we show that:

(9.5) There are a transversal $\overline{X}\,\overline{Y}$ of V, x, u in $\overline{X}$, and $f \in I\!K$ such that $\overline{X}$ is a transversal of V_G, $QXX^\square \subseteq \overline{X}$, $YY^\square \subseteq \overline{Y}$, $x \in X^e$, $u \in \overline{X} \setminus XX^\square Q$, $f(x) = u + v$ with $v \in \langle \overline{X} \setminus \{u\}\rangle$, and for all $x' \in XX^\square \setminus \{x\}$ we have $f(x') \in \langle \overline{X} \setminus \{u\}\rangle$. Furthermore $\beta(u, y) \neq 0$ for all $y \in YY^\square$.

The last assertion follows since all roots and handles of $YY^\square$ are in $\langle XX^\square Q\rangle$ and u is outside. To prove that we can assume (9.5) we distinguish several cases. f and x are given by (∗). We choose a transversal $\overline{X}\,\overline{Y}$ for V where $\overline{X}$ is a transversal of V_G, $XX^\square Q \subseteq \overline{X}$, and $YY^\square \subseteq \overline{Y}$. We will only vary X and $\overline{X}$ to get (9.5).

Case 1 $x \in X^e_{1^\nu}$.
We choose $\overline{X}$ in such a way that $f(x) \in \overline{X} \setminus \langle XX^\square Q\rangle$. Let u be $f(x)$. If tu with $t \neq 0$ occurs in the representation of $f(x')$ for $x' \neq x$ in $XX^\square$, then we replace x' by $x' - tx$. We call the result of this process $X^*X^{*\square}$. We have to show that $X^*X^{*\square}$ is again a part of a transversal. Note that $X_{1^\nu} = X^*_{1^\nu}$ since f induces a permutation of the $\approx$-classes of the elements of $\overline{X}_{1^\nu}$.
It remains to consider $x' \in X_{p,e}$ with tu in the representation of $f(x')$ over $\overline{X}$. Note that there is no $X^\square_{p,e}$. We have to show $\beta(x, e) = 0$. Since tu occurs in $f(x')$ we have $\beta(u, f(e)) = 0$. Hence $\beta(x, e) = 0$, as desired.
Hence $X^*X^{*\square}$ is a part of a transversal and $X^*_{1^\nu} = X_{1^\nu}$. If we replace X by X^*, $X^\square$ by $X^{*\square}$, and $\overline{X}$ by $(\overline{X} \setminus XX^\square) \cup X^*X^{*\square}$, then $(X^*, X^{*\square}, Y, Y^\square, D^\square)$ is a support for d, (a), $(\mathrm{a})^H$, (b), $(\mathrm{b})^H$, and (9.2) remain true. By (9.4) for X it follows (9.3) and (9.4) for X^*.

Case 2 $x \in X_{p,e}$ and (c) is true for all $x' \in X^e_{1^\nu}$.
Remember that for handles $v \in X_{1^\nu}$ with $Y_{p,v} \neq 0$ or $Y^\square_{p,v} \neq \emptyset$ we know $f(v) = v$ by (b).

Case 2.1 $f(x) \notin \langle XX^\square V_{1^\nu}\rangle$.
This case includes the case $f(e) \neq e$, since $f(e) \neq e$ and $(\mathrm{c})^{\mathrm{weak}}$ imply $f(e) \notin \langle XX^\square Q\rangle$. Then $f(x)$ is a proper element of type p with a handle $f(e) \notin \langle XX^\square Q\rangle$. The proper elements of type p in $\langle XX^\square V_{1^\nu}\rangle$ have a handle in X since they are of the form $w + v$ with $w \in X_p$ and $v \in V_{1^\nu}$. We define $u = f(x)$ and choose $\overline{X}$ such that $u \in \overline{X}$. The element u is proper of type p. It cannot occur in the representation of $f(x')$ for

$x' \in X_{1^\nu}$, because these elements are in $\bar{X}_{1^\nu}$. Let $x' \in XX^\square$ be such that u occurs in the representation of $f(x')$. Assume $x' \in X_{p,e'}$. Then $\beta(u, f(e')) = 0$. We have $\beta(u, f(e)) = 0$. Hence $f(e) \approx f(e')$ since u was a proper element of type p. We get $e \approx e'$ and $x' \in X_{p,e}$. A suitable substitution $x' - tx$ is possible. If $x' \in X_{1^\iota} X_{1^\iota}^\square$, then we have no difficulties with such a substitution.

Now we can assume that $f(x) = x$ for all handles x and therefore for all connections in X and $f \in I\!K$. Using (9.3) and (b) as above we have $f(x) = x$ for all $x \in X_{1^\nu}$ and all $f \in I\!K$.

Case 2.2 (c) is true for all $x' \in X_{1^\nu}$, $f(x') \in \langle XX^\square V_{1^\nu}\rangle$ for $x' \in X_p$, and there is some $x \in X_{p,e}$ with $f(x) \notin \langle XX^\square Q\rangle$ but $f(x) \in \langle XX^\square V_{1^\nu}\rangle$. By assumption we have $f(e) = e$. We choose u in the representation of $f(x)$ such that $u \in \overline{X}_{1^\nu} \setminus XX^\square Q$. By assumption $u \notin f(X_{1^\nu})$. If u occurs in the representation of $f(x')$ for some $x' \in X_{p,e'}$ with $e' \not\approx e$, then $\beta(u, e) = 0$ and $\beta(u, e') = 0$ since $f(e) = e$ and $f(e') = e'$ by assumption. Hence u is a connection of e and e' and $u \in X$, a contradiction. Therefore the occurrence of u in the representation of $f(x')$ implies $x' \in X_{p,e} \cup X_{1^\iota} \cup X_{1^\iota}^\square$. A suitable substitution $x' \rightsquigarrow x' - tx$ gives the desired conclusion (9.5).

Case 3 $x \in X_{1^\iota}$ and $f(x') = x'$ for all $x' \in X_{1^\nu,p} X_{1^\nu}^\square$.

Case 3.1 $f(x) \notin \langle XX^\square V_{1^\nu,p}\rangle$.

Again we set $f(x) = u \in \overline{X}$ w.l.o.g. If u occurs in the representation of $f(x')$, then $x' \in X_{1^\iota} \cup X_{1^\iota}^\square$. Hence suitable substitutions $x' \rightsquigarrow x' - tu$ give us (9.5).

Case 3.2 $f(x) \in \langle XX^\square V_{1^\nu,p}\rangle$.

Then we choose

$$f(x) = u + v \quad \text{where} \quad u \in \overline{X}_{1^\nu,p} \setminus XX^\square Q \quad \text{and} \quad v \in \langle \overline{X} \setminus \{u\}\rangle.$$

If u occurs in the representation of $f(x')$, then the assumption in Case 3 implies that $x' \in X_{1^\iota} \cup X_{1^\iota}^\square$. The substitutions as given above ensure (9.5).

We continue the proof of (c). We choose f, x, and u according to (9.5). There is some $y \in YD^\square$ with $s_{xy}\beta(x,y) \neq 0$ in (9.2) and $\beta(x,y) \neq 0$ for all $y \in YY^\square$ since $x \in X^e$. By $(9.4)^+$ we have $\sum\limits_{y \in YY^\square} r_{xy}\beta(x,y) \neq 0$ modulo $\langle \beta(V_G, V_G)\rangle$. We will modify the y's in $YY^\square$ that do not occur in (b) inside their cosets modulo V_G. This way we will get a contradiction to $(9.4)^+$.

By $(9.2)^{Y^\square}$ we obtain

$$f(d) = \sum_{\substack{x_1 \in XX^\square \\ y_1 \in YY^\square}} r_{x_1 y_1}\beta(f(x_1), f(y_1)) + \sum_{\substack{y_1 < y_2 \\ y_i \in YY^\square}} r_{y_1 y_2}\beta(f(y_1), f(y_2)) \text{ modulo } \langle \beta(V_G, V_G)\rangle.$$

Using the linear representations of $f(x_1)$ and $f(y_1)$ for all $x_1 \in XX^\square$ and $y_1 \in YY^\square$ over $\overline{X}\,\overline{Y}$ we compute the unique linear representation of $f(d)$ over the set of all basic commutators $\beta(a,b)$ with $a, b \in \overline{X}\,\overline{Y}$. For $y \in YY^\square$ let us summarize all summands of the form $r\beta(u,y)$. From $\sum\limits_{\substack{x_1\in XX^\square\\ y_1\in YY^\square}} r_{x_1y_1}\beta(f(x_1),f(y_1))$ we get only $r_{xy}\beta(u,y)$ since by (b) $f(y_1)=y_1$ modulo V_G and by (9.5) $f(x_1)$ for $x_1\neq x$ does not contain u in its linear representation over $\overline{X}\,\overline{Y}$. To find the summands $r\beta(u,y)$ in $\sum\limits_{\substack{y_1<y_2\\ y_i\in YY^\square}} r_{y_1y_2}\beta(f(y_1),f(y_2))$ we write $f(y_1)=y_1+t_{y_1}u+w_{y_1}$ for $y_1\in YY^\square$ where $w_{y_1}\in\langle\overline{X}\setminus\{u\}\rangle$. Then we obtain

$$\sum_{y_1<y} r_{y_1y}t_{y_1}\beta(u,y)-\sum_{y<y_1} r_{yy_1}t_{y_1}\beta(u,y)$$

as the desired sum. But since $f(d)=d$ modulo $\beta(V_G,V_G)$ there is no summand $r\beta(u,y)$ in the linear representation of $f(d)$. By linear independence

$$\sum_{y_1<y} r_{y_1y}t_{y_1}\beta(u,y)-\sum_{y<y_1} r_{yy_1}t_{y_1}\beta(u,y)=-r_{xy}\beta(u,y).$$

By (9.5) $\beta(u,y)\neq 0$ for $y\in YY^\square$. Hence we have

$$\text{(9.6)}\qquad \sum_{y_1<y} r_{y_1y}t_{y_1}-\sum_{y<y_1} r_{yy_1}t_{y_1}=-r_{xy}$$

for all y. Now we replace every $y\in YY^\square$ by $y^*=y+t_yx$ if $f(y)=y+t_yu+w_y$; this yields a substitution for $y\in D^\square$ as well. (9.6) and $(9.2)^{Y^\square}$ imply

$$d=\sum_{\substack{x'\in XX^\square\setminus\{x\}\\ y\in YY^\square}} r_{x'y}\beta(x',y^*)+\sum_{\substack{y_1<y_2\\ y_i\in YY^\square}} r_{y_1y_2}\beta(y_1^*,y_2^*)\text{ modulo }\langle\beta(V_G,V_G)\rangle.$$

Moving commutators from $\beta(V_G,V_G)$ to c we obtain a contradiction to $(9.4)^+$ if we can define a support $(X\setminus\{x\},X^{*\square},Y^*,Y^{*\square},D^{*\square})$ for this new "d".
Let us consider $y\in YY^\square$ or $y\in D^\square$ with $y\neq y^*$. By (b) $y\in Y_{p,e}Y_{p,e}^\square$ for $e\in X_{1^\nu}$ or $y\in Y_{1^\iota}$ or $y\in D^\square$.
If $y\neq y^*$ only for $y\in Y_{1^\iota}$ or $y\in D^\square$, then there is no problem. We obtain $X^{*\square}=X^\square$ and $Y^*Y^{*\square}D^{*\square}$ by replacing y by y^*.
For $y\in Y_{p,e}Y_{p,e}^\square$ with $y\neq y^*$ we have to be more careful. By (b) we know $f(e)=e$. We assume that there is some $y_0\in Y_p$ with $y_0^*\neq y$. To define $(X\setminus\{x\},X^{*\square},Y^*,Y^{*\square},D^{*\square})$ we distinguish the three cases of the proof of (9.5). In Case 3 we have $x\in X_{1^\iota}$. In this case we define

$$\begin{aligned}
D^{*\square} &= \{y^*: y\in D^\square \text{ or } (y\in Y_p \text{ and } y\neq y^*)\}\\
Y^* &= \{y^*: y\in (Y\setminus Y_p) \text{ or } (y\in Y_p \text{ and } y=y^*)\}\\
Y^{*\square} &= Y^\square\cup\{y\in Y_p: y\neq y^*\}\\
X^{*\square} &= X^\square\cup\{x\}\cup\{e\in X_{1^\nu}:\ \text{there is some } y\neq y^* \text{ in } Y_{p,e}\}
\end{aligned}$$

and obtain a support as desired. In the other cases we define

$$\begin{aligned}
D^{*\square} &= \{y^*: y\in D^\square\},\quad Y^*=\{y^*: y\in Y\}\\
Y^{*\square} &= \{y^*: y\in Y^\square\}\quad\text{and}\quad X^{*\square}=X^\square.
\end{aligned}$$

In these cases we have to show that $\beta(e,x) = 0$ for all e that are handles of some $y \in Y_{p,e}Y^{\Box}_{p,e}$ with $y \neq y^*$. Then $y^* = y + t_y x$ is again of type p with handle e. Since $y^* \neq y$ we have $f(y) = y + tu + w$ with $t \neq 0$ and $w \in \langle \overline{X} \setminus \{u\}\rangle$. $\beta(y,e) = 0$ implies $\beta(f(y), f(e)) = 0$. Since $e = f(e)$ we get $\beta(u,e) = 0$. In Cases 1 and 2.1 $u = f(x)$ and $\beta(u, f(e)) = 0$ implies $\beta(x,e) = 0$, as desired.
In Case 2.2 we have $x \in X_{p,e'}$ and therefore $\beta(x,e') = 0$, $u \in \overline{X}_{1^\nu}$ and $f(e') = e'$. Hence $\beta(u,e') = 0$. Above we have shown $\beta(u,e) = 0$. $e \not\approx e'$ would imply that u is a connection of e and e'. Then u would be an element of X, a contradiction. Hence $e' \approx e$ and $\beta(x,e) = 0$, as desired. □

Proof of (c)^H:
Our assumptions have been the following: d, $(X, X^{\Box}, Y, Y^{\Box}, D^{\Box})$, $I\!K$, and $I\!K^H$ are chosen such that $(X, X^{\Box}, Y, Y^{\Box}, D^{\Box})$ is a support of d, (a), (a)^H, (b), (b)^H, (9.2), (9.3), (9.4) and also (c) are true.
Using $\mathrm{acl}(aG) \cap H = G$, and $YY^{\Box} \subseteq \mathrm{acl}(aG)$ we obtain that $YY^{\Box}$ is linearly independent modulo V_H. Hence $YY^{\Box}$ is linearly independent modulo V_H. Now we choose again a platform $(X^H, X^{H\Box}, Y^H, Y^{H\Box}, D^{H\Box})$ for $\langle XYY^{\Box}\rangle$ with respect to H. We can choose this platform such that $X \subseteq X^H$ and $Y_{1^\nu} \cup \bigcup_{e \in Y_{1^\nu}} Y_{p,e} \subseteq Y^H$. To get this we apply first Lemma 6.7 to $\langle X\rangle \subseteq \langle X^H\rangle$. Since $\left\langle X^H X^{H\Box} Y_{1^\nu} \cup \bigcup_{e\in Y_{1^\nu}} Y_{p,e}\right\rangle$ is a good subspace of $\langle X^H X^{H\Box} Y^H\rangle$ we can apply Lemma 6.7 again.
Furthermore we can assume that

$$(+) \qquad YY^{\Box} \subseteq \langle (X^H \setminus X) X^{H\Box} Y^H Y^{H\Box}\rangle.$$

Elements of $Y^H_{1^\iota}$ are w.l.o.g elements of Y_{1^ι}. All elements of $Y^H_{1^\nu} Y^H_p Y^{H\Box}$ occur in a representation of elements of $YY^{\Box}$ as a linear combination over $Y^H Y^{H\Box}$ modulo V_H or as handles or connections as described in Lemma 6.10. Hence by (a)^H and that lemma we have w.l.o.g.:

(ab)$^{H^2}$ Let f be an element of $I\!K^H$. f fixes all elements of $Y^H Y^{H\Box}$ modulo V_H. Furthermore f fixes $Y^H_{1^\nu} \cup \bigcup_{e\in Y^H_{1^\nu}} Y^H_{p,e}$ pointwise and also all handles in X^H of elements of $Y^H_p Y^{H\Box}_p$ and all roots of elements of $Y^H_{1^\nu}$.

Similarly as above let Q^H be a part of a 1^ν-transversal for the subspace of roots of $Y^H_{1^\nu}$ in V_H such that $X^H_{1^\nu} \cap Q^H$ is a basis for $\langle X^H_{1^\nu}\rangle \cap \langle Q^H\rangle$.
Now we choose $I\!K^H$ such that in addition $(\mathrm{c}^H)^{\mathrm{weak}}$ is true:

$(\mathrm{c}^H)^{\mathrm{weak}}$ Let x be an element of X. If $f(x) \in \langle X^H X^{H\Box} Q^H\rangle$ for all $f \in I\!K^H$, then $f(x) = x$.

By (b)^H we already know that the automorphisms of $I\!K^H$ fix the elements of $Y_{1^\nu} \cup \bigcup_{e\in Y_{1^\nu}} Y_{p,e}$, the handles x of some $y \in Y_{p,x} \cup Y^{\Box}_{p,x}$ and the elements of Q^H.

We have to show $f(x) = x$ for $x \in X^e$ and $f \in I\!K^H$. Our main argument is $f(d) = d$ modulo $\beta(V_H, V_H)$ for $f \in I\!K^H$. We assume that there are $f \in I\!K^H$, $x \in X^e$ such that $f(x) \notin \langle X^H X^{H\Box} Q^H \rangle$ and obtain a contradiction. Similarly as in the proof of (c) we show:

(9.5^H) There are a transversal $\overline{X}^H \overline{Y}^H$ of V where $\overline{X}^H$ is a transversal of V_H, $X^H X^{H\Box} Q^H \subseteq \overline{X}^H$, and $Y^H Y^{H\Box} \subseteq \overline{Y}^H$ and some $u \in \overline{X}^H \setminus X^H X^{H\Box} Q^H$ such that $f(x) = u + v$ with $v \in \langle \overline{X}^H \setminus \{u\} \rangle$, $f(x') \in \langle \overline{X}^H \setminus \{u\} \rangle$ for $x' \in XX^\Box \setminus \{x\}$, and for $y \in Y^H Y^{H\Box}$ we have $\beta(u, y) \neq 0$.

Note $\beta(u, y) \neq 0$ for $y \in Y^H Y^{H\Box}$ is clear since $u \notin \langle X^H X^{H\Box} Q^H \rangle$ and all roots and handles of $Y^H Y^{H\Box}$ are in this subspace.

Case 1 We can choose x in X_{1^ν}. W.l.o.g. $f(x) \in \overline{X}^H_{1^\nu}$. We define $u = f(x)$ and replace $x' \in XX^\Box \setminus \{x\}$ by $x' - tx$ if tu occurs in the representation of $f(x')$. Again $t \neq 0$ implies $x' \in X_{p,e}$ or $x' \in X_{1^\iota} X^\Box_{1^\iota}$. For $x' \in X_{p,e}$ we have to show $\beta(x, e) = 0$. But $t \neq 0$ implies $\beta(u, f(e)) = 0$ and therefore $\beta(x, e) = 0$ as desired.

Case 2 $x \in X_{p,e}$ and $f(x') = x'$ for all $x' \in X^e_{1^\nu}$.

Case 2.1 $f(x) \notin \langle X^H X^{H\Box} V_{1^\nu} \rangle$.

We define $u = f(x)$ and choose $\overline{X}^H$ such that $u \in \overline{X}^H$. u is a proper element of type p with handle $f(e)$. u cannot occur in the representation of $f(x')$ for $x' \in X_{1^\nu}$.

Let $x' \in XX^\Box$ be such that u occurs in the representation of $f(x')$. Assume $x' \in X_{p,e'}$. Then $\beta(u, f(e')) = 0$. We have $\beta(u, f(e)) = 0$. Hence $f(e) \approx f(e')$ since u was a proper element of type p. We get $e \approx e'$ and $x' \in X_{p,e}$. A suitable substitution $x' - tx$ is possible. If $x' \in X_{1^\iota} X^\Box_{1^\iota}$, then we have no difficulties with such a substitution.

Case 2.1 provides us $f(x) = x$ for all handles x and therefore for all connections in X. Hence we can formulate

Case 2.2 $(c)^H$ is true for all $x' \in X_{1^\nu}$ and there is some $x \in X_{p,e}$ with $f(x) \notin \langle X^H X^{H\Box} Q^H \rangle$ but $f(x) \in \langle X^H X^{H\Box} V_{1^\nu} \rangle$.

By assumption we have $f(e) = e$. We choose u in the representation of $f(x)$ such that $u \in \overline{X}^H_{1^\nu} \setminus X^H X^{H\Box} Q^H$. Then $u \notin f(X_{1^\nu})$. If u occurs in the representation of $f(x')$ for some $x' \in X_{p,e'}$ with $e' \not\approx e$ then $\beta(u, e) = 0$ and $\beta(u, e') = 0$ since $f(e) = e$ and $f(e') = e'$. Since u is a connection of e and e', we get $u \in X$, a contradiction. Therefore the occurrence of u in the representation of $f(x')$ implies $x' \in X_{p,e} \cup X_{1^\iota} \cup X^\Box_{1^\iota}$. A suitable substitution $x' \rightsquigarrow x' - tx$ gives the desired conclusion.

Case 3^H $x \in X_{1^\iota}$ and $f(x') = x'$ for all $x' \in X_{1^\nu, p}$.

As above in Case 3 we choose u in the representation of $f(x)$ in $\langle \overline{X}^H \rangle$ such that $u \notin X^H X^{H\Box} Q^H$. Possible substitutions $x' - tx$ occur only for $x' \in X_{1^\nu} X^\Box_{1^\iota}$. Hence there are no problems.

Now we choose f, x, and u according to (9.5^H). We have $\beta(x, y) \neq 0$ for all $y \in Y^H Y^{\Box H}$ by $(ab)^{H^2}$ since $\beta(x, y) = 0$ would imply that x is a root or a handle for y. We use $(9.2)^{Y^\Box}$ to get a representation of $f(d)$ as a linear combination of commutators $\beta(v_1, v_1)$

modulo $\langle\beta(V_H,V_H)\rangle$ with $v_1,v_2\in\overline{X}^H\overline{Y}^H$. We consider the summand $r_{xy}\beta(f(x),f(y))$. If we want to write down $(9.2)^{Y^\square}$ with respect to $\overline{X}^H\overline{Y}^H$, then we have to replace the $y\in YY^\square$ by their representations as linear combinations over $(X^H\setminus X)X^{H\square}Y^HY^{H\square}$. As mentioned above $YY^\square$ is linearly independent modulo V_H.
We get

$$(9.2)^{Y^{H\square}}\qquad d=\sum_{\substack{x'\in X^HX^{H\square}\\ y\in Y^HY^{H\square}}} r^*_{x'y}\beta(x',y)+\sum_{\substack{y_1<y_2\\ y_i\in Y^HY^{H\square}}} r^*_{y_1y_2}\beta(y_1,y_2)\text{ modulo }\langle\beta(V_H,V_H)\rangle.$$

By $(\mathrm{ab})^{H^2}$ we have $f(y)=y$ modulo V_H for $y\in Y^HY^{H\square}$. Furthermore $d=f(d)$ modulo $\langle\beta(V_H,V_H)\rangle$. Since $u\in\overline{X}^H\setminus X^HX^{H\square}Q^H$, it does not occur in the unique representation as a linear combination of any $y\in YY^\square$ over $X^HX^{H\square}Y^HY^{H\square}\subseteq\overline{X}^H\overline{Y}^H$. Hence in the representation of d and therefore of $f(d)$ over $\overline{X}^H\overline{Y}^H$ any commutator $\beta(u,y)$ with $y\in Y^HY^{H\square}$ does not occur.
On the other side in the representation of $f(d)$ as a linear combination of basic commutators over $\overline{X}^H\overline{Y}^H$ we get $r^*_{xy}\beta(u,y)$ from $r^*_{xy}\beta(f(x),f(y))$ in $(9.2)^{Y^{H\square}}$. By $(9.5)^H$ the other $r^*_{x'y}\beta(f(x'),f(y))$ do not produce any $\beta(u,y)$ for $x'\in X^HX^{H\square}\setminus\{x\}$.
Let $f(y)$ be $y+t_yu+w_y$ for $y\in Y^HY^{H\square}$ where $w_y\in(\overline{X}\setminus\{u\})$. Then the second sum in $(9.2)^{Y^{H\square}}$ produces

$$\sum_{y_1<y} r^*_{y_1y}t_{y_1}\beta(u,y)-\sum_{y<y_1} r^*_{yy_1}t_{y_1}\beta(u,y)=\Big(\sum_{y_1<y} r^*_{y_1y}t_{y_1}-\sum_{y<y_1} r^*_{yy_1}t_{y_1}\Big)\beta(u,y).$$

Note that $\beta(u,y)\neq 0$ for all $y\in Y^HY^{H\square}$. $\beta(u,y)=0$ would imply that u is a handle or root of y and therefore an element of $\langle X^HX^{H\square}Q^H\rangle$, a contradiction. Hence we have

$$(9.6^H)\qquad \sum_{y_1<y} r^*_{y_1y}t_{y_1}-\sum_{y<y_1} r^*_{yy_1}t_{y_1}=-r^*_{xy}$$

for $y\in Y^HY^{H\square}$. Now we replace $y\in Y^HY^{H\square}$ according to this by $y^*=y+t_yx$ if $f(y)=y+t_yx+w_y$. If we use (9.6^H) in $(9.2)^{Y^{H\square}}$, then we have

$$(9.2)^*\quad d=\sum_{\substack{x'\in X^HX^{H\square}\setminus\{x\}\\ y\in Y^HY^{H\square}}} r^*_{x'y}\beta(x',y^*)+\sum_{\substack{y_1<y_2\\ y_i\in Y^HY^{H\square}}} r^*_{y_1y_2}\beta(y_1^*,y_2^*)\text{ modulo }\langle\beta(V_H,V_H)\rangle.$$

If $v\in YY^\square$ and

$$v=\sum_{y\in Y^HY^{H\square}} h_yy+\sum_{z\in X^HX^{H\square}\setminus X} h_zz,$$

then we define

$$v^*=v+t_vx\quad\text{ where }\ t_v=\sum_{y\in Y^HY^{H\square}} h_yt_y.$$

Hence

$$v^* = \sum_{y \in Y^H Y^{H\square}} h_y y^* + \sum_{z \in X^H X^{H\square} \setminus X} h_z z.$$

If we replace $v \in YY^\square$ by v^* in $(9.2)^{Y^\square}$, then we obtain

$$e = \sum_{\substack{x \in XX^\square \\ v \in YY^\square}} r_{xv} \beta(x, v^*) + \sum_{\substack{v_1 < v_2 \\ v_i \in YY^\square}} r_{v_1 v_2} \beta(v_1^*, v_2^*).$$

In the computation of $(9.2)^{Y^{H\square}}$ from $(9.2)^{Y^\square}$ we replaced $v \in YY^H$ by its linear representation over $Y^H Y^{H\square} X^H X^{H\square} \setminus X$. Now we do the same formal steps with e where y^* plays the role of $y \in Y^H Y^{H\square}$. That means we use

$$v^* = \sum_{y \in Y^H Y^{H\square}} h_y y^* + \sum_{z \in X^H X^{H\square} \setminus X} h_z z.$$

We obtain

$$e = \sum_{\substack{x' \in X^H X^{H\square} \\ y \in Y^H Y^{H\square}}} r^*_{x'y} \beta(x', y^*) + \sum_{\substack{y_1 < y_2 \\ y_i \in Y^H Y^{H\square}}} r^*_{y_1 y_2} \beta(y_1^*, y_2^*) \text{ modulo } \langle \beta(V_H, V_H) \rangle.$$

By $(9.2)^*$

$$e = d + \sum_{y \in Y^H Y^{H\square}} r^*_{xy} \beta(x, y^*) \text{ modulo } \langle \beta(V_H, V_H) \rangle.$$

If we consider $(9.2)^{Y^\square}$ modulo $\langle \beta(V_H, V_H) \rangle$ and compare it with $(9.2)^{Y^{\square H}}$, then we see

$$\begin{aligned} \sum_{y \in Y^H Y^{H\square}} r^*_{xy} \beta(x, y^*) &= \sum_{y \in Y^H Y^{H\square}} r^*_{xy} \beta(x, y) = \sum_{v \in YY^\square} r_{xv} \beta(x, v) \\ &= \sum_{v \in YY^\square} r_{xv} \beta(x, v^*) \text{ modulo } \langle \beta(V_H, V_H) \rangle. \end{aligned}$$

We use $YY^\square \subseteq \langle Y^H Y^{H\square} X^H X^{H\square} \setminus X \rangle$ and that any set of commutators $\beta(a, b) \neq 0$, where $a < b$ are elements of a given transversal, is linearly independent. Then we have

$$d = \sum_{\substack{x' \in X \setminus \{x\} \\ y \in YY^\square}} r_{x'y} \beta(x', y^*) + \sum_{\substack{y_1 < y_2 \\ y_i \in YY^\square}} r_{y_1 y_2} \beta(y_1^*, y_2^*) \text{ modulo } \langle \beta(V_H, V_H) \rangle.$$

As in the proof of (c) we find a support $(X \setminus \{x\}, X^{*\square}, Y^*, Y^{*\square}, D^{*\square})$ for d, such that $\{y^* : y \in YY^\square\} = Y^* Y^{*\square}$. We get the desired contradiction to $(9.4)^+$. $\square$

We repeat our assumptions:

(9.7) $(X, X^\square, Y, Y^\square, D^\square)$ is a support of d and $\mathbb{K} \subseteq \mathrm{Aut}_{\{G\}(a)}(\mathbb{C})$ and $\mathbb{K}^H \subseteq \mathrm{Aut}_{\{H\}(a)}(\mathbb{C})$ both of finite index are given such that (9.2), (9.3), (9.4), (a), (b), (c), $(a)^H$, $(b)^H$, and $(c)^H$ are true. All roots of Y are in X.

Furthermore remember that $YD^\square$ is linearly independent modulo V_H. We use again

$$(9.2) \qquad d = \sum_{\substack{x \in X \\ y \in YD^\square}} s_{xy}\beta(x,y) + \sum_{\substack{y_1 < y_2 \\ y_i \in YD^\square}} s_{y_1y_2}\beta(y_1,y_2).$$

We split $Y \cup D^\square$ into U^* and P^* where

$$P^* = Y_{1^\nu} \cup \bigcup_{e \in Y_{1^\nu}} Y_{p,e} \cup P$$

with

$$\begin{aligned} P \;=\; & \{y \in Y \cup D^\square : s_{yy'}\beta(y,y') \neq 0 \text{ or } s_{y'y}\beta(y',y) \neq 0 \\ & \text{for some } y' \in Y \cup D^\square\} \setminus \Big(Y_{1^\nu} \cup \bigcup_{e \in Y_{1^\nu}} Y_{p,e}\Big) \end{aligned}$$

and

$$U^* = (Y \cup D^\square) \setminus P^*.$$

Now we add to our assumptions that we have chosen a situation where $|P|$ is minimal. Then we have

(d) $\quad f(y) = y$ for $y \in P^*$ and $f \in I\!\!K$ and

(d)^H $\quad g(y) = y$ for $y \in P^*$ and $g \in I\!\!K^H$.

Proof of (d):
If $f(y) \neq y$ for some $f \in I\!\!K$ and $y \in P^*$, then $y \in P$ by (b). By (a) we have $f(y) = y$ modulo V_G for $y \in P$ and $f \in I\!\!K$. If some $y \in P$ satisfies $f(y) \in \langle XX^\square YY^\square \rangle$ for all $f \in I\!\!K$, then there is a subgroup $I\!\!K_0 \subseteq I\!\!K$ of finite index such that $f(y) = y$ for all $f \in I\!\!K_0$ and the considered y. Hence we can assume w.l.o.g.:

(d)$^{\text{weak}}$ For all $y \in P$ we have: If $f(y) \in \langle XX^\square YY^\square \rangle$ for all $f \in I\!\!K$, then $f(y) = y$ for all $f \in I\!\!K$.

To show (d) we assume that there are some $y_0 \in P$ and some $f \in I\!\!K$ such that $f(y_0) \notin \langle XX^\square YY^\square \rangle$. Let $\overline{X}\,\overline{Y}$ be a transversal of V where $\overline{X}$ is a transversal of V_G, $XX^\square \subseteq \overline{X}$, and $YY^\square \subseteq \overline{Y}$. Then w.l.o.g. $f(y_0) = y_0 + u + w$ by (a) where $u \in \overline{X} \setminus XX^\square$ and $w \in \langle \overline{X} \setminus \{u\} \rangle$. We show that we can modify Y, $Y^\square$, and $D^\square$ such that $f(y) \in \langle \overline{XY} \setminus \{u\} \rangle$ for $y \in YD^\square$ and $y \neq y_0$. We distinguish three cases.

Case 1 We can choose y_0 in $Y_{p,e} \cap P$. Since $y_0 \in P$ we have $e \in X$. By (c) $f(e) = e$. Then $\beta(f(e), f(y_0)) = 0$ implies $\beta(e,u) = 0$ and $\beta(e,w) = 0$.
For $y \in YY^\square$ we write $f(y) = y + t_y u + w_y$ where $w_y \in \langle \overline{X} \setminus \{u\} \rangle$. Note that $t_y \neq 0$ implies $y \in Y_{p,e} Y_{1^\nu} Y^\square_{p,e}$ by (b). We define for $y \in YY^\square$ and $y \neq y_0$

$$y' = y - t_y y_0 \quad \text{ and } \quad y'_0 = y_0.$$

We denote by $Y'Y^{\square'}$ the new basis of $\langle Y, Y^\square\rangle$. According to this we obtain $D^{\square'}$. We write $(YD^\square)'$ for $Y'D^{\square'}$. Then $(X, X^\square, Y', Y^{\square'}, D^{\square'})$ is again a support for d, for which (9.7) remains true.

Case 2 For all $f \in I\!K$ and $y \in Y_p$ we have $f(y) \in \langle XX^\square YY^\square\rangle$, but there is some $y_0 \in D^\square$ and $f \in I\!K$ such that w.l.o.g.

$$f(y_0) = y_0 + u + w \text{ where } u \in \overline{X} \setminus XX^\square,\ w \in \langle \overline{X} \setminus \{u\}\rangle.$$

By $(\mathrm{d})^{\mathrm{weak}}$ we can assume $f(y) = y$ for $y \in Y_p$. For $y \in YD^\square$ let $f(y)$ be $y + t_y u + w_y$. $t_y \neq 0$ implies $y \in D^\square Y_{1^\iota}$. For these $y \neq y_0$ we define $y' = y - t_y y_0$. For all other $y \in YD^\square$ we define $y' = y$. Let $Y'D^{\square'}$ be the new sets. We do not change $Y^\square$. But we have a new $X^{\square'}$. Then $(X, X^{\square'}, Y', Y^\square, D^{\square'})$ is a support of d and (9.7) is true.

By $(\mathrm{d})^{\mathrm{weak}}$ again we have the following remaining case

Case 3 For all $f \in I\!K$ and $y \in Y_p \cup D^\square$ we have $f(y) = y$ but there is some $y_0 \in Y_{1^\iota}$ and $f \in I\!K$ such that $f(y_0) \notin \langle XX^\square YY^\square\rangle$. We do the same substitution as in Case 2. It concerns only elements of Y_{1^ι}. Therefore it is easy to see that (9.7) remains true for the new support.

Now we assume w.l.o.g. that $f(y) \in \langle \overline{X}\,\overline{Y} \setminus \{u\}\rangle$ for all $y \in YD^\square$ with $y \neq y_0$. We use

$$(9.2) \qquad d = \sum_{\substack{x\in X\\ y\in YD^\square}} s_{xy}\beta(x, y) + \sum_{\substack{y_1<y_2\\ y_i\in P^*}} s_{y_1y_2}\beta(y_1, y_2)$$

and

$$f(9.2) \qquad f(d) = \sum_{\substack{x\in X\\ y\in YD^\square}} s_{xy}\beta(x, f(y)) + \sum_{\substack{y_1<y_2\\ y_i\in P^*}} s_{y_1y_2}\beta(f(y_1), f(y_2)).$$

If we rewrite (9.2) with respect to $\overline{X}\,\overline{Y}$, then this representation contains no commutator $\beta(u, w)$ with $w \in YY^\square$. But $f(9.2)$ produces

$$\sum_{\substack{y_0<y\\ y\in P^*}} s_{y_0y}\beta(u, y) - \sum_{\substack{y<y_0\\ y\in P^*}} s_{yy_0}\beta(u, y)$$

in the representation of $f(d)$ and this sum contains all commutators $\beta(u, w)$ with $w \in YY^\square$ in the representation of $f(d)$ over $\overline{X}\,\overline{Y}$. Since $u \notin \langle XX^\square\rangle$ the element u is not a root or handle of $YY^\square$ and therefore $\beta(u, w) \neq 0$. Since P^* is linearly independent over V_G and $s_{y_0y} \neq 0$ or $s_{yy_0} \neq 0$ for some y by minimality of $|P|$, we have a non-vanishing commutator $\beta(u, w)$ with $w \in YY^\square$ in the representation of $f(d)$ over $\overline{X}\,\overline{Y}$. This contradicts $d = f(d)$ modulo $\langle \beta(V_G, V_G)\rangle$. □

Proof of $(\mathrm{d})^H$:

By $(\mathrm{b})^H$ $g(y) \neq y$ for some $g \in I\!K^H$ implies $y \in P$. By $(\mathrm{a})^H$ $g(y) = y$ modulo V_H for $y \in P$ and $g \in I\!K^H$. We choose a platform $(X^H, X^{H\square}, Y^H, Y^{H\square}, D^{H\square})$ for $\langle XYD^\square\rangle$ with respect to V_H. We have w.l.o.g.

(d)$^{H\text{weak}}$ For all $y \in P$ we have: If $g(y) \in \langle X^H X^{H\square} Y^H Y^{H\square} \rangle$ for all $g \in \mathbb{K}^H$, then $g(y) = y$ for all $g \in \mathbb{K}^H$.

As in the proof of (c)^H we can assume that $X \subseteq X^H$ and

$$Y_{1^\nu} \cup \bigcup_{e \in Y_{1^\nu}} Y_{p,e} \subseteq Y^H.$$

We use a transversal $\overline{X}^H \overline{Y}^H$ of V where $\overline{X}^H$ is a transversal of V_H, $X^H X^{H\square} \subseteq \overline{X}^H$ and $Y^H Y^{H\square} \subseteq \overline{Y}^H$. As above we choose $y_0 \in YD^\square$ and $g \in \mathbb{K}^H$ with $g(y_0) = y_0 + u + w$ where $u \in \overline{X}^H \setminus X^H X^{H\square}$, $w \in \langle \overline{X}^H \setminus \{u\} \rangle$. Using the argumentation as in the proof of (d) with the three cases we can w.l.o.g. assume that for $y \in YD^\square$ with $y \neq y_0$ the image $g(y)$ does not contain u in its representation over $\overline{X}^H \overline{Y}^H$. If we rewrite (9.2) with respect to $\overline{X}^H \overline{Y}^H$ then commutators $\beta(u, w)$ with $w \in \overline{Y}^H$ do not occur. On the other hand we have

$$g(9.2) \qquad g(d) = \sum_{\substack{x \in X \\ y \in YD^\square}} s_{xy} \beta(x, g(y)) + \sum_{\substack{y_1 < y_2 \\ y_i \in P^*}} s_{y_1 y_2} \beta(g(y_1), g(y_2)).$$

Here we produce

$$\beta \left(u, \sum_{\substack{y_0 < y \\ y \in P^*}} s_{y_0 y} y - \sum_{\substack{y < y_0 \\ y \in P^*}} s_{y y_0} y \right)$$

as the sum of all commutators of the form $\beta(u, w)$ with $w \in Y^H Y^{H\square}$. As above $\beta(u, w) \neq 0$. By the minimality of $|P|$ the linear combination on the right side is non-trivial. Since P^* is linearly independent over V_H there are commutators $\beta(u, w)$ with $w \in Y^H Y^{H\square}$ in the representation of $g(d)$ over $\overline{X}^H \overline{Y}^H$. We have a contradiction to $d = g(d)$ modulo $\langle \beta(V_h, V_H) \rangle$. $\square$

Now it is easy to finish the proof of Theorem 9.1. We look at (9.2) and define

$$a' = \sum_{\substack{x \in X \\ y \in P^*}} s_{xy} \beta(x, y) + \sum_{\substack{y_1 < y_2 \\ y_i \in P^*}} s_{y_1 y_2} \beta(y_1, y_2).$$

By (c) and (d) we have $f(a') = a'$ for $f \in \mathbb{K}$. By (c)^H and (d)^H we have $g(a') = a'$ for $g \in \mathbb{K}^H$. If $y \in U^* \cap D^\square$, then $y = u_y + x_y$ where $u_y \in \langle Y^\square \rangle$ and $x_y \in X^\square$. In this case

$$s_{xy} \beta(x, y) = s_{xy} \beta(x, u_y) + s_{xy} \beta(x, x_y).$$

We move $s_{xy} \beta(x, x_y)$ into c. Let U be $U^* \setminus D^\square$ joint with all elements of $Y^\square$ that occur in some u_y above. Using $a = cd$ and (9.2) we have

$$a = a^G \prod_{\substack{x \in X \\ u \in U}} [x, u]^{s_{xu}} a'$$

where $a^G = c$. To check the remaining conditions in Theorem 9.1 we use (a), (a)^H, (c), (c)^H and the fact that $YY^\square$ is linearly independent modulo V_H. $\square$

10 Finale

We show Theorem 8.1. By Corollary 8.7 we can assume that $A \cap G = A \cap G'$ and $A \cap (G \cdot Z(\mathbb{C})) = A \cap (G \cdot \mathbb{C}')$. We compute $\mathrm{Cb}(\mathrm{tp}(A/G))$ up to the algebraic closure and show that it is included in $\mathrm{acl}(\mathrm{Cb}(\mathrm{tp}(A/H)))$. Let A_0 be a basis of $A \cap \mathbb{C}'$. Choose $A_1 \subseteq A$ such that $A_1/Z(\mathbb{C})$ is a basis of $A/Z(\mathbb{C})$ modulo $G/Z(\mathbb{C})$. By our assumption $\mathrm{acl}(AG) \cap H = G$ we have that $A_1/Z(\mathbb{C})$ is also linearly independent modulo $H/Z(\mathbb{C})$. Consider $A \cap G = A \cap G' \subseteq A \cap \mathbb{C}' \subseteq A \cap (G \cdot \mathbb{C}') = A \cap (G \cdot Z(\mathbb{C}))$. Then $A = \langle A_0 \cup A_1 \cup \{a_1, \ldots, a_m\}\rangle$ where $a_1, \ldots, a_m$ are mixed elements. We can choose $a_1, \ldots, a_m$ in such a way that there is some ℓ with $0 \leq \ell \leq m$, $a_1, \ldots, a_\ell$ are linearly independent modulo $Z(\mathbb{C})$, and $a_{\ell+1}, \ldots, a_m \subseteq Z(\mathbb{C})$ are linearly independent modulo $\mathbb{C}'$. By Theorem 9.1 there are $a_i^G \in G$, $a_i' \in \mathbb{C}'$, parts of a transversal $X^i U^i$ $(1 \leq i \leq m)$ that respect V_G with $\langle X^i U^i\rangle \cap V_G = \langle X^i \rangle$ and $U^i_{1^\nu} \cup \bigcup_{e \in U^i_{1^\nu}} U^i_{p,e} = \emptyset$, and subgroups $\mathbb{K} \subseteq \mathrm{Aut}_{\{G\}(A)}(\mathbb{C})$ and $\mathbb{K}^H \subseteq \mathrm{Aut}_{\{H\}(A)}(\mathbb{C})$ both of finite index such that

$$(10.1) \qquad a_i = a_i^G \prod_{\substack{x \in X^i \\ u \in U^i}} [x, u]^{s^i_{xu}} a_i' ,$$

and every $f \in \mathbb{K}$ fixes all a_i' and X^i pointwise, and fixes U^i pointwise modulo V_G, and every $g \in \mathbb{K}^H$ fixes all a_i' and X^i pointwise, and fixes U^i pointwise modulo V_H.
Since $f(a_i') = a_i'$ for $f \in \mathbb{K}$ and $g(a_i') = a_i'$ for $g \in \mathbb{K}^H$ we have

$$\begin{aligned} \mathrm{acl}(\mathrm{Cb}(\mathrm{tp}(A \cup \{a_1', \ldots, a_m'\}/G))) &= \mathrm{acl}(\mathrm{Cb}(\mathrm{tp}(A/G))) \qquad \text{and} \\ \mathrm{acl}(\mathrm{Cb}(\mathrm{tp}(A \cup \{a_1', \ldots, a_m'\}/H))) &= \mathrm{acl}(\mathrm{Cb}(\mathrm{tp}(A/H))). \end{aligned}$$

Hence we can w.l.o.g. assume that $a_i' \in \langle A_0 \rangle \subseteq A$ for $1 \leq i \leq m$. We can choose $a_1, \ldots, a_m$ in such a way that

$$(10.2) \qquad a_i = a_i^G \prod_{\substack{x \in X^i \\ u \in U^i}} [xu]^{s^i_{xu}} .$$

Using Theorem 6.9 and Corollary 6.4 there is a good subspace D such that $A \cap \mathbb{C}' \subseteq \beta(D, D)$, $A/Z(\mathbb{C}) \subseteq D$, $X^i \subseteq D$ for $1 \leq i \leq m$ and conversely $D \subseteq \mathrm{acl}\Big((A \cap \mathbb{C}') \cup A/Z(\mathbb{C}) \cup \bigcup_{1 \leq i \leq m} X_i\Big)$. W.l.o.g. we can assume that D is fixed pointwise by all automorphisms in $\mathbb{K}$ and in $\mathbb{K}^H$.
By Theorem 7.4 there is a platform $(X, X^\square, Y, Y^\square, D^\square)$ for D with respect to V_G. By Theorem7.4iii) $\mathrm{Aut}_{\{G\}(D)}(\mathbb{C})$ and $\mathrm{Aut}_{\{G\}(XYD^\square)}(\mathbb{C})$ are commensurable, as are analogously $\mathrm{Aut}_{\{H\}(D)}(\mathbb{C})$ and $\mathrm{Aut}_{\{H\}(XYD^\square)}(\mathbb{C})$ by Theorem 7.4iv). Hence $XYD^\square$ is w.l.o.g. fixed pointwise by all automorphisms $f \in \mathbb{K}$ and $g \in \mathbb{K}^H$. Furthermore we have w.l.o.g. $f(y) = y$ modulo V_G for $y \in Y^\square$ and $f \in \mathbb{K}$, and similarly $g(y) = y$ modulo V_H for $y \in Y^\square$ and $g \in \mathbb{K}^H$.
To unify (10.2) we will enlarge D. Let D^* be the smallest good subspace that contains

$XX^\Box YY^\Box$ and $\bigcup_{1\le i\le m} U^i$. Again by Theorem 7.4 there is a platform $(X^*, X^{*\Box}, Y^*, Y^{*\Box}, D^{*\Box})$ for D^* with respect to V_G. W.l.o.g. $XX^\Box \subseteq X^*$ and $YY^\Box \subseteq Y^*$. $\langle V_G Y^* Y^{*\Box}\rangle$ is the smallest good subspace that contains V_G, D and $\bigcup_{1\le i\le m} U^i$.
Therefore this space and also V_G are fixed setwise by all $f \in I\!K$. We can replace $I\!K$ in such a way that furthermore $f(y) = y$ modulo V_G for all $y \in Y^*Y^{*\Box}$ and $f \in I\!K$. $\langle V_H Y^* Y^{*\Box}\rangle$ is contained in the smallest good subspace that contains V_H, D, and $\bigcup_{1\le i\le m} U_i$. This subspace is finite modulo V_H. Again we can w.l.o.g. assume that $g(y) = y$ modulo V_H for $y \in Y^*Y^{*\Box}$ and $g \in I\!K^H$. Let X^+ be the extension of X by handles of $Y^*Y^{*\Box}$ in V_G. Let Y^+ be $Y \cup Y^*_{1\nu} \cup \bigcup_{e\in Y^*_{1\nu}} Y^*_{p,e}$ and $D^+ = \langle D \cup X^+ \cup Y^+\rangle$.
Note $Y^\Box \cap Y^+ = \emptyset$. $(X^+, X^\Box, Y^+, Y^\Box, D^\Box)$ is a platform for D^+. Again w.l.o.g. D^+, X^+, Y^+, $D^\Box$ are fixed pointwise by automorphisms of $I\!K$ and $I\!K^H$. Therefore $\mathrm{Cb}(\mathrm{tp}(D^+/G)) \subseteq \mathrm{acl}(\mathrm{Cb}(\mathrm{tp}(A/G)))$ and $\mathrm{Cb}(\mathrm{tp}(D^+/H)) \subseteq \mathrm{acl}(\mathrm{Cb}(\mathrm{tp}(A/H)))$. By Corollary 7.5 we have

$$\mathrm{acl}(\mathrm{Cb}(\mathrm{tp}(D^+/G))) = \mathrm{acl}(\mathrm{Cb}_\Gamma(\mathrm{tp}_\Gamma(Y^{+\approx}_{1\nu}/\Gamma(G)) \cup X^+ \cup \underline{\Box}))$$

and by Corollary 7.6 we have

$$\mathrm{Cb}(\mathrm{tp}(D^+/G)) \subseteq \mathrm{acl}(\mathrm{Cb}(\mathrm{tp}(D^+/H))).$$

We will show that we obtain $\mathrm{acl}(\mathrm{Cb}(\mathrm{tp}(A/G)))$ from $\mathrm{acl}(\mathrm{Cb}(\mathrm{tp}(D^+/G)))$ if we replace $\underline{\Box}$ by another imaginary element over X^+ and that this imaginary element is also contained in $\mathrm{acl}(\mathrm{Cb}(\mathrm{tp}(A/H)))$. Then CM-triviality is shown. Note $Y^+ \subseteq Y^*$ and $Y^\Box \subseteq (Y^*)\setminus Y^+$. Let U be $(Y^*Y^{*\Box})\setminus Y^+$. We can write $u \in U^i$ as a linear combination over $X^*X^{*\Box}Y^*Y^{*\Box}$. If $u = u_0 + u_1 + u_2$ with $u_0 \in \langle X^*X^{*\Box}\rangle$, $u_1 \in Y^+$ and $u_2 \in U$, then for $x \in X$

$$[x, u] = [x, u_0][x, u_1][x, u_2].$$

Let us consider (10.1) again. Since $[x, u_1]$ is fixed by all automorphisms in $I\!K$ and in $I\!K^H$ we can assume w.l.o.g. that $[x, u_1]$ is an element of $A \cap \mathbb{C}'$. $[x, u_0]$ is in G and can be moved to a_i^G. Hence we can rewrite (10.2) as

$$a_i = a_i^G \prod_{\substack{x\in X\\ u\in U}} [x, u]^{s^i_{xu}}. \qquad (10.3)$$

As assumed above $a_1^G, \ldots, a_\ell^G$ are linearly independent modulo $Z(\mathbb{C})$ and $a_{\ell+1}^G, \ldots, a_m^G \subseteq Z(G)$ are linearly independent modulo G'.
By definition of D all $a_i^G/Z(\mathbb{C})$ for $1 \le i \le \ell$ are in $D \cap V_G$ and therefore in $\langle X\rangle$. Let $X^u \subseteq X^+_{1\nu}$ be the set of handles for those elements in U that are of type p and have a handle in V_G. According to the special triple $(X^\Box, Y^\Box, D^\Box)$ let $d = x_d + y_d$ for $d \in D^\Box$, where $x_d \in X^\Box$ and $y_d \in \langle Y^\Box\rangle$. Using (10.3) we extend the definition of $\theta_\Box$. Note $Y^\Box \subseteq U$.

Definition Let $\bar{v}^i = (v^i_d : d \in D^\square)^\wedge(v^i_1, \ldots, v^i_m)$ $(i = 0, 1)$ be two sequences in $V^{|D^\square|} \times \mathbb{C}^m$, where $v^i_d \in V$ and $v^i_j \in \mathbb{C}$. We say $\theta_\Delta(\bar{v}^0, \bar{v}^1)$, if there is a $\mathbb{B}_p$-homomorphism h of $\langle X^u U\rangle$ into $\mathcal{F}(\mathbb{C})$ with $h(x) = x$ for $x \in X^u$ such that

$$\begin{aligned} v^0_d &= v^1_d + h(y_d) \quad \text{for } \ d \in D^\square \ \text{ and} \\ v^0_i &= v^1_i \prod_{\substack{x \in X \\ u \in U}} [x, h(u)]^{s^i_{xu}} \quad \text{for } \ 1 \le i \le m. \end{aligned}$$

The intention of this definition is that we are interested in the θ_Δ-class of $(x_d : d \in D^\square)^\wedge(a^G_1, \ldots, a^G_m)$.
Note:

- In $\langle X^u U\rangle$ $\beta(e, y) = 0$ for $e \in X^u$ and $y \in U_{p,e}$ are the only relations. Then we have $\beta(e, h(y)) = 0$ in this case. Hence in the definition above we can characterize h as an vector space homomorphism with $h(x) = x$ for $x \in X^u$ and $\beta(e, h(y)) = 0$ for $e \in X^u$ and $u \in U_{p,e}$.
- $\theta_\Delta(\bar{v}^0, \bar{v}^1)$ defines an equivalence relation.
- θ_Δ is definable over X.
- If X is fixed pointwise, then we consider the θ_Δ-classes as elements of $\mathbb{C}^{\mathrm{eq}}$.
- Under this assumption let $\underline{\Delta}$ be the θ_Δ-class of $(x_d : d \in D^\square)^\wedge(a^G_1, \ldots, a^G_m)$.
- $\underline{\square}$ is in $\mathrm{dcl}(\underline{\Delta} X)$.

For $f \in \mathbb{K}$ we have $f(d) = d$ for $d \in D^\square$ and $f(a_i) = a_i$. For $u \in U$ we have $f(u) = u + v_u$ where $v_u \in V_G$. Let k be the vector space homomorphism of $\langle X^u U\rangle$ into V_G defined by $k(u) = v_u$ for $u \in U$ and $k(x) = x$ for $x \in X^u$. It can be extended to an $\mathbb{B}_p$-homomorphism that we call again k. Since $f(d) = d$ for $d \in D^\square$ implies

$$f(x_d + y_d) = f(x_d) + f(y_d) = x_d + y_d,$$

we obtain

$$f(x_d) = x_d + (y_d - f(y_d)) = x_d + v_{y_d} = x_d + k(y_d).$$

Furthermore

$$a_i = f(a_i) = f(a^G_i) \prod_{\substack{x \in X \\ u \in U}} \beta(x, u)^{s^i_{xu}} \prod_{\substack{x \in X \\ u \in U}} \beta(x, v_u)^{s^i_{xu}}$$

and therefore

$$a^G_i = f(a^G_i) \prod_{\substack{x \in X \\ u \in U}} \beta(x, k(u))^{s^i_{xu}}.$$

We have shown that

$$\theta_\Delta((x_d : d \in D^\Box)^\wedge(a_1^G, \ldots, a_m^G),\ (f(x_d) : d \in D^\Box)^\wedge(f(a_1^G), \ldots, f(a_m^G)))$$

holds. Hence $\underline{\Delta}$ is preserved by all $f \in I\!K$. Therefore $\underline{\Delta} \in \mathrm{acl}(\mathrm{Cb}(\mathrm{tp}(A/G)))$. Since all $g \in K^H$ fix $D^\Box$, X and all a_i pointwise, we can use the same steps to prove $\underline{\Delta} \in \mathrm{acl}(\mathrm{Cb}(\mathrm{tp}(A/H)))$.
Now we know that $\mathrm{acl}[\mathrm{Cb}(\mathrm{tp}(D^+(G))) \cup \underline{\Delta})]$ is contained in $\mathrm{acl}(\mathrm{Cb}(\mathrm{tp}(A/G)))$ and in $\mathrm{acl}(\mathrm{Cb}(\mathrm{tp}(A/H)))$. Hence it is sufficient to show that

$$\mathrm{acl}(\mathrm{Cb}(\mathrm{tp}(D^+/G)) \cup \underline{\Delta}) = \mathrm{acl}(\mathrm{Cb}(\mathrm{tp}(A/G))).$$

Consider $f^G \in \mathrm{Aut}(G)$ fixing $\mathrm{Cb}(\mathrm{tp}(D^+/G)) \cup \underline{\Delta}$ pointwise. We will extend f^G to $f \in \mathrm{Aut}_{\{G\}}(\mathbb{C})$ that fixes A pointwise. By the assumption we can extend $\mathcal{F}(f^G)$ to $(g, h) \in \mathrm{Aut}_{\{\mathcal{F}(G)\}}(\mathcal{F}(\mathbb{C}))$ with $g(d) = d$ for all $d \in D^+$. Since f^G fixes $\underline{\Delta}$ we have a $I\!B_p$-homomorphism k of $\langle X^u U\rangle$ into V_G such that $g^G(x_d) = x_d + k(y_d)$ for $x_d \in X^\Box$, and $f^G(a_i^G) = a_i^G \prod_{\substack{x\in X\\ u\in U}} [x, k(u)]^{s^i_{xu}}$ for $1 \le i \le m$.
Now we change (g, h) into (g^*, h^*). Let $\overline{X}\,\overline{Y}$ be a transversal of V such that $\overline{X}$ is a transversal of V_G, $X^+X^\Box \subseteq \overline{X}$ and $Y^+U \subseteq \overline{Y}$. Since $g(d) = d$ for $d \in D^\Box$ we have $d = g(d) = x_d + k(y_d) + g(y_d)$ and therefore $g(y_d) = y_d - k(y_d)$ for $y_d \in \langle Y^\Box\rangle$ with $d = x_d + y_d$. We obtain g^* from g if we define $g^*(v) = g(v)$ for $v \in \overline{X}\,\overline{Y} \setminus U$ and $g^*(u) = u - k(u)$ for $u \in U$. Note $g^*(u) = g(u)$ for $u \in Y^\Box$. Therefore $g^*(y) = g(y)$ for $y \in D^\Box$. Since k is a $I\!B_p$-homomorphism of $\langle X^u U\rangle$ into V_G with $k(x) = x$ for $x \in X^u$ the map g^* induces an automorphism (g^*, h^{**}) of $\langle V, \langle\beta(V, V)\rangle, \beta\rangle$.
Then $g^*(v) = g(v) = v$ for $v \in \langle X^+Y^+D^\Box\rangle$ and therefore for $v \in D^+$. (g^*, h^{**}) extends $\mathcal{F}(f^G)$ and we can extend it to an automorphism (g^*, h^*) of $\langle V, W, \beta\rangle$. Since A_1 is linearly independent modulo $\langle Z(\mathbb{C}) \cup G\rangle$, by Corollary 3.3 there is an automorphism f in $\mathrm{Aut}_{\{G\}}(\mathbb{C})$ that extends f^G with $\mathcal{F}(f) = (g^*, h^*)$ and $f(a) = a$ for $a \in A_1$.
Furthermore f fixes $A \cap \mathbb{C}'$ pointwise since $(A \cap \mathbb{C}') \subseteq \beta(D, D)$. Finally for $1 \le i \le m$

$$\begin{aligned} f(a_i) &= f(a_i^G) \prod_{\substack{x\in X\\ u\in U}} [x, f(u)]^{s^i_{xu}} \\ &= a_i^G \prod_{\substack{x\in X\\ u\in U}} [x, k(u)]^{s^i_{xu}} \prod_{\substack{x\in X\\ u\in U}} ([x, u]^{s_{xu}} [x, k(u)]^{-s^i_{xu}}) \\ &= a_i^G \prod_{\substack{x\in X\\ u\in U}} [x, u]^{s^i_{xu}} = a_i. \end{aligned}$$

We have shown CM-triviality and

$$\mathrm{acl}(\mathrm{Cb}(\mathrm{tp}(A/G))) = \mathrm{acl}[\mathrm{Cb}_\Gamma(\mathrm{tp}_\Gamma(Y_{1^\nu}^{+\approx}/\Gamma(G))) \cup X^+ \cup \underline{\Delta}]. \qquad \Box$$

11 CM-triviality of $\mathrm{Th}(F_2(p,\omega))$

Let p be a prime greater than 2. Let $F_2(p,\kappa)$ be the free group in the variety of all nilpotent groups of class 2 and exponent p with κ free generators. The elementary theory of $F_2(p,\omega)$ is ω-stable [1]. The models G of $\mathrm{Th}(F_2(p,\omega))$ could be considered "Mekler-groups, where V_G contains only elements of type 1^ι". Hence the proof of the preservation of CM-triviality provides us a proof of the following result. We will give some details in this chapter.

Theorem 11.1 $\mathrm{Th}(F_2(p,\omega))$ *is ω-stable and CM-trivial.*

Proof. Again we consider the following situation. Assume $G \preceq H \preceq \mathbb{C}$ are saturated models of $\mathrm{Th}(F_2(p,\omega))$ where $\mathbb{C}$ is big. Let A be a finite subgroup of $\mathbb{C}$ such that $\mathrm{acl}(AG) \cap H = G$. We have to show

$$\mathrm{Cb}(\mathrm{tp}(A/G)) \subseteq \mathrm{acl}(\mathrm{Cb}(\mathrm{tp}(A/H))).$$

As above we use $\mathcal{F}(\mathbb{C}) = \langle V, W, \beta \rangle$ and its substructures. If U is a subspace of V, then a transversal of U is just a basis of this vectorspace. (Every subspace is good!) It is not a problem to find a transversal X of U that respects V_G. That means $X \cap V_G$ is a basis of $U \cap V_G$.

Analogously to Theorem 6.9 it is much easier to show:

Lemma 11.2 *For every $a \in \beta(V,V)$ there is a smallest finite subspace $U(a) \subseteq V$ such that $a \in \beta(U(a),U(a))$. $U(a)$ is definable over a.*

By Corollary 8.7 we can w.l.o.g. assume that $A \cap G = A \cap G'$ and $A \cap (G \cdot Z(\mathbb{C})) = A \cap (G\mathbb{C}')$. The critical case are the mixed elements a in A, where $a \in G \cdot \mathbb{C}'$, $a \notin G$, and $a \notin \mathbb{C}'$.
We can show the following version of Theorem 9.1.

Lemma 11.3 *Let a be a mixed element of A. Then there are $a^G \in G$, $a' \in \mathbb{C}$ and a transversal XU with $\langle XU \rangle \cap V_G = \langle X \rangle$ such that*

$$a = a^G \prod_{\substack{x \in X \\ u \in U}} [x,u]^{s_{xu}} a'.$$

Furthermore there are subgroups $\mathbb{K}$ of $\mathrm{Aut}_{\{G\}(a)}(\mathbb{C})$ and $\mathbb{K}^H$ of $\mathrm{Aut}_{\{H\}(a)}(\mathbb{C})$ both of finite index such that all $f \in \mathbb{K}$ and $g \in \mathbb{K}^H$ fix a' and X pointwise, and U pointwise modulo V_G and V_H respectively.

Sketch of a proof.
Assume $a = cd$ with $c \in G$ and $d \in \mathbb{C}'$. Then d is uniquely determined modulo G'. By Lemma 11.2 there is a smallest finite good subspace $D \subseteq V$ such that $d \in \beta(D,D)$

and D is definable over a. Let XY be a transversal for D with $D \cap V_G = \langle X \rangle$. Then V_G and $\langle V_G \cup D \rangle$ are fixed setwise by all $f \in \mathrm{Aut}_{\{G\}(a)}(\mathfrak{C})$. Hence there is a subgroup $\mathbb{K}$ of $\mathrm{Aut}_{\{G\}(a)}(\mathfrak{C})$ of finite index such that

(a) $f(y) = y$ modulo V_G for every $y \in Y$ and $f \in \mathbb{K}$.

V_H and $\langle V_H \cup D \rangle$ are fixed setwise by all $g \in \mathrm{Aut}_{\{H\}(a)}(\mathfrak{C})$. Since $cd = g(c)g(d)$, we have $c^{-1}g(c) = dg(d)^{-1} \in \mathfrak{C}' \cap H = H'$. We get a subgroup $\mathbb{K}^H$ of $\mathrm{Aut}_{\{H\}(a)}(\mathfrak{C})$ of finite index such that

(a)^H $g(y) = y$ modulo V_H for every $y \in Y$ and $g \in \mathbb{K}^H$.

We can assume

$$(11.1) \qquad d = \sum_{\substack{x \in X \\ y \in Y}} s_{xy}\beta(x,y) + \sum_{\substack{y_1 < y_2 \\ y_i \in Y}} s_{y_1 y_2}\beta(y_1, y_2)$$

and for every $x \in X$ there is some $y \in Y$ with $s_{xy} \neq 0$.

We choose $a = cd$, D, and XY in such a way that (11.1) is true and $|X|$ is minimal. We show that we can choose $\mathbb{K} \subseteq \mathrm{Aut}_{\{G\}(a)}(\mathfrak{C})$ of finite index in such a way that furthermore

(c) $f(x) = x$ for all $x \in X$ and $f \in \mathbb{K}$.

Suppose not. Then there are some $x_0 \in X$ and $f \in \mathbb{K}$ such that $f(x_0) = u \notin \langle X \rangle$. Let $\overline{X} \supseteq X \cup \{u\}$ be a transversal for V_G and $\overline{Y} \supseteq Y$ such that $\overline{X}\,\overline{Y}$ is a transversal for V. We can change X and therefore $\overline{X}$ in such a way that $f(x) \in \langle \overline{X} \setminus \{u\} \rangle$ for all $x \in X \setminus \{x_0\}$.
Assume $f(y) = y + t_y u + w_y$ for $y \in Y$. Using (11.1) and $f(d) = d$ modulo V_G we obtain for every $y \in Y$

$$s_{xy}\beta(u,y) + \left[\sum_{y_1 < y} s_{y_1 y}\beta(t_{y_1}u, y) - \sum_{y < y_1} s_{yy_1}(t_{y_1}u, y) \right] = 0.$$

Since $\beta(u,y) \neq 0$

$$s_{xy} + \left[\sum_{y_1 < y} s_{y_1 y} t_{y_1} - \sum_{y < y_1} s_{yy_1} t_{y_1} \right] = 0$$

for all $y \in Y$.
If we set $y' = y + t_y x_0$, then we obtain (11.1) using only $X \setminus \{x_0\}$ a contradiction to the minimality of $|X|$. As $\mathrm{acl}(AG) \cap H = G$ we have $\langle XY \rangle \cap V_H = \langle X \rangle$. We can do the same steps and choose $\mathbb{K}^H$ in such a way that furthermore

(c)^H $g(x) = x$ for $x \in X$ and $g \in \mathbb{K}^H$.

Let Y be the disjoint union of P and U, where $P = \{y \in Y :$ there is some y' with $s_{yy'} \neq 0$ or $x_{y'y} \neq 0\}$.
Now we choose $a = cd$, D, and X, U, P such that (11.1), (a), and (c) are true and $|P|$ is minimal. We show that we can choose $I\!K$ such that

(d) $\quad f(y) = y$ for $y \in P$ and $f \in I\!K$.

By (a) $f(y) = y$ modulo V_G. If (d) is not true, then there are $y_0 \in P$ and $u \notin \langle X\rangle$ such that $f(y_0) = y_0 + u$.
We can choose another basis of $\langle P\rangle$ that we call again P such that for $y \in P \setminus \{y_0\}$ $f(y) = y + w$ where $w \in \langle \overline{X} \setminus \{u\}\rangle$. But then we have a contradiction to the minimality of $|P|$, since summands $s_{yy_0}\beta(y,u)$ or $s_{y_0y}\beta(u,y_0)$ cannot occur in the representation of $f(d) = d$ modulo G' as linear combination of basic commutators over $\overline{X}\,\overline{Y}$. Note that $s_{yy_0} \neq 0$ or $s_{y_0y} \neq 0$ for some y since $y_0 \in P$. Similarly we can assume

(d)^H $\quad g(y) = y$ for $y \in P$ and $g \in I\!K^H$.

(a), (a)^H, (c), (c)^H, (d), (d)^H give the desired result. □

Now we can prove the theorem.
By Corollary 8.7 we have assumed that $A \cap G = A \cap G'$ and $A \cap (G \cdot Z(\mathbb{C})) = A \cap (G \cdot \mathbb{C}')$. Then $A = \langle (A \cap \mathbb{C}') \cup A_1 \cup \{a_1, \ldots, a_m\}\rangle$ where A_1 is a basis of A modulo $Z(\mathbb{C})$ and $a_1, \ldots, a_m$ are mixed elements. We assume

$$a_1, \ldots, a_\ell \text{ are linearly independent modulo } Z(\mathbb{C})$$

and

$$a_{\ell+1}, \ldots, a_m \subseteq Z(G) \text{ are linearly independent modulo } G'.$$

We apply Lemma 11.3 and obtain a part of a transversal X, U with $\langle X\rangle = \langle XU\rangle \cap V_G$, and $I\!K \subseteq \mathrm{Aut}_{\{G\}(A)}(\mathbb{C})$ and $I\!K^H \subseteq \mathrm{Aut}_{\{H\}(A)}(\mathbb{C})$ both of finite index such that

$$(11.2) \qquad a_i = a_i^G \prod_{\substack{x \in X \\ u \in U}} [x,u]^{s^i_{xu}} a_i'$$

where $a_i^G \in G$, $a_i' \in \mathbb{C}'$,
all $f \in I\!K$ fix all a_i' and X pointwise, and U pointwise modulo V_G, and
all $g \in I\!K^H$ fix all a_i' and X pointwise, and U pointwise modulo V_H.

Hence we can assume w.l.o.g. $a_i' \in A \cap \mathbb{C}'$. We choose $D \subseteq V$ definable over $A \cup X$ such that $(A \cap \mathbb{C}') \subseteq \beta(D,D)$, $A/Z(\mathbb{C}) \subseteq D$, and $X \subseteq D$. W.l.o.g. D is fixed pointwise by the automorphisms of $I\!K$ and $I\!K^H$. We choose a basis X^+Y^+ of D with $D \cap V_G = \langle X^+\rangle$. Then we choose U^+ such that $X^+Y^+U^+$ is a basis of $\langle D \cup U\rangle$. Note that U^+ is linearly independent over V_GY^+.
We have that X^+Y^+ is fixed pointwise by the automorphism of $I\!K$ and $I\!K^H$ and $f(u) = u$ modulo V_G for $u \in U^+$ and $f \in I\!K$ and $g(u) = u$ modulo V_H for $u \in U^+$ and

$g \in I\!K^H$. We can write $u \in U$ as $u = u_0 + u_1 + u_2$ with $u_0 \in \langle X^+ \rangle$, $u_1 \in \langle Y^+ \rangle$, and $u_2 \in \langle U^+ \rangle$. In (11.2) we can multiply $[xu_0]^{s^i_{xu}}$ with a_i^G and w.l.o.g. consider $[xu_1]$ as an element of $A \cap \mathbb{C}'$. Hence w.l.o.g.

$$a_i = a_i^G \prod_{\substack{x \in X \\ u \in U^+}} [x, u]^{s^i_{xu}}. \tag{11.3}$$

Note that

$$X^+ \subseteq \mathrm{acl}(\mathrm{Cb}(\mathrm{tp}(A/G))) \quad \text{and} \quad X^+ \subseteq \mathrm{acl}(\mathrm{Cb}(\mathrm{tp}(A/H))).$$

To describe $\mathrm{acl}(\mathrm{Cb}(\mathrm{tp}(A/G)))$ completely we need:

Definition Let $\bar{v}^i = (v_1^i, \ldots, v_m^i)$ $(i = 0, 1)$ be two sequences in $\mathbb{C}$. We say $\theta_\Delta(\bar{v}^0, \bar{v}^1)$ holds if there is an vector space homomorphism h of $\langle U^+ \rangle$ into V such that

$$v_i^0 = v_i^1 \prod_{\substack{x \in X \\ u \in U^+}} [x, h(u)]^{s^i_{xu}} \quad \text{for } 1 \le i \le m.$$

This is an equivalence relation definable over X. Let $\underline{\Delta}$ be the θ_Δ-class of $(a_1^G, \ldots, a_m^G)$. If X is fixed pointwise we can consider $\underline{\Delta}$ as an imaginary. Since $a_i = f(a_i)$ for $f \in I\!K$ we can show that $\underline{\Delta} \in \mathrm{acl}(\mathrm{Cb}(\mathrm{tp}(A/G)))$ by the same proof as in Chapter 10. Analogously we have $\underline{\Delta} \in \mathrm{acl}(\mathrm{Cb}(\mathrm{tp}(A/H)))$. Hence it remains to show that

$$\mathrm{acl}(\mathrm{Cb}(\mathrm{tp}(A/G))) = \mathrm{acl}(X^+ \cup \underline{\Delta}).$$

Let f^G be an automorphism of G that fixes X^+ pointwise and $\underline{\Delta}$. Let $\overline{Y}$ with $Y^+U^+ \subseteq \overline{Y}$ be a basis for V modulo V_G. $\underline{\Delta}$ is the θ_Δ-class of $(a_1^G, \ldots, a_m^G)$. Since $f^G(\underline{\Delta}) = \underline{\Delta}$ there is some linear map k of U^+ into V_G such that

$$f(a_i^G) = a_i^G \prod_{\substack{x \in X \\ u \in U^+}} [x, k(u)]^{s^i_{xu}}.$$

Now we define an extension $(g, h) \in \mathrm{Aut}(\mathcal{F}(\mathbb{C}))$ of $\mathcal{F}(f^G)$ by $g(y) = y$ for $y \in \overline{Y} \setminus U^+$ and $g(u) = u - k(u)$ for $u \in U^+$. The automorphism h of $\beta(V, V)$ is determined by g. Then h can be any extension on W. By Corollary 3.3 there is an automorphism f in $\mathrm{Aut}_{\{G\}}(\mathbb{C})$ that extends f^G with $\mathcal{F}(f) = (g, h)$ and $f(a) = a$ for $a \in A_1$. Since $D = \langle X^+ Y^+ \rangle$ D is fixed pointwise by f.
Hence f fixes A_1 and D and therefore $A \cap \mathbb{C}'$. Finally we show $f(a_i) = a_i$. We have

$$\begin{aligned} f(a_i) &= f(a_i^G) \cdot f\left(\prod_{\substack{x \in X \\ u \in U^+}} [x, u]^{s^i_{xu}} \right) \\ &= a_i^G \prod_{\substack{x \in X \\ u \in U^+}} [x, k(u)]^{s^i_{xu}} \cdot \prod_{\substack{x \in X \\ u \in U^+}} [x, u(k(u))^{-1}]^{s^i_{xu}} \\ &= a_i^G \prod_{\substack{x \in X \\ u \in U^+}} [x, u]^{s^i_{xu}} = a_i. \qquad \square \end{aligned}$$

The CM-triviality of $\mathrm{Th}(F_2(p,\omega))$ is shown. It follows that it is impossible to interpret a field in this theory.

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