

More Fraïssé limits of nilpotent groups of finite exponent

Andreas Baudisch

Abstract

The class of nilpotent groups of class c and prime exponent $p > c$ with additional predicates $P_c \subseteq P_{c-1} \subseteq \dots \subseteq P_1$ for suitable subgroups has the amalgamation property. Hence the Fraïssé limit D of the finite groups of this class exists. $\langle 1 \rangle \subseteq P_c(D) \subseteq \dots \subseteq P_2(D) \subseteq P_1(D) = D$ is the lower and the upper central series of D . In this extended language D is ultrahomogeneous. The elementary theory of D allows the elimination of quantifiers and it is \aleph_0 -categorical. For $c = 2$ this was proved in [2].

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1 Introduction

We generalize the results of the first part of [2]. Let $\mathcal{G}_{c,p}$, where $c < p$ and p prime, be the variety of the nilpotent groups of class c and of exponent p . As in the case $c = 2$ ([2]) it is necessary to extend the language L of group theory to prove the amalgamation property.

Let $\langle 1 \rangle = \Gamma_{c+1}(G) \subseteq \Gamma_c(G) \subseteq \dots \subseteq \Gamma_2(G) \subseteq \Gamma_1(G) = G$ be the lower central series and $Z_0(G) = \langle 1 \rangle \subseteq Z_1(G) \subseteq \dots \subseteq Z_c(G) = G$ be the upper central series of $G \in \mathcal{G}_{c,p}$. If $X \subseteq G$, then let $\langle X \rangle$ be the subgroup of G generated by X . In general $\langle X \rangle$ is the substructure of the structure G generated by X . $[X, Y]$ is used to denote $\{[x, y] : x \in X, y \in Y\}$, where $[x, y] = x^{-1}y^{-1}xy$. Let G' be the commutator subgroup $\langle [G, G] \rangle$.

Definition Let G be a group in $\mathcal{G}_{c,p}$. We often write Q_n, P_n, Z_n instead of $Q_n(G), P_n(G),$ and $Z_n(G)$ respectively. Let Q_n and P_n be subgroups of G such that

$Q_1 = P_1 = G$ and for $1 < n \leq c + 1$ we have $Q_n = \left\langle \bigcup_{\ell+k=n} [P_\ell, P_k] \right\rangle$ and

P_n is a subgroup of G with $Q_n \subseteq P_n \subseteq Z_{c+1-n}$ and $P_n \subseteq P_{n-1}$.

Let $\mathcal{G}_{c,p}^P$ be the class of all groups in $\mathcal{G}_{c,p}$ with additional predicates $P_c(G) \subseteq \dots \subseteq P_2(G)$ as described above. Let L^+ be the corresponding language that extends L .

Note that $P_n(G)$ is a normal subgroup of G . It is easily seen that $Q_n(G) \subseteq Q_{n-1}(G)$. We also use $Q_1(G) = P_1(G) = G$ and $Q_{c+1}(G) = P_{c+1}(G) = \langle 1 \rangle$. Furthermore $[G, P_n] \subseteq Q_{n+1} \subseteq P_{n+1}$. Hence P_n/P_{n+1} is abelian; in fact central in G/P_{n+1} . The structures in $\mathfrak{G}_{c,p}^P$ we often call L^+ -groups.

Note that every group $G \in \mathfrak{G}_{c,p}$ has an L^+ -expansion in $\mathfrak{G}_{c,p}^P$. We define $P_n(G) = Q_n(G) = \Gamma_n(G)$.

As in [2] we use R. Fraïssé's construction [3]. We follow the presentation of W. Hodges in ([4], Theorem 7.1.2). The main result of this paper is the following:

Theorem 1.1 $\mathfrak{G}_{c,p}^P$ has the amalgamation property (AP).

Then (AP) implies the joint embedding property (JEP) of $\mathfrak{G}_{c,p}^P$. Furthermore $\mathfrak{G}_{c,p}^P$ has the hereditary property (HP): Every substructure of a L^+ -group in $\mathfrak{G}_{c,p}^P$ is again in $\mathfrak{G}_{c,p}^P$. Note that (HP) implies that a definition $P_n(G) = \Gamma_n(G)$ for all $G \in \mathfrak{G}_{c,p}^P$ is impossible. By R. Fraïssé's result quoted above we have:

Corollary 1.2 The Fraïssé limit D of the finite L^+ -groups in $\mathfrak{G}_{c,p}^P$ exists. Every countable or finite group in $\mathfrak{G}_{c,p}$ can be embedded into $D \upharpoonright L$.

Corollary 1.3 For the Fraïssé limit D we have that $\Gamma_n(D) = Q_n(D) = P_n(D) = Z_{c+1-n}(D)$ for $1 \leq n \leq c$. D is ultrahomogeneous in the language L^+ . The elementary L^+ -theory of D with this interpretation of the predicates P_n allows the elimination of quantifiers.

Corollary 1.4 The elementary theories of D in L and L^+ are \aleph_0 -categorical.

In [2] the well-known functor from $\mathfrak{G}_{2,p}^P$ into the category of alternating bilinear maps is used to prove Theorem 1.1 in the case $c = 2$. For $\mathfrak{G}_{c,p}^P$ with $c > 2$ the corresponding functor into the category of graded Lie-algebras loses some information. Therefore we need a new strategy to prove the result:

Let $F_c(p, \kappa)$ be the free group in $\mathfrak{G}_{c,p}$ with κ free generators. In [1] the subgroups F of $F_c(p, \kappa)$ are investigated. Their algebraic structure is very similar to the structure of $F_c(p, \kappa)$. For $F \subseteq F_c(p, \kappa)$ we can define $P_n(F) = Z_{c+1-n}(F_c(p, \kappa)) \cap F$. The corresponding L^+ -structure we denote again by F . We call it a quasi-free group in $\mathfrak{G}_{c,p}^P$. For $G \in \mathfrak{G}_{c,p}^P$ we call F a quasi-free group with the same dimensions as G , if

$$\dim(P_n(G)/\langle P_{n+1}(G) \cup Q_n(G) \rangle) = \dim(P_n(F)/\langle P_{n+1}(F) \cup Q_n(F) \rangle).$$

From results in [1] follows that the isomorphism type of F is fixed by G via the condition above. We show that there is a strong L^+ -homomorphism f of F onto G . f is strong if $P_n(f(a))$ in G implies $P_n(b)$ in F for some b in F with $f(a) = f(b)$. Let N be the normal subgroup of F such that $G \cong F/N$. We can show that the isomorphism type

of the pair (F, N) is uniquely determined by G . We call (F, N) the canonical pair for G . These canonical pairs are the main tool in the proof of Theorem 1.1.

In the next section we prepare results of [1] for the purposes of this paper. In Section 3 we prove (AP) for $\mathcal{G}_{c,p}^P$.

In [2] the results are proved for $c = 2$. There non-forking is characterized for $\text{Th}(D)$. It does not have symmetry and transitivity. I want to mention here, that also independence over a model fails.

2 Subgroups of $F_c(p, \kappa)$

All results which are stated here you will find in [1], Section 3. The background is the work of P. Hall, W. Magnus, E. Witt, and M. Hall. But the main result used in [1] is the theorem of Širšov-Witt that every subalgebra of a free Lie algebra over a field is again free.

We use \mathbb{Z} to denote the ring of the integers. All considered groups G are nilpotent of class c and of exponent p .

Let A be a subset of G . The elements of A are basic commutators of A -weight 1. By induction on the A -weight n we define basic commutators b of A -weight $w_A(b) = n$ on A . Every definition of basic commutators involves a choice of total order $<$ on them, such that $w_A(b_1) < w_A(b_2)$ implies $b_1 < b_2$. Hence the elements of A are the smallest elements in this order. Now we assume that the basic commutators on A of A -weight less than n have already been defined and ordered. The A -weight of a basic commutator b is n if $b = [b_1, b_2]$ where b_1 and b_2 are basic commutators, $w_A(b_1) + w_A(b_2) = n$, $b_1 > b_2$, and if $b_1 = [b_3, b_4]$, then $b_2 \geq b_4$.

There are several procedures to obtain basic commutators on A depending on the order chosen for each A -weight. If we speak about basic commutators over A , then we assume one fixed choice. We assume that the signature of L is $\{., ^{-1}, 1\}$. The following fact is well-known:

Fact 2.1 *For every term $t(x_1, \dots, x_n)$ of L there is a sequence $y_0(x_1, \dots, x_n) < \dots < y_{m-1}(x_1, \dots, x_n)$ of basic commutators over $\{x_1, \dots, x_n\}$ such that*

$$\mathcal{G}_{c,p} \models t(x_1, \dots, x_n) = \prod_{i < m} y_i(x_1, \dots, x_n)^{r_i}$$

where $0 \leq r_i < p$. We get the equality using only identities of the form $cd = dc[c, d]$ and $c^p = 1$.

To prove Fact 2.1 we can use the following procedure. Let g be $t(x_1, \dots, x_n)$.

After n steps we have $g = y_1^{r_1} \dots y_k^{r_k} g_1 \dots g_\ell$ where the y_i and g_j are basic commutators, $y_1 < y_2 < \dots < y_k$ and $y_k < g_j$ for $1 \leq j \leq \ell$. (At the beginning there is no y_i and $g_j \in \{x_1, \dots, x_n\}$. x_i^{-1} is represented by a product of $(p - 1)$ many x_i .) Let

$g_{i_1} = g_{i_2} = \dots = g_{i_t}$ with $i_1 < i_2 < \dots < i_t$ be the smallest basic commutator on the right side. Using $cd = dc[c, d]$ we move first g_{i_1} then g_{i_2} and so on to the place after $y_k^{r_k}$. By this procedure we produce only basic commutators of $\{x_1, \dots, x_n\}$ -weight greater than $w_{\{x_1, \dots, x_n\}}(y_k)$. Since in $\mathcal{G}_{c,p}$ commutators of weight $> c$ are 1 we can stop if $w_{\{x_1, \dots, x_n\}}(g_j) > c$ for all g_j .

Now we consider the free groups $F_c(p, \kappa)$ in $\mathcal{G}_{c,p}$.

Theorem 2.2 (*Second Basis Theorem of P. Hall*) *Let $A = \{a_\alpha : \alpha < \kappa\}$ be a set of free generators of $F_c(p, \kappa)$. Then $\Gamma_i(F_c(p, \kappa))/\Gamma_{i+1}(F_c(p, \kappa))$ is a free module over the field $\mathbb{Z}/p\mathbb{Z}$ for $1 \leq i \leq c$. The cosets of all basic commutators of A -weight i form a basis of that module.*

Corollary 2.3 *Assume $\{b_\alpha : \alpha < \lambda\}$ is a sequence of all basic commutators on A ordered according to A -weight. Then every element g of $F_c(p, \kappa)$ can be uniquely expressed as*

$$g = \prod_{\alpha} b_{\alpha}^{r_{\alpha}} \quad \text{where } r_{\alpha} \in \mathbb{Z}/p\mathbb{Z}$$

and $r_{\alpha} = 0$ up to finitely many α .

Corollary 2.4 $\Gamma_i(F_c(p, \kappa)) = Z_{c+1-i}(F_c(p, \kappa))$ for $1 \leq i \leq c+1$.

Theorem 2.2 provides us a free nilpotent Lie algebra $L[F_c(p, \kappa)]$ of class c over the field $\mathbb{Z}/p\mathbb{Z}$. As a $\mathbb{Z}/p\mathbb{Z}$ -module let $L[F_c(p, \kappa)]$ be $\bigoplus_{1 \leq i \leq c} L[F_c(p, \kappa)]^i$ where $L[F_c(p, \kappa)]^i = \Gamma_i(F_c(p, \kappa))/\Gamma_{i+1}(F_c(p, \kappa))$. For $\bar{a} = \sum_{1 \leq i \leq c} \bar{a}_i$ and $\bar{b} = \sum_{1 \leq i \leq c} \bar{b}_i$ where $\bar{a}_i, \bar{b}_i \in L[F_c(p, \kappa)]^i$ choose elements $a_i, b_i \in F_c(p, \kappa)$ in the cosets \bar{a}_i resp. \bar{b}_i . We define $(\bar{a}, \bar{b}) = \sum_{1 \leq i \leq c} \bar{c}_i$ where $\bar{c}_i \in L[F_c(p, \kappa)]^i$ and \bar{c}_i is the coset of $\prod_{r+s=i} [a_r, b_s]$. Then (\bar{a}, \bar{b}) is a well-defined

Lie multiplication. It is not hard to prove that $L[F_c(p, \kappa)]$ is a free nilpotent Lie algebra over $\mathbb{Z}/p\mathbb{Z}$ using Theorem 2.2 and similar results for free nilpotent Lie algebras.

Let G be a L^+ -group in $\mathcal{G}_{c,p}^P$ and $A \subseteq G$ where $A = \{a_{\alpha}^i : 1 \leq i \leq c, \alpha < \lambda_i\}$ and $a_{\alpha}^i \in P_i(G)$. We set $a_{\alpha}^i < a_{\beta}^j$ if $i < j$ or $i = j$ and $\alpha < \beta$. We define basic commutators over A as above starting with this order. In addition to A -weight we introduce the A -degree $d_A(b)$ of commutators b on A : $d_A(a_{\alpha}^i) = i$, $d_A([b_1, b_2]) = d_A(b_1) + d_A(b_2)$. Note that any commutator of A -degree i is in $P_i(G)$.

Definition Let G be a group in $\mathcal{G}_{c,p}^P$ and $A = \{a_{\alpha}^i : 1 \leq i \leq c, \alpha < \lambda_i\} \subseteq G$. A is an (o) -system, if $a_{\alpha}^i \in P_i(G)$ for every i with $1 \leq i \leq c$, and $\{a_{\alpha}^i : \alpha < \lambda_i\}$ is linearly independent modulo the normal subgroup generated by $P_{i+1}(G)$ and all other basic commutators on A of A -degree i . A is a $(*)$ -system, if $a_{\alpha}^i \in P_i(G)$ and the basic commutators over A of A -degree i are linearly independent modulo $P_{i+1}(G)$ for all i with $1 \leq i \leq c$.

In [1] these notions were defined for groups $G \in \mathcal{G}_{c,p}$ with the property $[Z_{c+1-i}(G), Z_{c+1-j}(G)] \subseteq Z_{c+1-(i+j)}(G)$. If we define $P_i(G) = Z_{c+1-i}(G)$ for these groups, then the definitions in the two papers coincide. As in the introduction we call the L^+ -subgroups F of the $F_c(p, \kappa)$ with $P_i(F) = Z_{c+1-i}(F_c(p, \kappa)) \cap F$ the *quasi-free groups of $\mathcal{G}_{c,p}^P$* . Then a L^+ -subgroup of a quasi-free group is quasi-free.

Analogously to [1] we have

Lemma 2.5 *Let G be in $\mathcal{G}_{c,p}^P$.*

- i) *For every subgroup of G there exists a generating (o)-system.*
- ii) *Every (*)-system in G is an (o)-system.*
- iii) *A is an (o)-system ((*)-system) in G if and only if every finite subset of A is an (o)-system ((*)-system) in G .*

In our terminology we can formulate Theorem 3.6 in [1]:

Theorem 2.6 *Let F be a quasi-free group. Then every (o)-system in F is a (*)-system.*

The proof of Theorem 2.6 is described in [1]. We consider $F \subseteq F_c(p, \kappa)$ and move to $L[F_c(p, \kappa)]$ to apply the key lemma behind the Theorem of Širšov-Witt.

Corollary 2.7 *Let F be a quasi-free group and $A = \{a_\alpha^i : 1 \leq i \leq c, \alpha < \lambda_i\}$ be an (o)-system in F . Let $\{b_\alpha : \alpha < \kappa\}$ be an enumeration of the basic commutators on A according to A -degree. Then every element g of $\langle A \rangle$ can be uniquely expressed as $g = \prod_{\alpha} b_\alpha^{r_\alpha}$ where $r_\alpha \in \mathbb{Z}/p\mathbb{Z}$ and $r_\alpha = 0$ for all but finitely many α . There is an algorithm to compute this representation of g using only identities of the form $cd = dc[c, d]$ and $c^p = 1$.*

Proof. By Theorem 2.6 A is a (*)-system. Let $\{b_\alpha : \alpha < \kappa_n\}$ be the set of all basic commutators on A of A -degree $\leq n$ for $1 \leq n \leq c$ with the given order. Let $\kappa_0 = 0$. We have $\kappa_c = \kappa$. We show the assertion for $g \in P_n$ by induction on n . Let g be an element of $P_n(F) \setminus P_{n+1}(F)$. Using only $ab = ba[a, b]$ we obtain

$$g = \prod_{\alpha < \kappa} c_\alpha^{s_\alpha}$$

where $\{c_\alpha : \alpha < n\}$ is an enumeration of the basic commutators on A according to A -weight (Fact 2.1).

Let $c_{\alpha_1} = b_{\beta_1}, \dots, c_{\alpha_m} = b_{\beta_m}$ be the basic commutators in this representation of g of minimal A -degree t such that $s_{\alpha_i} \neq 0$. We have $\kappa_{t-1} \leq \beta_1 < \dots < \beta_m < \kappa_t$. Let $r_{\beta_1} = s_{\alpha_1}, \dots$, and $r_{\beta_m} = s_{\alpha_m}$. Then $g = \prod_{1 \leq i \leq m} b_{\beta_i}^{r_{\beta_i}} g'$ where $g' \in P_{t+1}$. Since A is a (*)-system we have $t = n$. We obtain this representation using only identities $cd = dc[c, d]$ and $c^p = 1$. If we apply the induction hypothesis to g' , the assertion follows. \square

Corollary 2.8 *Let F be a quasi-free group and $A = \{a_\alpha^i : 1 \leq i \leq c, \alpha < \lambda_i\}$ an (o)-system in F . Then $\{a_\alpha^i : \alpha < \lambda_i\}$ is linearly independent modulo $\langle P_{i+1}(F) \cup Q_i(F) \rangle$ and $P_i(F)$ is generated by the basic commutators on A of A -degree $\geq i$.*

3 Canonical pairs and amalgamation

Let G and H be L^+ -groups in $\mathcal{G}_{c,p}^P$. f is a L^+ -homomorphism of G into H , if f is a group homomorphism and $P_n(a)$ implies $P_n(f(a))$ for all $a \in G$ and for all $2 \leq n < c$. f is strong, if for every $a \in G$ with $P_n(f(a))$ there exists some $b \in G$ such that $P_n(b)$ and $f(b) = f(a)$.

Lemma 3.1 *Let F be a quasi-free group in $\mathcal{G}_{c,p}^P$ and $G \in \mathcal{G}_{c,p}^P$. Assume $A = \{a_\alpha^i : 1 \leq i \leq c, \alpha < \lambda_i\}$ is an (o) -system in F and $C = \{c_\alpha^i : 1 \leq i \leq c, \alpha < \lambda_i, c_\alpha^i \in P_i(G)\}$ is a subset of G . Then $f(a_\alpha^i) = c_\alpha^i$ extends to an L^+ -homomorphism of $\langle A \rangle$ onto $\langle C \rangle$.*

Proof. By Theorem 2.6 A is a $(*)$ -system. W.l.o.g. we can assume that A generates F . Let $\{b_\alpha : \alpha < \kappa\}$ be an enumeration of the basic commutators on A according to A -degree. Let d_α be the basic commutator over C that we obtain if we replace a_β^i by c_β^i in b_α . If b_α is in $P_n(F)$, then $d_A(b_\alpha) = n$ and d_α is in $P_n(G)$. If $g = \prod_\alpha b_\alpha^{r_\alpha}$ is the unique representation of g according to Corollary 2.7, then we define

$$f(g) = \prod_\alpha d_\alpha^{r_\alpha}.$$

By uniqueness of the representation of g we have that f is well-defined. If g is in $P_n(F)$, then all b_α with $r_\alpha \neq 0$ are in $P_n(F)$ since A is a $(*)$ -system. Then all d_α with $r_\alpha \neq 0$ are in $P_n(G)$ and $f(g)$ is in $P_n(G)$.

It remains to show that f is a group homomorphism onto $\langle C \rangle$. Let $g = \prod_\alpha b_\alpha^{r_\alpha}$ and $h = \prod_\alpha b_\alpha^{s_\alpha}$ be two elements of $\langle A \rangle$. By Corollary 2.7 there is an algorithm to compute the unique representation of $g \cdot h = \prod_\alpha b_\alpha^{t_\alpha}$ using only identities of the form $cd = dc[c, d]$ and $c^p = 1$. If we apply the same steps to $f(g) \cdot f(h) = \prod_\alpha d_\alpha^{r_\alpha} \prod_\alpha d_\alpha^{s_\alpha}$, where a_α^i is replaced by c_α^i , then we obtain $f(g) \cdot f(h) = \prod_\alpha d_\alpha^{t_\alpha} = f(gh)$ as desired. Now it follows

$$f(a_{\alpha_1}^{i_1} a_{\alpha_2}^{i_2} \dots a_{\alpha_n}^{i_n}) = c_{\alpha_1}^{i_1} c_{\alpha_2}^{i_2} \dots c_{\alpha_n}^{i_n}.$$

Hence f is surjective onto $\langle C \rangle$. □

Definition Let G and H be groups in $\mathcal{G}_{c,p}^P$. We define

$$\dim_i(G) = \dim(P_i(G)/\langle P_{i+1}(G) \cup Q_i(G) \rangle).$$

We say that G and H have the same dimensions if

$$\dim_i(G) = \dim_i(H) \quad \text{for } 1 \leq i \leq c.$$

If F is a quasi-free group in $\mathcal{G}_{c,p}^P$ and $A = \{a_\alpha^i : \alpha < \lambda_i, 1 \leq i \leq c\}$ is a generating (o) -system, then A is a $(*)$ -system. Hence $\dim_i(F) = \lambda_i$. Using Theorem 3.12 in [1] we have

Lemma 3.2 *Quasi-free groups in $\mathbb{G}_{c,p}^P$ with the same dimensions are isomorphic.*

Lemma 3.3 *Let G and F be groups in $\mathbb{G}_{c,p}^P$, where F is quasi-free and $\dim_i(G) = \dim_i(F)$ for $1 \leq i \leq c$.*

- i) *Let $A = \{a_\alpha^i : A \leq i \leq c, \alpha < \lambda_i\}$ be a generating (o)-system of F . Let f be a L^+ -homomorphism of F onto G such that $\{f(a_\alpha^n) : \alpha < \lambda_n\}$ is a basis of $P_n(G)$ modulo $\langle P_{n+1}(G) \cup Q_n(G) \rangle$ for $1 \leq n \leq c$. Then f is strong.*
- ii) *There is a strong L^+ -homomorphism of F onto G .*
- iii) *Let H be another quasi-free group with $\dim_i(H) = \dim_i(G)$ for $1 \leq i \leq c$. If f is a strong L^+ -homomorphism of F onto G and h is a strong L^+ -homomorphism of H onto G , then there is a L^+ -isomorphism g of F onto H such that*

$$\begin{array}{ccc} F & \xrightarrow{g} & H \\ & \searrow f & \swarrow h \\ & & G \end{array} .$$

Proof. i) We define $f(a_\alpha^i) = c_\alpha^i$ and $C = \{c_\alpha^i : 1 \leq i \leq c, \alpha < \lambda_i\}$. We have to show that f is strong. By induction on i we show:

If $e \in P_{c+1-i}(G)$, then there is some $a \in P_{c+1-i}(F)$ with $f(a) = e$. For $i = 0$ we obtain $e = 1$ and $a = 1$. Assume the assertion is true for $j < i$. Let $c + 1 - i = n$. Let $e \in P_n(G)$. By the assumption of i)

$$e = \prod_{1 \leq j \leq m} (c_{\alpha_j}^n)^{r_j} q(e) w(e)$$

where $w(e) \in P_{n+1}(G)$, $q(e) \in Q_n(G)$. W.l.o.g. we can assume that $q(e)$ is a product of commutators of C -degree n over $\{c_\alpha^j : 1 \leq j < n, \alpha < \lambda_j\}$ since $q(e)$ is such a product modulo $P_{n+1}(G)$ by the commutator identities of Hall-Witt.

Let $q(a)$ be the element of F that we obtain if we replace c_α^j by a_α^j in $q(e)$. By induction there is some $w(a) \in P_{n+1}(F)$ with $f(w(a)) = w(e)$. If $a = \prod_{1 \leq j \leq m} (a_{\alpha_j}^n)^{r_j} q(a) w(a)$, then

$a \in P_n(F)$ and $f(a) = e$, as desired.

ii) Let $A = \{a_\alpha^i : 1 \leq i \leq c, \alpha < \lambda_i\}$ be a generating (o)-system of F . Then A is a (*)-system and $\lambda_i = \dim_i(F)$. Choose $C = \{c_\alpha^i : 1 \leq i \leq c, \alpha < \lambda_i\}$ such that $\{c_\alpha^i : \alpha < \lambda_i\}$ is a basis of $P_i(G)/\langle P_{i+1}(G) \cup Q_i(G) \rangle$ for $1 \leq i \leq c$. This is possible since $\dim_i(F) = \dim_i(G)$. By Lemma 3.1 there exists a L^+ -homomorphism f of F onto $G = \langle C \rangle$ with $f(a_\alpha^i) = c_\alpha^i$. By i) f is strong.

iii) Choose a subset $E \subseteq G$ such that $E = \{e_\alpha^i : 1 \leq i \leq c, \alpha < \lambda_i\}$ and for $1 \leq i \leq c$ $\{e_\alpha^i : \alpha < \lambda_i\}$ is a basis for $P_i(G)/\langle P_{i+1}(G) \cup Q_i(G) \rangle$. Then $\lambda_i = \dim_i(G)$. For every pair (i, α) with $1 \leq i \leq c$ and $\alpha < \lambda_i$ choose $a_\alpha^i \in P_i(F)$ and $c_\alpha^i \in P_i(H)$ such that $f(a_\alpha^i) = e_\alpha^i = h(c_\alpha^i)$. This is possible since f and h are strong. Let A be $\{a_\alpha^i : 1 \leq i \leq$

$c, \alpha < \lambda_i\}$ and C be $\{c_\alpha^i : 1 \leq i \leq c, \alpha < \lambda_i\}$. Since $\dim_i(F) = \dim_i(G)$ for $1 \leq i \leq c$ we can show by induction on i that $\{a_\alpha^i : \alpha < \lambda_i\}$ is a basis of $P_i(F)/\langle P_{i+1}(F) \cup Q_i(F) \rangle$. Hence A is a generating (o)-system and therefore a $(*)$ -system of F . Analogously C is a generating $(*)$ -system of H .

Hence $g(a_\alpha^i) = c_\alpha^i$ provides a L^+ -isomorphism by Lemma 3.2. Since $h(g(a_\alpha^i)) = e_\alpha^i = f(a_\alpha^i)$ we obtain the desired diagram. \square

Definition Let N be a L^+ -subgroup of $G \in \mathfrak{G}_{c,p}^P$ such that N is normal in G . Let $(G/N)^+$ be the L^+ -group that we obtain from the L -factor group G/N if we define $P_i(aN)$ if there is some $b \in N$ with $P_i(ab)$ in G .

Note that $P_i(a_1b_1)$ and $P_i(a_2b_2)$ implies $P_i(a_1b_1a_2b_2)$ and $P_i(a_1a_2b_1[b_1, a_2]b_2)$ where $b_1[b_1, a_2]b_2$ is an element of N . Hence $P_i(G/N)$ is a subgroup of G/N . By definition $P_n((G/N)^+) \subseteq P_{n-1}((G/N)^+)$. If $P_n(aN)$, then $P_n(ab)$ for some $b \in N$. Hence $a \cdot b \in Z_{c+1-n}(G)$ and $aN \in Z_{c+1-n}(G/N)$. To show $Q_n((G/N)^+) \subseteq P_n((G/N)^+)$ let $aN \in P_\ell((G/N)^+)$ and $eN \in P_k((G/N)^+)$ where $\ell + k = n$. Then $ab \in P_\ell(G)$ and $ed \in P_k(G)$ for some $b, d \in N$. Hence $[ab, ed] \in P_n(G)$. Then $[ab, ed] = [a, e]u$ for some $u \in N$ and $[aN, bN] \in P_n((G/N)^+)$.

We have shown the first part of the following

Lemma 3.4 Let G, H, N be in $\mathfrak{G}_{c,p}^P$.

- i) If N is a normal L^+ -subgroup of G , then $(G/N)^+$ is in $\mathfrak{G}_{c,p}^P$.
- ii) Let f be a strong L^+ -homomorphism of G onto H . Let N be the L -kernel of f . Define $P_n(N) = P_n(G) \cap N$. Let j be the canonical strong L^+ -homomorphism of G onto $(G/N)^+$. Then there is a L^+ -isomorphism g of H onto $(G/N)^+$ with

$$\begin{array}{ccc} & G & \\ f \swarrow & & \searrow j \\ H & \xrightarrow{g} & (G/N)^+ \end{array}$$

Proof of ii) If we consider the situation in L , then the assertion is clear. Since f and j are strong the assertion follows. \square

If $G \in \mathfrak{G}_{c,p}^P$, then Lemma 3.3ii) and 3.4 provides us a pair (F, N) such that

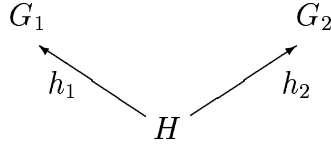
- i) F is quasi-free,
- ii) $\dim_i(F) = \dim_i(G)$ for $1 \leq i \leq c$, and
- iii) N is a normal L^+ -subgroup of F such that $G \cong_{L^+} (F/N)^+$.

By Lemma 3.3iii) the pair (F, N) is unique up to isomorphisms. We call (F, N) the canonical pair for G . Note that iii) can be replaced by the statement that there exists a strong L^+ -homomorphism f of F onto G with kernel N .

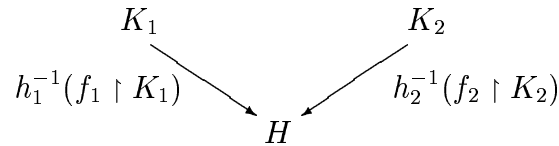
Lemma 3.5 *Assume $H \subseteq G$ and let (F, N) be the canonical pair for G . Then there is a L^+ -subgroup K of F such that $(K, K \cap N)$ is the canonical pair for H .*

Proof. Let $f : F \rightarrow G$ be the strong L^+ -homomorphism that corresponds to (F, N) . We choose $E = \{e_\alpha^i : 1 \leq i \leq c, \alpha < \lambda_i\}$ in H such that $\{e_\alpha^i : \alpha < \lambda_i\}$ is a basis of $P_i(H)$ modulo $\langle P_{i+1}(H) \cup Q_i(H) \rangle$ for $1 \leq i \leq c$. For each pair (i, α) there is some $a_\alpha^i \in P_i(F)$ with $f(a_\alpha^i) = e_\alpha^i$ since f is strong. Let A be $\{a_\alpha^i : 1 \leq i \leq c, \alpha < \lambda_i\}$. Let K be $\langle A \rangle$ and $P_i(K) = K \cap P_i(F)$. Then $\{a_\alpha^i : \alpha < \lambda_i\}$ is a basis of $P_i(K)$ modulo $\langle P_{i+1}(K) \cup Q_i(K) \rangle$. Hence $\dim_i(K) = \dim_i(H)$. Lemma 3.3i) implies that $f \upharpoonright K$ is strong. As a L^+ -subgroup of F the L^+ -group K is also quasi-free. This shows us, that $(K, K \cap N)$ is a canonical pair for H . \square

Now we are ready to prove the amalgamation property for $\mathcal{G}_{c,p}^P$. Assume H can be L^+ -embedded into G_1 and G_2 , where all groups are in $\mathcal{G}_{c,p}^P$.



Let (F_i, N_i) be the canonical pair for G_i ($i = 1, 2$) and f_i be the corresponding strong L^+ -homomorphism of F_i onto G_i with kernel N_i . Furthermore we denote $h_i(H)$ by H_i . Then $H_i \subseteq_{L^+} G_i$. According to Lemma 3.5 we find $K_i \subseteq_{L^+} F_i$ such that $(K_i, K_i \cap N_i)$ is a canonical pair for H_i . We apply Lemma 3.3iii) to the situation



and obtain a L^+ -isomorphism k of K_1 onto K_2 with the following property: If $f_1(v) = h_1(a)$ for $a \in H$ and $v \in K_1$, then $f_2(k(v)) = h_2(a)$.

Let F be the free product of F_1 and F_2 in $\mathcal{G}_{c,p}^P$. That means if $A = \{a_\alpha^i : 1 \leq i \leq c, \alpha < \lambda_i\}$ and $C = \{c_\alpha^i : 1 \leq i \leq c, \alpha < \kappa_i\}$ are generating $(*)$ -systems of F_1 and F_2 respectively, then $A \cup C$ is a generating $(*)$ -system of F .

Let J be the L^+ -group that we obtain if we factorize F by the normal subgroup N where N is the smallest normal subgroup of F that contains $N_1 \cup N_2 \cup \{a(k(a))^{-1} : a \in K_1\}$. The following claim implies that J is an amalgam of G_1 and G_2 over H with respect to L^+ .

Claim. $N \cap F_i = N_i$ ($i = 1, 2$).

There is a well-known functor $G \rightsquigarrow \bar{G}$ from $\mathcal{G}_{c,p}^P$ into the class of c -nilpotent graded Lie algebras over the field \mathbb{F}_p with p elements. (Using the Baker-Campbell-Hausdorff formula it can be shown that the functor is onto.)

\bar{G} is build on $\bigoplus_{1 \leq i \leq c} P_i(G)/P_{i+1}(G)$ using the commutator to define the Lie multiplication as described for $G = F_c(p, w)$ and $P_i(G) = \Gamma_i(G)$ below Corollary 2.4. It is sufficient to prove the claim (and therefore the amalgamation) for the corresponding Lie algebras:

$$\bar{F}_i \cap \bar{N} = \bar{N}_i, \text{ where } \bar{N}, \bar{N}_i \text{ are ideals in } \bar{F} \text{ and } \bar{F}_i \text{ respectively.}$$

We construct some Lie algebra L with embeddings ℓ_i of graded Lie algebras such that

$$(*) \quad \begin{array}{ccc} & L & \\ \ell_1 \nearrow & & \nwarrow \ell_2 \\ \bar{G}_1 & & \bar{G}_2 \\ \bar{h}_1 \nwarrow & & \nearrow \bar{h}_2 \\ & \bar{H} & \end{array}$$

By construction of \bar{F}/\bar{N} (*) implies the same amalgamation with \bar{F}/\bar{N} instead of L . Denote $P_i(G_1)/P_{i+1}(G_1)$ by $\bar{P}_i(G_1)$. To construct L let $\{e_\alpha^i : \alpha < \mu_i\}$ be a vector space basis of $\bar{P}_i(G_1)$ and $\{d_\alpha^i : \alpha < \nu_i\}$ be a vector space basis of $\bar{P}_i(G_2)$ ($1 \leq i \leq c$) such that there are χ_i with

$$e_\alpha^i = d_\alpha^i \quad \text{for } \alpha < \chi_i \text{ and these elements form a basis of } \bar{H}.$$

Then we have equations

$$(**) \quad [e_\alpha^i, e_\beta^j] = \sum_\gamma r_\gamma^{i,j} e_\gamma^{i+j} \quad \text{and} \quad [d_\alpha^i, d_\beta^j] = \sum_\gamma s_\gamma^{i,j} d_\gamma^{i+j}$$

that describe the Lie multiplication in \bar{G}_1 and \bar{G}_2 respectively. Let L be a vector space with the basis

$$\{e_\alpha^i : \alpha < \mu_i, 1 \leq i \leq c\} \cup \{d_\beta^j : \beta < \nu_j, 1 \leq j \leq c\}$$

with the identifications

$$e_\alpha^i = d_\alpha^i \quad \text{for } \alpha < \chi_i.$$

The Lie multiplication in L we define by (**) and

$$[e_\alpha^i, d_\beta^j] = 0 \quad \text{for } \chi_i \leq \alpha \text{ and } \chi_j \leq \beta. \quad \square$$

In the claim above N , F_i , and N_i are considered as L^+ -subgroups of F . Especially $P_n(F) \cap F_i = P_n(F_i)$. The claim implies

$$F_i N / N \cong F_i / F_i \cap N \cong F_i / N_i \cong G_i.$$

This gives us a L -embedding j_i of G_i in $J = F/N$. Furthermore we have

$$N \cap P_n(F_i) = N \cap F_i \cap P_n(F) = N_i \cap P_n(F_i)$$

by the claim. Hence

$$(P_n(F) \cap F_i)N/N \cong P_n(F_i)/P_n(F_i) \cap N \cong P_n(F_i)/P_n(F_i) \cap N_i \cong P_n(G_i).$$

Therefore j_i is a L^+ -embedding of G_i into $J = F/N$.

Let a be an element of H . We choose $v \in F_1$ as above such that $f_1(v) = h_1(a)$. Since $h_1(a) \in H_1$ we can choose v in K_1 . By definition of k we have $f_2(k(v)) = h_2(a)$. Hence $j_1(h_1(a)) = vN$ and $j_2(h_2(a)) = k(v)N$. Since $vk(v)^{-1} \in N$ we have $j_1h_1(a) = j_2h_2(a)$ for all $a \in H$, as desired.

We have shown:

Theorem 1.1 $\mathcal{G}_{c,p}^P$ has the amalgamation property.

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Institut für Mathematik
Humboldt-Universität zu Berlin
10099 Berlin
baudisch@mathematik.hu-berlin.de