THE UNIRATIONALITY OF $\overline{\mathcal{S}}_9^-$ AND MODULI SPACES OF POINTED SPIN CURVES

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ABSTRACT. We show that the moduli space $\overline{\mathcal{S}}_9^-$ of odd spin curves of genus 9 is unirational. This is the highest genus for which such a result is known. This is achieved by realizing birationally the moduli space $\overline{\mathcal{S}}_g^-$ when $g \leq 9$ as a locally trivial projective bundle over a certain (finite quotient of the) moduli space $\overline{\mathcal{S}}_{g',n}^-$ of n-pointed odd stable spin curves of genus $g' < g$. We then present general results on the Kodaira dimension of both moduli spaces $\overline{\mathcal{S}}_{g,n}^-$ and $\overline{\mathcal{S}}_{g,n}^+$.

1. INTRODUCTION

The moduli space $\mathcal{S}_g$ of spin curves of genus g parametrizes pairs $[C, \vartheta]$ consisting of a smooth projective curve C of genus g and a theta-characteristic ϑ, that is, a line bundle $\vartheta \in \mathrm{Pic}^{g-1}(C)$ such that $\vartheta^{\otimes 2} \cong \omega_C$. Using classical results [Mu], it is well-known that $\mathcal{S}_g$ has two connected components $\mathcal{S}_g^-$ and $\mathcal{S}_g^+$ depending on the parity of $h^0(C, \vartheta)$. Both components admit Deligne-Mumford compactifications $\overline{\mathcal{S}}_g^-$ and $\overline{\mathcal{S}}_g^+$ by means of stable spin curves which were constructed by Cornalba [Cor]. The spin moduli spaces (as well as their pointed counterparts $\overline{\mathcal{S}}_{g,n}^-$ and $\overline{\mathcal{S}}_{g,n}^+$) have been intensely studied either from a geometric [Lu], tropical [CMP], or string-theoretic point of view [DW]. One has a complete classification by Kodaira dimension of both $\overline{\mathcal{S}}_g^-$ and $\overline{\mathcal{S}}_g^+$ for all genera, see [F1], [FV1], [FV3]. This situation contrasts the case of the moduli space $\overline{\mathcal{M}}_g$, where in spite of the results in [EH], [HM], [FJP], the Kodaira dimension of $\overline{\mathcal{M}}_g$ when $17 \leq g \leq 21$ remains unknown.

It is known that the odd spin moduli space $\overline{\mathcal{S}}_g^-$ is of general type for $g \geq 12$, uniruled for $g \leq 11$, and unirational for $g \leq 8$, see [FV3]. The main result of this paper is the following:

Theorem 1.1. *The odd spin moduli space $\overline{\mathcal{S}}_9^-$ is unirational.*

Note that this is the highest genus g for which $\overline{\mathcal{S}}_g^-$ is known to be unirational. It has recently established that the Prym moduli space $\overline{\mathcal{R}}_9$ is uniruled [FV4]. The even spin moduli space $\overline{\mathcal{S}}_9^+$ is of general type [F2]; this is the lowest g for which $\overline{\mathcal{S}}_g^+$ is of general type. Therefore genus 9 can be regarded as the transition case for both the Prym and spin moduli spaces.

The proof of Theorem 1.1 uses, on the one hand Mukai's celebrated structure theorem [M2] for curves of genus 9, on the other hand a fundamental construction in [FV3] applying to all odd spin moduli spaces $\overline{\mathcal{S}}_g^-$ for $g \leq 9$. Precisely, denoting by

$$\mathfrak{V} := \mathrm{Sp}(3, 6) \subseteq \mathbf{P}^{13}$$

the 6-dimensional symplectic Grassmannian, since $K_{\mathfrak{V}} = \mathcal{O}_{\mathfrak{V}}(-4)$, linear 1-dimensional sections of $\mathfrak{V}$ are canonical curves of genus 9. It is the main result of [M2] that *every* non-pentagonal curve $[C] \in \mathcal{M}_9$ appears in this way, that is, $C = \mathfrak{V} \cap H_1 \cap \ldots \cap H_5 \subseteq \mathbf{P}^8$, where $H_i \in |\mathcal{O}_{\mathbf{P}^{13}}(1)|$ are hyperplanes. Following [FV3] we denote by $\Xi \subseteq \mathrm{Hilb}_{16}(\mathfrak{V})$ the variety of length 16 *clusters*, consisting of 0-dimensional curvilinear subschemes $Z \in \mathrm{Hilb}_{16}(\mathfrak{V})$ with $\mathrm{supp}(Z) = \{p_1, \ldots, p_8\}$ such that $\mathrm{mult}_{p_i}(Z) = 2$, for $i = 1, \ldots, 8$. Introducing the variety

$$\mathcal{U} := \Big\{(C, Z) : C = H_1 \cap \cdots \cap H_5 \cap \mathfrak{V} \text{ is a smooth curve section, } \ Z \subseteq C\Big\},$$

and retaining the previous notation, one has an $\mathrm{Aut}(\mathfrak{V})$-equivariant map $\mathcal{U} \dashrightarrow \overline{\mathcal{S}}_9^-$ given by $\big(C, Z\big) \mapsto [C, \mathcal{O}_C(p_1 + \cdots + p_8)]$. It is shown in [FV3, Proposition 3.7] that this construction induces a birational isomorphism

$$\mathfrak{s}_9 := \mathcal{U}//\mathrm{Aut}(\mathfrak{V}) \overset{\cong}{\dashrightarrow} \overline{\mathcal{S}}_9^-. \tag{1}$$

We regard $\mathfrak{s}_9$ as being the *Mukai model* of $\overline{\mathcal{S}}_9^-$. For every $1 \leq \delta \leq g$, we denote by $\mathcal{U}_\delta$ the subvariety of $\mathcal{U}$ consisting of pairs (Γ, Z) such that Γ is a δ-nodal curve with $\mathrm{sing}(\Gamma) \subseteq \mathrm{supp}(Z)$. We write $\mathrm{supp}(Z) = \{p_1, \ldots, p_8\}$ and we may assume that $\{p_1, \ldots, p_\delta\} = \mathrm{sing}(\Gamma)$. If $\nu \colon N \to \Gamma$ is the normalization of Γ and $\{x_i, y_i\} = \nu^{-1}(p_i)$ for $i = 1, \ldots, \delta$, then

$$\big[N, x_1 + y_1, \ldots, x_\delta + y_\delta, \ \mathcal{O}_N(p_{\delta+1} + \cdots + p_8)\big]$$

can be regarded as an odd 2δ-pointed spin curve of genus $9 - \delta$. Letting $B_{g,\delta}^- := \overline{\mathcal{S}}_{g-\delta,2\delta}^- / \mathbb{Z}_2^{\oplus \delta}$ to be the moduli space of odd spin curves of genus $g - \delta$ with δ pairs of points, one has a birational isomorphism

$$\mathfrak{s}_{9,\delta} := \mathcal{U}_\delta//\mathrm{Aut}(\mathfrak{V}) \overset{\cong}{\dashrightarrow} B_{9,\delta}^-. \tag{2}$$

One can combine the isomorphisms (1) and (2) into the following incidence correspondence

$$\begin{array}{ccc} & \mathcal{P}_{9,\delta} := \Big\{(C, \Gamma, Z) \in \mathcal{U} \times_{\Xi} \mathcal{U}_\delta : Z \subseteq C \cap \Gamma\Big\}//\mathrm{Aut}(\mathfrak{V}) & \\ \swarrow^{\bar{\alpha}} & & \searrow^{\bar{\beta}} \\ \overline{\mathcal{S}}_9^- & & B_{9,\delta}^- \end{array} \tag{3}$$

where the maps $\bar{\alpha}$ and $\bar{\beta}$ are induced by the birational isomorphisms (1) and (2) respectively. Note that $\bar{\beta}$ is birational to a Zariski locally trivial projective bundle, whereas $\bar{\alpha}$ is surjective if and only if $\delta \leq \dim(\mathfrak{V}) - 1$, see [FV3, Proposition 3.13]. This construction has been used in [FV3, Theorem 0.2] to show that $\overline{\mathcal{S}}_g^-$ is unirational for $g \leq 8$ by taking $\delta = g - 1$, in which case $B_{g,g-1}^-$ is a moduli space of pointed elliptic curves. Due to the small dimension of the Mukai variety $\mathfrak{V}$, this strategy does not work for the moduli space $\overline{\mathcal{S}}_9^-$, therefore this case remained open. It is one of the main tasks of this paper to complete this program in genus 9. We take $\delta = \dim(\mathfrak{V}) - 1$ and then Theorem 1.1 will follow from the following result:

Theorem 1.2. *The moduli space $\overline{\mathcal{S}}_{4,n}^-$ of n-pointed odd spin curves of genus 4 is unirational for $n \leq 10$.*

One has a finite surjective map $\overline{\mathcal{S}}_{4,10}^- \twoheadrightarrow B_{9,5}^-$. Therefore, from the correspondence (3) applied in the case $\delta = 5 = \dim(\mathfrak{V}) - 1$, we obtain via Theorem 1.2 that $\mathcal{P}_{9,5}$ is unirational. Since $\bar{\alpha}$ is a dominant morphism, we conclude that the spin moduli space $\overline{\mathcal{S}}_9^-$ is unirational. It is an interesting open question in the spirit of [CL] whether the Chow wing $CH^*(\overline{\mathcal{S}}_g^-)$ is tautological for $g \leq 9$.

We now discuss the main ideas in the proof of Theorem 1.2. We start with a general triple $[C, \vartheta, p]$, where $[C, \vartheta] \in \mathcal{S}_4^-$ and $p \in \mathrm{supp}(\vartheta)$. Observe that such a triple induces a plane quintic model $\Gamma \subseteq \mathbf{P}^2$ with a fixed bitangent line. Precisely, the canonical curve $C \subseteq \mathbf{P}^3$ lies on a unique quadric $Q \subseteq \mathbf{P}^3$ and we denote by ℓ' and ℓ'' the two rulings of Q passing through $p \in C$, therefore $\mathbf{T_p}(Q) = \langle \ell', \ell'' \rangle \subseteq \mathbf{P}^3$ is the tangent plane to the quadric Q. Then if

$$\phi = \phi_{|\omega_C(-p)|} \colon C \longrightarrow \mathbf{P}^2 = \mathbf{P}H^0\big(\omega_C(-p)\big)^{\vee}$$

is the corresponding projection, we set $n := \phi(p)$ and denote by $n' := \phi(\ell')$ and $n'' := \phi(\ell'')$ the two nodes of the projected canonical curve $\Gamma := \phi(C)$. We set $F := \langle n', n'' \rangle \subseteq \mathbf{P}^2$. Writing

$$\mathrm{supp}(\vartheta) = p + t_1 + t_2,$$

we consider a second line $L = \langle \phi(t_1), \phi(t_2) \rangle \subseteq \mathbf{P}^2$. The lines F and L meet in the point n and the curve Γ is tangent to L at both points $\phi(t_1)$ and $\phi(t_2)$. Observe that the union of the two lines $L + F$ can be viewed as a (degenerate) totally tangent conic to the quintic curve Γ.

We now reverse this construction and start with two fixed lines F and L in $\mathbf{P}^2$. We denote by $\{n\} = F \cap L$ and fix two further general points $n', n'' \in F$. We let

$$\mathbf{P} = \mathbf{P}^{13} = \big|\mathcal{O}_{\mathbf{P}^2}(5)(-n - 2n' - 2n'')\big| \subseteq \big|\mathcal{O}_{\mathbf{P}^2}(5)\big| \tag{4}$$

be the linear system of quintics nodal at n' and n'' and passing through n. One has a map

$$\rho \colon \mathbf{P} \dashrightarrow L^{[4]} \cong \mathbf{P}^4 \tag{5}$$

given by assigning to each quintic curve $\Gamma \in \mathbf{P}$ the cycle $\Gamma \cdot L \setminus \{n\}$. Note that we identify the a-th symmetric product $L^{[a]}$ of L with the projective space $\mathbf{P}^a$.

The key observation is, that under this identification, the image of the *squaring map*

$$L^{[2]} \longrightarrow L^{[4]}, \;\; x + y \mapsto 2(x + y),$$

can be identified with the *projected Veronese surface* $\mathbf{V} \subseteq \mathbf{P}^4$. In this way the cone $\rho^*(\mathbf{V}) \subseteq \mathbf{P}$ over $\mathbf{V}$ can be regarded as the family of plane quintics passing through n, nodal at n' and n'', which are bitangent to the line L at two unspecified points.

In order to parametrize $\overline{\mathcal{S}}_{4,10}^-$, we start with a general point $\overline{p} = (p_1, \ldots, p_{10}) \in (\mathbf{P}^2)^{10}$ and let $\mathbf{P}^3_{\overline{p}} = \big|\mathcal{O}_{\mathbf{P}^2}(5)(-n - 2n' - 2n'' - p_1 - \cdots - p_{10})\big|$ be the 3-dimensional space of quintic curves $\Gamma \in \mathbf{P}$ passing through the points in $\overline{p}$. We then have a rational quartic curve

$$R_{\overline{p}} := \mathbf{P}^3_{\overline{p}} \cdot \rho^*(\mathbf{V}) \subseteq \mathbf{P}^3_{\overline{p}} \tag{6}$$

parametrizing those quintics in $\mathbf{P}^3_{\overline{p}}$ which are bitangent to L. One has two fibrations

$$\begin{array}{ccccc} & & \mathcal{R} := \Big\{(\overline{p}, \Gamma) : \overline{p} = (p_1, \ldots, p_{10}) \in (\mathbf{P}^2)^{10}, \quad \Gamma \in R_{\overline{p}}\Big\} & & \\ & \swarrow^{m} & & \searrow^{\mathfrak{q}} & \\ \overline{\mathcal{S}}_{4,10}^- & & & & \big(\mathbf{P}^2\big)^{10} \end{array} \tag{7}$$

where $\mathfrak{q}^{-1}(\overline{p}) = R_{\overline{p}}$, whereas m is the moduli map assigning to $\big(\overline{p}, \Gamma\big)$ the pointed spin curve

$$\big[C = \phi^{-1}(\Gamma), \vartheta = \mathcal{O}_C(p + t_1 + t_2), \; p_1, \ldots, p_{10}\big] \in \overline{\mathcal{S}}_{4,10}^-,$$

where $\phi \colon C \to \Gamma$ denotes the normalization map, $p = \phi^{-1}(n)$ and $L \cdot \Gamma = n + 2\big(\phi(t_1) + \phi(t_2)\big)$. Observe that $\mathfrak{q}$ is a birational to a fibration in rational (quartic curves), whereas m is dominant. Now Theorem 1.2 follows by constructing a *rational bisection* of $\mathfrak{q}$ which is a rational variety.

Via a standard base change argument, this immediately implies that $\mathcal{R}$, and therefore also $\overline{\mathcal{S}}^-_{4,10}$, are unirational, which finishes the proof.

The Kodaira dimension of $\overline{\mathcal{S}}_{g,n}$. Theorem 1.2 presented above invites the more general question which moduli spaces $\overline{\mathcal{S}}^{\pm}_{g,n}$ of n-pointed spin curves of genus g are of general type. For similar results in the much more studied case of $\overline{\mathcal{M}}_{g,n}$, we refer to [Log], [F1] and [AB]. Note also the progress [Ba], [Bu], [CCM] on determining the Kodaira dimension of the strata of abelian differentials, which can be thought of as generalizations of the spaces $\overline{\mathcal{S}}^-_{g,n}$. Introducing the quotient $\overline{\mathcal{S}}_{g,[n]} := \overline{\mathcal{S}}_{g,n}/\mathfrak{S}_n$, we have an isomorphism

$$B_0 \cong \overline{\mathcal{S}}_{g-1,[2]},$$

where B_0 is the ramification divisor of the map $\overline{\mathcal{S}}_g \to \overline{\mathcal{M}}_g$. We prove the following result:

Theorem 1.3. *1) The even pointed spin moduli space $\overline{\mathcal{S}}^+_{g,n}$ is of general type for $n \geq f(g)$:*

g	4	5	6	7	8	≥ 9
$f(g)$	9	7	7	4	1	0

Furthermore, $\overline{\mathcal{S}}^+_{7,3}$ has non-negative Kodaira dimension.
2) The odd spin pointed moduli space $\overline{\mathcal{S}}^-_{g,n}$ is of general type for $n \geq h(g)$:

g	4	5	6	7	8	9	10
$h(g)$	12	11	10	7	5	4	2

Furthermore, both spaces $\overline{\mathcal{S}}^-_{8,4}$ and $\overline{\mathcal{S}}^-_{9,3}$ have non-negative Kodaira dimension.

It is an interesting open question whether the moduli spaces $\overline{\mathcal{S}}^+_{7,3}$, $\overline{\mathcal{S}}^-_{8,4}$ and $\overline{\mathcal{S}}^-_{9,3}$ are indeed of Kodaira dimension zero, and if so, to find Calabi-Yau models for them. The proof of Theorem 1.3 consists of two parts. First, by using Ludwig's results [Lu], we show that as long as $g \geq 4$, any pluricanonical form on the coarse moduli spaces $\overline{\mathcal{S}}^{\pm}_{g,n}$ extends to any resolution of singularities. Then we show that the canonical class $K_{\overline{\mathcal{S}}^+_{g,n}}$ (respectively $K_{\overline{\mathcal{S}}^-_{g,n}}$) is big for $n \geq f(g)$ (respectively for $n \geq h(g)$). To that end we use our extensive knowledge of the cones of effective divisors of $\overline{\mathcal{M}}_{g,n}$ provided by [Log], [F1], respectively of the unpointed spin moduli spaces $\overline{\mathcal{S}}^{\pm}_g$. An important new divisor class calculation provided in this paper is that of the *universal odd spin structure*, that is, the closure $\overline{\Theta}_{g,1}$ in $\overline{\mathcal{S}}^-_{g,1}$ of the locus

$$\Theta_{g,1} := \Bigl\{[C, \vartheta, p] \in \mathcal{S}^-_{g,1} : H^0\bigl(C, \vartheta(-p)\bigr) \neq 0\Bigr\}$$

of points in the support of odd spin structures. The class of $\overline{\Theta}_{g,1}$ is computed in Theorem 3.6.

Particularly interesting is the case $g = 11$. It is shown in [FV3] that $\overline{\mathcal{S}}^-_{11}$ is uniruled, this being the highest g for hich $\overline{\mathcal{S}}^-_g$ is not of general type. We have the following result:

Theorem 1.4.
1) The moduli space $\overline{\mathcal{S}}^-_{11,1}$ is not of general type; it has Kodaira dimension at least 19.
2) The moduli space $\overline{\mathcal{S}}^-_{11,[2]}$ has Kodaira dimension at least 19.

We show that $K_{\overline{\mathcal{S}}_{11,1}^-}$ is a linear combination of $[\overline{\Theta}_{11,1}^-]$, the pull-back of the Brill-Noether divisor under the forgetful map $\overline{\mathcal{S}}_{11,1}^- \to \overline{\mathcal{M}}_{11}$ and the pull-back under the forgetful map $\overline{\mathcal{S}}_{11,1}^- \to \overline{\mathcal{S}}_{11}^-$ of the branch divisor of the generically finite map $\overline{\Theta}_{11,1}^- \to \overline{\mathcal{S}}_{11}^-$. Furthermore, we show in Theorem 3.7 that through a general point of $\overline{\Theta}_{11,1}^-$ there passes a rational curve $\Gamma \subseteq \overline{\mathcal{S}}_{11,1}^-$ such that $\Gamma \cdot K_{\overline{\mathcal{S}}_{11,1}^-} = 0$, which immediately implies (the first part of) Theorem 1.4. We are tempted to conjecture that both $\overline{\mathcal{S}}_{11,1}^-$ and $\overline{\mathcal{S}}_{11,[2]}^-$ have Kodaira dimension 19, this being the dimension of the moduli space $\mathcal{F}_{11}$ of polarized $K3$ surfaces of genus 11. Note that it has been shown in [FV2] that the universal Jacobian over $\overline{\mathcal{M}}_{11}$ has Kodaira dimension 19, though we cannot see a direct connection to $\overline{\mathcal{S}}_{11,1}^-$. There are other interesting moduli spaces of curves of genus 10 and 11 that have submaximal Kodaira dimension [BM].

The last topic we mention concerns the moduli space $\overline{\mathcal{S}}_{1,n}^+$. It is a well known result of Deligne that $H^{11,0}(\overline{\mathcal{M}}_{1,11}) \neq 0$, see [FP, §3.5], or [BF] for various presentations. This has been one of the sources for producing non-tautological cohomology classes on $\overline{\mathcal{M}}_{g,n}$, see [CLP] and references therein. We have the following result[1] on the space $\overline{\mathcal{S}}_{1,n}^+$:

Theorem 1.5. *One has $H^{7,0}(\overline{\mathcal{S}}_{1,7}^+) \neq 0$. Furthermore, the Kodaira dimension of $\overline{\mathcal{S}}_{1,7}^+$ is equal to zero, whereas the Kodaira dimension of $\overline{\mathcal{S}}_{1,n}^+$ is equal to one for $n \geq 8$.*

Note that Krug [Kr, Corollary 4.25] showed that $\overline{\mathcal{S}}_{1,n}^+$ is rational for $n \leq 6$. Coupled with Theorem 1.5, one has a complete description of the Kodaira dimension of $\overline{\mathcal{S}}_{1,n}^+$ for every n.

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2. Spin curves of genus 4 and the projected Veronese surface

We begin by recalling some of the basic properties of the projected Veronese surface, whic are essential for the proof of Theorem 1.2. We let $L \cong \mathbf{P}^1$ be the smooth rational curve and denote by $L^{[n]} \cong \mathrm{Hilb}^n(L)$ its n-th symmetric product. We consider the Veronese embedding

$$\nu \colon \mathbf{P}^2 \cong \big|\mathcal{O}_{\mathbf{P}^2}(1)\big| \longrightarrow \big|\mathcal{O}_{\mathbf{P}^2}(2)\big| \cong \mathbf{P}^5, \quad [v] \mapsto [v^2],$$

identifying the Veronese surface $\nu(\mathbf{P}^2)$ with the space of rank one conics in $\mathbf{P}^2$. We then introduce the *squaring map*

$$\mathrm{sq}\colon L^{[2]} \longrightarrow L^{[4]}, \quad \mathrm{sq}(x+y) := 2(x+y). \tag{8}$$

Under the identification $L^{[2]} \cong \mathbf{P}^2$, the map sq can be given in coordinates as

$$[a,b,c] \mapsto \big[a^2, b^2, c^2+2ab, 2ac, 2bc\big] \in \mathbf{P}^4.$$

In particular, the smooth surface $\mathbf{V} := \mathrm{Im}(\mathrm{sq}) \subseteq L^{[4]} \cong \mathbf{P}^4$ can be identified with an isomorphic projection of the Veronese surface $\nu(\mathbf{P}^2)$. Severi showed that $\mathbf{V} \subseteq \mathbf{P}^4$ is the only smooth non-degenerate surface that can be obtained as a linear projection.

[1]This contradicts [BF, Lemma 2], which is incorrect.

It is a classical result [Dol, Theorem 2.1.4] that the variety $\mathbf{D} := \big\{\ell \in \mathbf{G}(1,4) : \ell\cdot \mathbf{V} \geq 3\big\}^-$ representing the closure in the Grassmannian of lines $\mathbf{G}(1,4) \subseteq \mathbf{P}^9$ of the family of trisecant lines to $\mathbf{V}$ is a de Pezzo threefold being a smooth linear section of $\mathbf{G}(1,4)$. Furthermore, the universal line over $\mathbf{D}$, that is,

$$\mathcal{X} := \Big\{(x,\ell) : \ell \in \mathbf{D},\ x \in \ell \subseteq \mathbf{P}^4\Big\} \subseteq \mathbf{P}^4 \times \mathbf{G}(1,4), \qquad \pi_1 : \mathcal{X} \to \mathbf{P}^4, \quad \pi_2 : \mathcal{X} \to \mathbf{D}$$

realizes π_1 as a birational map, that is, for a general point $o \in \mathbf{P}^4$ there is a unique trisecant line $\ell \in \mathbf{D}$ passing through o.

2.1. **Plane quintic models of odd spin curves of genus** 4. Let $[C, \vartheta, p] \in \Theta^-_{4,1}$ be a general point, consisting of a theta characteristic ϑ with $h^0(C,\vartheta) = 1$ and write $\mathrm{supp}(\vartheta) = p + t_1 + t_2$. We denote by $A', A'' \in W^1_3(C)$ the two minimal pencils on C, thus $A' \otimes A'' = \omega_C$.

Proposition 2.1. *For a general point $[C, \vartheta, p] \in \Theta^-_{4,1}$, for both $A', A'' \in W^1_3(C)$, we have*

$$H^0(C, A'(-2p)) = 0,\ H^0(C, A''(-2p)) = 0.$$

Furthermore, both divisors $A'(-p)$ and $A''(-p)$ are reduced.

Proof. We use that $\Theta^-_{4,1}$ is irreducible, see also Remark 3.5. Assuming that for a general point $[C, \vartheta, p] \in \Theta^-_{4,1}$, the point p is either a ramification, or an antiramification point[2] of a pencil $A' \in W^1_3(C)$, it follows that $A', A'' \in W^1_3(C)$ have in total at least $(g-1)\cdot 2^{g-1}(2^g+1) = 408$ ramification or antiramification points. But the total number of ramification and antiramification points of A' and A'' equals $4\cdot\big(2g + 2\cdot \deg(A') - 2\big) = 56$, a contradiction. □

We denote by $\phi = \phi_{|\omega_C(-p)|} \colon C \to \mathbf{P}^2$ the projected canonical curve and by $\Gamma := \mathrm{Im}(\phi)$ the corresponding quintic plane model. We write

$$A' = \mathcal{O}_C(p + n_1' + n_2') \ \text{ and } A'' = \mathcal{O}_C(p + n_1'' + n_2''),$$

where by Proposition 2.1 the points n_1', n_2', n_1'', n_2'' and p are pairwise distinct. It follows that the curve Γ has two nodes at the points $n' = \phi(n_1') = \phi(n_2')$ respectively $n'' = \phi(n_1'') = \phi(n_2'')$. As explained in the Introduction, we denote by

$$F = \big\langle n', n''\big\rangle \subseteq \mathbf{P}^2$$

the line spanned by the nodes of Γ. Set also $n := \phi(p)$. Since $n_1' + n_2' + n_1'' + n_2'' + p \in \big|\omega_C(-p)\big|$, it follows that $n \in F$.

We set $b_1 := \phi(t_1)$ and $b_2 := \phi(t_2)$ and introduce the line $L := \big\langle b_1, b_2\big\rangle \subseteq \mathbf{P}^2$. Note that

$$\Gamma \cdot L = \phi_*(C)\cdot L = \phi_*(2t_1 + 2t_2 + p) = 2b_1 + 2b_2 + n,$$

showing, on the one hand that $n \in L$, on the other that Γ is bitangent to L at the points b_1 and b_2. The data contained in L and F is that of a rank 1 conic which is totally tangent to the quintic curve Γ, that is,

$$(L + F)\cdot \Gamma = 2\big(n' + n'' + n + b_1 + b_2\big). \tag{9}$$

[2]We say that $p \in C$ is an antiramification point of a pencil $f \colon C \to \mathbf{P}^1$ if $f^{-1}(p)$ contains a ramification point of f, or equivalently, $f(p) \in \mathbf{P}^1$ is a branch point.

As explained in the Introduction, in order to parametrize the moduli space of odd spin curves of genus 4, we now turn this construction around and fix a line $F \subseteq \mathbf{P}^2$ and distinct points $n', n'', n \in F$. We also fix a general line $L \subseteq \mathbf{P}^2$ passing through n. Let

$$S := \mathrm{Bl}_{\{n,n',n''\}}(\mathbf{P}^2) \stackrel{\epsilon}{\longrightarrow} \mathbf{P}^2$$

be the blow-up of $\mathbf{P}^2$ at n, n' and n'' and denote by E, E', E'' the exceptional divisors over the points n, n' respectively n''. Let $L' \subseteq S$ be the proper transform of L.

We introduce the linear system of 2-nodal plane quintics $\mathbf{P} := \bigl|5h - E - 2E' - 2E''\bigr|$, where $h \in |\epsilon^* \mathcal{O}_{\mathbf{P}^2}(1)|$ and consider the map $\rho\colon \mathbf{P} \dashrightarrow L^{[4]}$ described by (5) and which is the projectivization of the surjective map ρ in the following exact sequence:

(10)

$$0 \longrightarrow H^0\bigl(S, \mathcal{O}_S(4h{-}2E'{-}2E'')\bigr) \stackrel{\cdot L'}{\longrightarrow} H^0\bigl(S, \mathcal{O}_S(5h{-}2E'{-}2E''{-}E)\bigr) \stackrel{\rho}{\longrightarrow} H^0\bigl(L', \mathcal{O}_{L'}(4)\bigr) \longrightarrow 0.$$

In this way, the map $\rho\colon \mathbf{P} \dashrightarrow L^{[4]}$ is a linear projection.

Definition 2.2. Let $\rho\colon \mathcal{C} := \rho^*(\mathbf{V}) \to \mathbf{V}$ denote the cone over the projected Veronese surface $\mathbf{V}$.

Clearly $\mathcal{C}$ can be identified with the parameter space of those plane quintics $\Gamma \in \mathbf{P}$ which are bitangent to the line L. Note that $\mathcal{C}$ being a cone over a rational surface, is a rational 11-dimensional variety.

Proposition 2.3. *One has a dominant map $m^-\colon \mathcal{C} \twoheadrightarrow \overline{\mathcal{S}}_4^-$.*

Proof. We take a general point $[\Gamma] \in \mathbf{P}$, where the plane quintic $\Gamma \subseteq \mathbf{P}^2$ satisfies the relation (9). Retaining the same notation, we may assume the points n', n'', b_1 and b_2 are pairwise distinct and we assume Γ is tangent to L at the points b_1 and b_2. Let

$$\nu\colon C \longrightarrow \Gamma$$

be the normalization map and set $p := \nu^{-1}(n)$, where $\{n\} = L \cap F$. Let $t_i := \nu^{-1}(b_i)$, for $i = 1, 2$. Setting $\vartheta = \mathcal{O}_C(p{+}t_1{+}t_2)$, from (9), we obtain that $\vartheta^2 \cong \omega_C$, therefore $[C, \vartheta, p] \in \Theta_{4,1}^-$. We then define $m^-\bigl([\Gamma]\bigr) = [C, \vartheta]$. Our discussion above shows this map is dominant. □

2.2. **The unirational parametrization of $\overline{\mathcal{S}}_{4,10}^-$.** Let us now fix a general point

$$\overline{p} = (p_1, \ldots, p_{10}) \in \bigl(\mathbf{P}^2\bigr)^{10},$$

and we do not distinguish whether we regard these points in $\mathbf{P}^2$, or in S. We denote by $\mathbf{P}^3_{\overline{p}} := \bigl\{\Gamma \in \mathbf{P} : p_1, \ldots, p_{10} \in \Gamma\bigr\} = \bigl|\mathcal{I}_{\{p_1,\ldots,p_{10}\}/S}(5h - E - 2E' - 2E'')\bigr|$. We consider the intersection $R_{\overline{p}} := \mathcal{C} \cdot \mathbf{P}^3_{\overline{p}}$ already introduced in (6). We have the following:

Proposition 2.4. *If $\overline{p} \in (\mathbf{P}^2)^{10}$ is general, then $R_{\overline{p}} \subseteq \mathbf{P}^3_{\overline{p}}$ is a smooth quartic rational curve.*

Proof. We claim that when $\overline{p}$ is general as above, then $\mathbf{P}^3_{\overline{p}} \cong \rho_*\bigl(\mathbf{P}^3_{\overline{p}}\bigr)$ is a general hyperplane in $L^{[4]} \cong \mathbf{P}^4$. Indeed, we use the sequence (10) and observe that by the generality assumption of the points $p_1, \ldots, p_{10}$, we have

$$H^0\bigl(S, \mathcal{I}_{\{p_1,\ldots,p_{10}\}/S}(4h - 2E' - 2E'')\bigr) = 0,$$

therefore $\rho_*\bigl(\mathbf{P}^3_{\overline{p}}\bigr)$ is a hyperplane in $L^{[4]}$. As $\overline{p}$ varies in $\bigl(\mathbf{P}^2\bigr)^{10}$, then $R_{\overline{p}}$ is a general hyperplane section of $\mathbf{V}$. By Bertini's theorem, it is therefore a smooth rational quartic curve. □

We now globalize this construction and ultimately obtain a variety dominating $\overline{\mathcal{S}}^-_{4,10}$. We denote by $\mathcal{U} \subseteq (\mathbf{P}^2)^{10}$ the dense open subset of configurations $\overline{p} \in (\mathbf{P}^2)^{10}$ such that $\dim \left|\mathcal{I}_{\{p_1,\ldots,p_{10}\}/S}(5h - E - 2E' - 2E'')\right| = 3$. We then consider the projective bundle

$$q\colon \mathcal{P} \longrightarrow \mathcal{U}, \ \text{ with } \ q^{-1}(\overline{p}) = \mathbf{P}^3_{\overline{p}}, \text{ for each } \overline{p} \in \mathcal{U}.$$

We introduce the subvariety $\mathcal{R} = \{(\overline{p}, \Gamma) : \overline{p} \in \mathcal{U}, \ \Gamma \in R_{\overline{p}}\} \subseteq \mathcal{P}$. Using Proposition 2.4, we conclude that the projection

$$\mathfrak{q}\colon \mathcal{R} \longrightarrow \mathcal{U} \tag{11}$$

is birationally a fibration in rational quartic curves over the rational variety $\mathcal{U}$. In particular, using [GHS], we obtain that $\mathcal{R}$ is rationally connected, though we shall soon prove that it is in fact unirational.

There is a moduli map $m\colon \mathcal{R} \dashrightarrow \overline{\mathcal{S}}^-_{4,10}$, which we now describe. Since there is a dominant map $\mathcal{R} \twoheadrightarrow \mathcal{C}$, a general point $(\overline{p}, \Gamma) \in \mathcal{R}$ corresponds to an irreducible plane quintic $\Gamma \subseteq \mathbf{P}^2$, which is nodal at n' and n'' and satisfies the equations, cf. (9):

$$\Gamma \cdot F = 2n' + 2n'' + n \quad \text{and} \quad \Gamma \cdot L = 2b_1 + 2b_2 + n.$$

Choosing the points $p_1, \ldots, p_{10}$ generally, we may also assume they are smooth points of Γ. As in the proof of Proposition 2.3, we denote by $\nu\colon C \to \Gamma$ the normalization map and set $p = \nu^{-1}(n)$, $t_1 = \nu^{-1}(b_1)$ and $t_2 = \nu^{-1}(b_2)$. We then define the map

$$m\big([\overline{p}, \Gamma]\big) = \big[C, \mathcal{O}_C(p + t_1 + t_2), \ p_1, \ldots, p_{10}\big] \in \overline{\mathcal{S}}^-_{4,10},$$

where we identify p_i and $\nu^{-1}(p_i)$. We summarize what has been achieved so far:

Proposition 2.5. *The moduli map $m\colon \mathcal{R} \longrightarrow \overline{\mathcal{S}}^-_{4,10}$ is dominant.*

Proof. Follows immediately by combining Propositions 2.1 and 2.3. □

The canonical way to show that $\mathcal{R}$ is unirational, would be by constructing a rational section of the fibration $\mathfrak{q}\colon \mathcal{R} \to \mathcal{U}$ defined in (11). It is however unclear whether such a section exists. Instead, we first construct a natural unirational *bisection* of $\mathfrak{q}$. To that end, we fix a conic

$$Q \subseteq \mathbf{V} \subseteq \mathbf{P}^4.$$

Note that $\mathbf{V}$ has a 2-dimensional family of such conics which are image of lines in $\mathbf{P}^2$ under the map sq described in (8). We denote by $\mathrm{Hilb}^2(\mathfrak{q}) \to \mathcal{U}$ the relative Hilbert scheme of length 2 subschemes of the fibration $\mathfrak{q}$.

Definition 2.6. The rational section $\sigma\colon (\mathbf{P}^2)^{10} \dashrightarrow \mathrm{Hilb}^2(\mathfrak{q})$ is defined by setting $\sigma(\overline{p}) := R_{\overline{p}} \cdot Q$.

Note that $\sigma(\overline{p})$ corresponds to the intersection in $L^{[2]}$ of a conic corresponding to $R_{\overline{p}}$ with the fixed line whose image under sq defines the conic Q, therefore $\sigma(\overline{p})$ is indeed a length 2 subscheme of $R_{\overline{p}}$. Observe that σ also gives rise to a rational morphism

$$f\colon \mathcal{U} \dashrightarrow Q^{[2]}, \ \ \overline{p} \mapsto R_{\overline{p}} \cdot Q. \tag{12}$$

Proposition 2.7. *The fibration f is birational to a Zariski locally trivial $\mathbf{G}(1,10)$-bundle over $Q^{[2]}$.*

Proof. We choose points $x, x' \in Q$ and let $\ell = \langle x, x' \rangle \subseteq \langle Q \rangle$ be the line spanned by x and x' in $\mathbf{P}^4$. Then $f^*(x + x')$ corresponds to those points $\overline{p} \in \mathcal{U}$ such that $x, x' \in \rho(\mathbf{P}^3_{\overline{p}})$. Identifying

$x, x' \in \mathbf{V}$ with the divisors $2(a+b)$ respectively $2(a'+b')$, where $a, a', b, b' \in L$, then $f^*(x+x')$ corresponds to those points $\overline{p}$ for which there exists nodal quintics $\Gamma, \Gamma' \in \mathbf{P}^3_{\overline{p}}$ such that

$$\Gamma \cdot L = n + 2a + 2b, \quad \text{and} \quad \Gamma \cdot L' = n + 2a' + 2b'.$$

Then recalling that the map ρ defined in (5) is a linear projection, this immediately implies that $\dim \mathbf{P}^3_{\overline{p}} \cdot \rho^*(\ell) \geq 1$. Conversely, if the intersection $\rho^*(\ell) \cdot \mathbf{P}^3_{\overline{p}}$ is positive dimensional, there exist two points $x, x' \in \ell \cdot Q \subseteq \langle Q \rangle$ such that $f(\overline{p}) = x + x'$. We conclude that the general fibre $f^*(x+x')$ is isomorphic to the Grassmannian of lines $\mathbf{G}(1, \rho^*\ell)$. Varying now $x + x' \in Q^{[2]}$, we obtain that f is birational to a locally trivial $\mathbf{G}(1,10)$-bundle over $Q^{[2]}$. $\square$

We now denote by $\tau\colon Q \times Q \to Q^{[2]}$ the double cover given by $\tau(x,y) = x + y$ and set

$$\tilde{f}\colon \widetilde{\mathcal{U}} := \mathcal{U} \times_{Q^{[2]}} \big(Q \times Q\big) \longrightarrow Q \times Q.$$

Therefore $\widetilde{\mathcal{U}}$ is the parameter space of pairs $(\overline{p}, x)$, where $\overline{p} \in \mathcal{U}$ and $x \in \mathbf{P}^3_{\overline{p}} \cdot Q$.

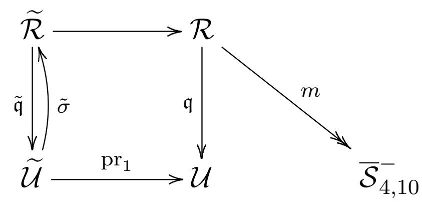
Corollary 2.8. *The morphism $\tilde{f}\colon \widetilde{\mathcal{U}} \to Q \times Q$ is birational to a locally trivial $\mathbf{G}(1,10)$-bundle. In particular, $\widetilde{\mathcal{U}}$ is a rational variety.*

Proof. Follows immediately by base change from Proposition 2.7. $\square$

We now base change the fibration in rational curves $\mathfrak{q}$, to obtain the family

$$\tilde{\mathfrak{q}}\colon \widetilde{\mathcal{R}} = \mathcal{R} \times_{\mathcal{U}} \widetilde{\mathcal{U}} \longrightarrow \widetilde{\mathcal{U}}.$$

The general fibre of $\tilde{\mathfrak{q}}$ is still a rational quartic curve. Note that there exists a rational section $\tilde{\sigma}\colon \widetilde{\mathcal{U}} \dashrightarrow \widetilde{\mathcal{R}}$ given by $\tilde{\mathfrak{q}}\big(\overline{p}, x\big) := \big(\overline{p}, R_{\overline{p}}, \Gamma_x\big)$. Here, $\Gamma_x \in R_{\overline{p}}$ denotes the nodal quintic curve corresponding to the point $x \in Q \cdot \mathbf{P}^3_{\overline{p}}$. The situation is summarized by the following diagram:



Proof of Theorem 1.2. Since $\widetilde{\mathcal{U}}$ is rational by Corollary 2.8 and $\tilde{\mathfrak{q}}$ is generically a fibration in rational varieties which possesses a rational section, it follows that $\widetilde{\mathcal{R}}$ is a rational curve over the function field of $\widetilde{\mathcal{U}}$ having a rational point. We conclude that $\widetilde{\mathcal{R}}$ is a rational variety. Using Proposition 2.5 and the diagram above, we have a dominant map $\widetilde{\mathcal{R}} \dashrightarrow \overline{\mathcal{S}}_{4,10}^-$, therefore we obtain that $\overline{\mathcal{S}}_{4,10}^-$ is unirational. $\square$

3. MODULI SPACES OF POINTED SPIN CURVES

We recall basic facts concerning the moduli space $\overline{\mathcal{S}}_{g,n}$ of n-pointed spin curves of genus g, largely following [Cor], [F2]. If $(X, p_1, \ldots, p_n)$ is an n-pointed nodal curve, a smooth rational component $E \subseteq X$ is said to be *exceptional* if $\big|E \cap \overline{(X \setminus E)}\big| = 2$ and no marked point p_i lies on E. The curve X is said to be *quasi-stable* if any two exceptional components of X are disjoint. A quasi-stable curve is thus obtained from a stable pointed curve by blowing-up each node at most once.

Definition 3.1. An n-pointed *stable spin curve* of genus g consists of a triple $(X, p_1, \ldots, p_n, \vartheta, \beta)$, where $(X, p_1, \ldots, p_n)$ is an n-pointed genus g quasi-stable curve, $\vartheta \in \mathrm{Pic}^{g-1}(X)$ is a line bundle of total degree $g-1$ such that $\vartheta_E = \mathcal{O}_E(1)$ for every exceptional component $E \subseteq X$, and $\beta\colon \vartheta^{\otimes 2} \to \omega_X$ is a sheaf homomorphism which is generically non-zero along each non-exceptional component of X.

We denote by $\overline{\mathcal{S}}_{g,n}$ the moduli stack of n-pointed spin curves of genus g, which splits into two connected components $\overline{\mathcal{S}}^-_{g,n}$ and $\overline{\mathcal{S}}^+_{g,n}$ depending on the parity of $h^0(X, \vartheta)$. One has a finite morphism $\pi\colon \overline{\mathcal{S}}_{g,n} \to \overline{\mathcal{M}}_{g,n}$ of degree 2^{2g}. We also set $\overline{\mathcal{S}}_{g,[n]} := \overline{\mathcal{S}}_{g,n}/\mathfrak{S}_n$, where the action is by permuting the marked points. Let $\lambda := \pi^*(\lambda)$ and $\psi_i := \pi^*(\psi_i) \in CH^1(\overline{\mathcal{S}}_{g,n})$ be the standard codimension one tautological classes; we refer to [AC] for background on divisor classes on $\overline{\mathcal{M}}_{g,n}$.

3.1. **Boundary divisors on $\overline{\mathcal{S}}_{g,n}$.** We introduce the standard notation for the boundary divisors on both components of $\overline{\mathcal{S}}^+_{g,n}$ as follows. For $0 \le i \le g$ and a subset $S \subseteq [n]$, we denote by $A_{i:S} \subseteq \overline{\mathcal{S}}^+_{g,n}$ (respectively $B_{i:S} \subseteq \overline{\mathcal{S}}^+_{g,n}$) the closure of the locus consisting of spin curves $[C \cup_y D, p_1, \ldots, p_n, \vartheta_C, \vartheta_D]$, where C and D are smooth curves of genera i and $g-i$ meeting at a point y, both theta-characteristics ϑ_C and ϑ_D are even (respectively odd), and the marked points lying on the component C are precisely those labelled by the set S. We set $\alpha_{i:S} := [A_{i:S}]$ and $\beta_{i:S} := [B_{i:S}]$. Note that $B_{0:S} = \emptyset$ for all S. For $2 \le s \le n$, we set

$$\alpha_{0:s} := \sum_{S \in \binom{[n]}{s}} \alpha_{0:S} \in CH^1(\overline{\mathcal{S}}^+_{g,n}). \tag{13}$$

Similarly, we let $A_{i:S} \subseteq \overline{\mathcal{S}}^-_{g,n}$ be the closure of the locus consisting of those spin curves $[C \cup_y D, p_1, \ldots, p_n, \vartheta_C, \vartheta_D]$, where C and D are smooth curves of genera i and $g-i$ meeting at a point y, the theta-characteristic ϑ_C is odd, whereas ϑ_D is even, and the marked points lying on C are precisely those labelled by S. We set $\alpha_{i:S} := [A_{i:S}]$. To ensure uniformity with the notation from [FV3], we also set $B_{i:S} := A_{g-i:S^c}$. Note that in this case, $A_{0:S} = \emptyset$ for all S. Then for $s \ge 2$, we can define the class

$$\beta_{0:s} := \sum_{S \in \binom{[n]}{s}} \beta_{0:S} \in CH^1(\overline{\mathcal{S}}^-_{g,n}). \tag{14}$$

We recall the description of the pull-back of the boundary divisor Δ_{irr} of $\overline{\mathcal{M}}_{g,n}$ under the finite cover $\pi\colon \overline{\mathcal{S}}_{g,n} \to \overline{\mathcal{M}}_{g,n}$. For a point $[X, p_1, \ldots, p_n, \vartheta, \beta] \in \overline{\mathcal{S}}_{g,n}$ corresponding to a stable model $\mathrm{st}(X) = C_{yq} := C/y \sim q$, with $[C, y, q] \in \mathcal{M}_{g-1,2}$, there are two possibilities for the spin structure ϑ, depending on whether X has an exceptional component or not. If X has no exceptional component and $\vartheta_C := \nu^*(\vartheta)$, where $\nu\colon C \to X$ denotes the normalization map, then $\vartheta_C^{\otimes 2} = \omega_C(y+q)$. For each choice of ϑ_C as above, there is precisely one choice of gluing the fibres $\vartheta_C(y)$ and $\vartheta_C(q)$ such that $h^0(X, \eta)$ has a prescribed parity. We denote by A_0 the closure in $\overline{\mathcal{S}}_{g,n}$ of the locus of spin curves as above.

If $X = C \cup_{\{y,q\}} E$, where E is an exceptional component, then the restriction ϑ_C is a theta-characteristic on C. Let $B_0 \subseteq \overline{\mathcal{S}}_{g,n}$ be the closure of the locus of spin curves

$$\big[C \cup_{\{y,q\}} E, p_1, \ldots, p_n,\ \vartheta_C \in \sqrt{\omega_C},\ \vartheta_E = \mathcal{O}_E(1)\big] \in \overline{\mathcal{S}}_{g,n}.$$

Remark 3.2. One has an isomorphism $B_0 \cong \overline{\mathcal{S}}_{g-1:[2]}$ valid for both components of $\overline{\mathcal{S}}_g$.

If $\alpha_0 := [A_0], \beta_0 := [B_0] \in CH^1(\overline{\mathcal{S}}_{g,n})$, then $\pi^*\big(\delta_{\mathrm{irr}}\big) = \alpha_0 + 2\beta_0$. In particular, B_0 is the ramification divisor of the map π. Coupled with the formula [Log] for $K_{\overline{\mathcal{M}}_{g,n}}$, this yields

$$K_{\overline{\mathcal{S}}_{g,n}} = 13\lambda - 2\alpha_0 - 3\beta_0 + \sum_{i=1}^{n} \psi_i - 2\sum_{i,S\subseteq[n]} \big(\alpha_{i:S} + \beta_{i:S}\big) - \alpha_{1:\emptyset} - \beta_{1:\emptyset} \in CH^1(\overline{\mathcal{S}}_{g,n}). \tag{15}$$

3.2. **Extension of pluricanonical forms.** The Kodaira dimension of $\overline{\mathcal{S}}_{g,n}$, being an invariant of the coarse moduli space $\overline{\mathcal{S}}_{g,n}$ rather than of the stack, is defined by passing to a resolution of singularities $\epsilon\colon \widehat{\mathcal{S}}_{g,n} \to \overline{\mathcal{S}}_{g,n}$. For a $\mathbb{Q}$-factorial normal projective variety X, we denote by $\kappa(X)$ its Kodaira dimension and by $\kappa(X, K_X)$ the Kodaira–Iitaka dimension of its canonical bundle. In order to work directly on the space $\overline{\mathcal{S}}_{g,n}$ whose geometry we can control, we need to know that pluricanonical forms on $\overline{\mathcal{S}}_{g,n}$ extend to any resolution, therefore $\kappa(\overline{\mathcal{S}}_{g,n}) = \kappa(\overline{\mathcal{S}}_{g,n}, K_{\overline{\mathcal{S}}_{g,n}})$. Such a result has been established for $\overline{\mathcal{M}}_g$ in [HM], for $\overline{\mathcal{S}}_g$ in [Lu], and for $\overline{\mathcal{M}}_{g,n}$ in [Log].

Proposition 3.3. *We fix $g \geq 4$ and $n \geq 0$. Then for any $\ell \geq 0$ one has an isomorphism*

$$\epsilon^*\colon H^0\big(\overline{\mathcal{S}}_{g,n}, K_{\overline{\mathcal{S}}_{g,n}}^{\otimes \ell}\big) \xrightarrow{\cong} H^0\big(\widehat{\mathcal{S}}_{g,n}, K_{\widehat{\mathcal{S}}_{g,n}}^{\otimes \ell}\big).$$

Proof. We explain the local description of the map $\pi\colon \overline{\mathcal{S}}_{g,n} \to \overline{\mathcal{M}}_{g,n}$ around a spin curve $[X, p_1, \ldots, p_n, \vartheta, \beta]$. We denote by $E_1, \ldots, E_r$ the exceptional components of X, by $\nu\colon X \to C$ the map contracting $E_1, \ldots, E_r$ and we set $q_i := \nu(E_i) \in C_{\mathrm{sing}}$, for $i = 1, \ldots, r$. Let $\mathbb{C}_\tau^{3g-3+n}$ be the versal deformation space of $(X, p_1, \ldots, p_n, \vartheta, \beta)$, where the coordinates $(\tau_1, \ldots, \tau_{3g-3+n})$ are chosen in such a way that for $1 \leq i \leq r$ the locus $(\tau_i = 0)$ corresponds to those deformations which preserve the components E_i. Then $\mathbb{C}_t^{3g-3+n} \cong \mathrm{Ext}^1\big(\Omega_C(x_1 + \cdots + x_n), \mathcal{O}_C\big)$ is the versal defomation space of $[C, \nu(p_1), \ldots, \nu(p_n)]$, where the coordinates $(t_1, \ldots, t_{3g-3+n})$ are chosen in such a way that for $1 \leq i \leq r$ the divisor $(t_i = 0) \subseteq \mathbb{C}_t^{3g-3+n}$ corresponds to the deformation failing to smooth the node q_i.

The morphism $\pi\colon \overline{\mathcal{S}}_{g,n} \to \overline{\mathcal{M}}_{g,n}$ is locally given by the following map

$$\frac{\mathbb{C}^{3g-3+n}}{\mathrm{Aut}(X, p_1, \ldots, p_n, \vartheta, \beta)} \longrightarrow \frac{\mathbb{C}^{3g-3+n}}{\mathrm{Aut}\big(C, \nu(p_1), \ldots, \nu(p_n)\big)}, \quad t_i = \tau_i^2,\ i \leq r, \quad t_i = \tau_i, i \geq r+1.$$

The singularities of the quotient $\mathbb{C}_\tau^{3g-3+n}/\mathrm{Aut}(X, p_1, \ldots, p_n, \vartheta)$ are studied via the *Reid-Tau criterion* [HM, p.27]. For an automorphism $\phi \in \mathrm{Aut}(X, p_1, \ldots, p_n, \vartheta, \beta)$ having order m, we define its *age* as the quantity $\mathrm{age}(\phi) = \frac{a_1}{m} + \cdots + \frac{a_{3g-3+n}}{m}$, where $\zeta^{a_1}, \ldots, \zeta^{a_{3g-3+n}}$ are the eigenvalues of ϕ, with $\zeta = e^{\frac{2\pi i}{m}}$. An automorphism ϕ with $\mathrm{age}(\phi) \geq 1$ leads to a canonical singularity. Every automorphism of $\phi \in \mathrm{Aut}(X, p_1, \ldots, p_n, \vartheta, \beta)$ induces an automorphism ϕ_{up} of the *unpointed* spin curve (X, ϑ, β). Clearly, one has $\mathrm{age}(\phi_{\mathrm{up}}) \leq \mathrm{age}(\phi)$. Note also that contracting a rational curve under the stabilization map $\nu\colon X \to C$ does not change the age of the automorphism. We now use [Lu, Theorem 3.1] stating that if (X, ϑ, β) admits an automorphism ϕ with $\mathrm{age}(\phi) < 1$, then necessarily C has an elliptic tail J_0 with j-invariant $j(J_0) = 0$ and that $\vartheta_{J_0} \cong \mathcal{O}_{J_0}$ and ϕ acts non-trivially on J_0. Therefore the same conclusion applies for $\phi \in \mathrm{Aut}(X, p_1, \ldots, p_n, \vartheta, \beta)$.

We are left with the exceptional case singled out in [Lu], that is, when X has an ellipic tail J_0 as above. If one of the marked points p_i lies on J_0, then the resulting singularity is canonical, cf. [Log, Theorem 2.5]. If $p_1, \ldots, p_n \in X \setminus J_0$, the argument in [Lu, Theorem 4.1] or in [HM, p.43-44] applies mutatis mutandis to show that the non-canonical singularity of

$\overline{\mathcal{S}}_{g,n}$ induced by ϕ does not impose global adjunction conditions, that is, every pluricanonical form on $\overline{\mathcal{S}}_{g,n}$ can be lifted to a resolution. $\Box$

Remark 3.4. A result analogous to Proposition 3.3 has been established in [Kr, Theorem 5.61] for the space $\overline{\mathcal{S}}^+_{1,n}$. Probably the analysis in Proposition 3.3 can be extended to cover the case $g = 2, 3$ as well.

3.3. **The universal odd theta-characteristic.** We now compute the class of an important effective divisor on $\overline{\mathcal{S}}^-_{g,1}$, namely the closure $\overline{\Theta}^-_{g,1}$ of the locus

$$\Theta^-_{g,1} := \Big\{[C, p, \vartheta] \in \mathcal{S}^-_{g,1} : H^0(C, \vartheta(-p)) \neq 0\Big\}$$

of points in the supports of odd theta-characteristics. The forgetful map $\Theta^-_{g.1} \to \mathcal{S}^-_g$ is a generically finite cover branched along the divisor Z_g considered in [FV3] and consisting of odd spin curves $[C, \vartheta] \in \mathcal{S}^-_g$ such that ϑ is non-reduced.

Remark 3.5. The divisor $\Theta^-_{g,1}$ is intimately related to the stratum of abelian differentials

$$\mathcal{H}_g\big(2^{g-1}\big) := \Big\{[C, p_1, \ldots, p_{g-1}] \in \mathcal{M}_{g,2g-2} : \omega_C = \mathcal{O}_C\big(2p_1 + \cdots + 2p_{g-1}\big)\Big\}.$$

There is a dominant forgetful morphism $\mathcal{H}_g\big(2^{g-1}\big) \to \Theta^-_{g,1}$. Since $\mathcal{H}_g\big(2^{g-1}\big)$ is irreducible, see [KZ], it follows that $\Theta^-_{g,1}$ is also irreducible for $g \geq 2$.

Theorem 3.6. *The following formula holds for $g \geq 2$:*

$$[\overline{\Theta}^-_{g,1}] = \frac{\lambda}{4} + \frac{\psi}{2} - \frac{\alpha_0}{16} - \frac{1}{2}\sum_{i=1}^{g-1} \alpha_{i:\emptyset} \in CH^1(\overline{\mathcal{S}}^-_{g,1}).$$

Proof. We expand the class of the divisor $\overline{\Theta}^-_{g,1}$ in the standard basis of $CH^1(\overline{\mathcal{S}}^-_{g,1})$

$$[\overline{\Theta}^-_{g,1}] = \bar{\lambda}\cdot\lambda + \bar{\psi}\cdot\psi - \bar{\alpha}_0\cdot\alpha_0 - \bar{\beta}_0\cdot\beta_0 - \sum_{i=1}^{g-1}\big(\bar{\alpha}_{i:1}\cdot\alpha_{i:1} + \bar{\alpha}_{i:\emptyset}\cdot\alpha_{i:\emptyset}\big) \in CH^1(\overline{\mathcal{S}}^-_{g,1}), \tag{16}$$

and determine the coefficients by intersecting both sides of the equality (16) with standard test curves lying in the boundary of $\overline{\mathcal{S}}^-_{g,1}$.

For $1 \leq i \leq g-1$, we fix general spin curves $[C, \vartheta^+_C] \in \mathcal{S}^+_i$ and $[D, q, y, \vartheta^-_D] \in \mathcal{S}^-_{g-i,2}$ and we form the test curve

$$F_i := \Big\{[C \cup_y D, q \in D, \vartheta^+_C,\ \vartheta^-_D] : y \in C\Big\} \subseteq \overline{\mathcal{S}}^-_{g,1}.$$

Note that $F_i \cdot \alpha_{g-i:1} = -\deg(\omega_C) = 2 - 2i$ and that F_i has intersection number zero with all remaining generators of $CH^1(\overline{\mathcal{S}}^-_{g,1})$. Then we fix general spin curves $[C, \vartheta^-_C] \in \mathcal{S}^-_i$ and $[D, q, y, \vartheta^+_D] \in \mathcal{S}^+_{g-i,2}$ and consider the following test curve

$$G_i := \Big\{[C \cup_y D, q \in D, \vartheta^-_C,\ \vartheta^+_D] : y \in C\Big\} \subseteq \overline{\mathcal{S}}^-_{g,1}.$$

Then $G_i \cdot \alpha_{i:\emptyset} = 2-2i$ and the other intersections with the generators of $CH^1(\overline{\mathcal{S}}^-_{g,1})$ equal zero.

We claim that F_i is disjoint from $\overline{\Theta}^-_{g,1}$, in particular $\bar{\alpha}_{g-i:1} = 0$, for $i = 2, \ldots, g-1$. Indeed, assume there exists a point $t \in \overline{\Theta}^-_{g,1} \cap F_i$ corresponding to a point $y \in C$. Then there exists a pair of non-zero sections (σ_C, σ_D) corresponding to a limit theta-characteristic on $C \cup_y D$ such that $\sigma_D(q)=0$. We may assume that $h^0(D,\vartheta_D^-)=1$ and that neither y nor q are zeroes of the unique section in $H^0(D,\vartheta_D^-)$. One has $\sigma_C\in H^0\big(C,\vartheta_C^+((g-i)\cdot y)\big)$, $\sigma_D\in H^0\big(D,\vartheta_D^-(i\cdot y)\big)$ and, by the definition of a limit linear series, that $\mathrm{ord}_y(\sigma_C)+\mathrm{ord}_y(\sigma_D)\geq g-1$. Since $\sigma_D(q)=0$, it follows that $\mathrm{ord}_y(\sigma_D)\leq i-1$, therefore $\mathrm{ord}_y(\sigma_C)\geq g-i$, from which it follows that $H^0(C,\vartheta_C^+)\neq 0$, which contradicts the generality of $[C,\vartheta_C^+]\in\mathcal{S}_{i,1}^+$. This proves that $F_i\cdot\overline{\Theta}_{g,1}^-=0$.

Next we claim that $G_i\cdot\overline{\Theta}_{g,1}^-=i-1$, therefore $\bar{\alpha}_{i:\emptyset}=\frac{1}{2}$, for $i=2,\ldots,g-1$. Let $t\in\overline{\Theta}_{g,1}^-\cap G_i$ be a point corresponding to a point $y\in C$. Then we have $\sigma_C\in H^0\big(C,\vartheta_C^-((g-i)\cdot y)\big)$ and $\sigma_D\in H^0\big(D,\vartheta_D^+(i\cdot y)\big)$ and, again, $\mathrm{ord}_y(\sigma_C)+\mathrm{ord}_y(\sigma_D)\geq g-1$. Since, by generality, we may assume $H^0\big(D,\vartheta_D(y-q)\big)=0$, it follows that $\mathrm{ord}_y(\sigma_D)\leq i-2$, therefore $\mathrm{ord}_y(\sigma_C)\geq g-i+1$, that is, $y\in\mathrm{supp}(\eta_C^-)$. This yields $i-1$ points for the intersection $G_i\cdot\overline{\Theta}_{g,1}^-$, each corresponding to the points in the support of ϑ_C^-. To see that each of them counts with multiplicity one, we reason along the lines of [FV3, Proposition 5.1].

Now we argue that $\overline{\Theta}_{g,1}^-$ is disjoint from two elliptic pencils in the boundary of $\overline{\mathcal{S}}_{g,1}^-$. Let $f\colon \mathrm{Bl}_9(\mathbf{P}^2)\to\mathbf{P}^1$ be the family of elliptic curves induced by blowing-up a pencil of plane cubics at its 9 base points and denote by $\tau\colon\mathbf{P}^1\to\mathrm{Bl}_9(\mathbf{P}^2)$ a section corresponding to one of the base points of the pencil. We fix a general spin curve $[C,y,q,\vartheta_C^+]\in\mathcal{S}_{g-1,2}^+$ and set

$$F_0:=\Big\{\big[C\cup_{y\sim\tau(t)}f^{-1}(t),\ \vartheta_C^+,\ \vartheta_{f^{-1}(t)}=\mathcal{O}_{f^{-1}(t)}\big]:t\in\mathbf{P}^1\Big\}\subseteq\overline{\mathcal{S}}_{g,1}^-.$$

We find $F_0\cdot\alpha_{1:\emptyset}=-1$ and $F_0\cdot\lambda=1$. For each of the 12 points $t_\infty\in\mathbf{P}^1$ corresponding to singular fibres of f, the spin structure is locally free on $C\cup f^{-1}(t_\infty)$, therefore we obtain a point lying in the divisor A_0. We therefore conclude that $F_0\cdot\beta_0=0$ and, accordingly, $F_0\cdot\alpha_0=12$.

A second elliptic pencil is obtained by choosing a general element $[C,y,q,\vartheta_C^-]\in\mathcal{S}_{g-1,2}^-$. On $f^{-1}(t)$ one takes each of the three even theta-characteristics. This induces the family

$$G_0:=\Big\{\big[C\cup_{y\sim\tau(t)}f^{-1}(t),\ \vartheta_C^-,\ ,\vartheta_{f-1(t)}^+\in\gamma^{-1}[f^{-1}(t)]\big]:t\in\mathbf{P}^1\Big\}\subseteq\overline{\mathcal{S}}_{1,1}^-,$$

with $\gamma\colon\overline{\mathcal{S}}_{1,1}^+\to\overline{\mathcal{M}}_{1,1}$ being the degree 3 map forgetting the spin structure. We obtain $G_0\cdot\lambda=3$ and $G_0\cdot\beta_{1:\emptyset}=-3$. The map γ being simply ramified over the point corresponding to j-invariant ∞, we conclude that $G_0\cdot\alpha_0=12$ and therefore $G_0\cdot\beta_0=12$.

It is a simple exercise in limit linear series to show that $\overline{\Theta}_{g,1}^-$ is disjoint from both F_0 and G_0. Accordingly, we obtain the following relations

$$\bar{\lambda}-12\bar{\alpha}_0+\bar{\alpha}_{1:\emptyset}=0\quad\text{and}\quad\bar{\lambda}-4\bar{\alpha}_0-4\bar{\beta}_0+\bar{\beta}_{1:\emptyset}=0. \tag{17}$$

Next we fix a general pointed even spin curve $[C,y,\vartheta_C^+]\in\mathcal{S}_{g-1,1}^+$ and consider the map $\nu\colon\overline{\mathcal{M}}_{1,2}\to\overline{\mathcal{S}}_{g,1}^-$ given by $[J,y,q]\mapsto\big[J\cup_y C,q,\vartheta_J^-=\mathcal{O}_J,\ \vartheta_C^+\big]\in\overline{\mathcal{S}}_{g,1}^-$, where J denotes an elliptic curve. The following relations are immediate:

$$\nu^*(\lambda)=\lambda,\ \nu^*(\psi)=\psi_q,\nu^*(\alpha_0)=\delta_{\mathrm{irr}},\ \nu^*(\beta_0)=0,\ \nu^*(\alpha_{1:1})=-\psi_x,\ \ \nu^*(\alpha_{1:\emptyset})=\delta_{0:yq}.$$

Again, it is an easy exercise to show that $\nu^*(\overline{\Theta}_{g,1}^-)=0$. Inside $CH^1(\overline{\mathcal{M}}_{1,2})\cong\mathbb{Q}\langle\psi_y,\delta_{0:yq}\rangle$ one has the following well-known relations [AC, Proposition 1.9]:

$$\psi_y=\psi_q,\ \lambda=\psi_y-\delta_{0:yq},\ \ \delta_{\mathrm{irr}}=12(\psi_y-\delta_{0:yq}).$$

We obtain that $0 = \big[\nu^*(\overline{\Theta}^-_{g,1})\big] = \big(\bar{\lambda} + \bar{\psi} - 12\bar{\alpha}_0\big) \cdot \psi_y - \big(\bar{\lambda} - 12\bar{\alpha}_0 + \bar{\alpha}_{1:\emptyset}\big) \cdot \delta_{0:yq} \in CH^1(\overline{\mathcal{M}}_{1,2})$. It is a consequence of [F3, Proposition 4.3] that $\bar{\alpha} = \frac{1}{4}$, while obviously $\bar{\psi} = \frac{1}{2}$. It follows that $\bar{\alpha}_0 = \frac{1}{12}\big(\frac{1}{2} + \frac{1}{4}\big) = \frac{1}{16}$. It then follows from the first relation in (17) that $\bar{\alpha}_{1:\emptyset} = \frac{1}{2}$.

We are now in a position to determine the coefficient $\bar{\beta}_0$ and to that end, we use that in [F3, Theorem 0.3] the class of the pushforward $\pi_*\big([\overline{\Theta}^-_{g,1}]\big) \in CH^1(\overline{\mathcal{M}}_{g,1})$ has been determined. In particular, its δ_{irr}-coefficient is equal to -2^{2g-6}. From the set-theoretic description of both divisors α_0 and β_0, clearly $\pi_*(\alpha_0) = 2^{2g-2}\delta_{\mathrm{irr}}$ and $\pi_*(\beta_0) = 2^{g-2}(2^{g-1}-1)\delta_{\mathrm{irr}}$, therefore $2^{2g-2}\bar{\alpha}_0 + 2^{g-2}(2^{g-1}-1)\bar{\beta} = 2^{2g-6}$. Since $\bar{\alpha}_0 = \frac{1}{16}$, we obtain $\bar{\beta}_0 = 0$. Finally, from the second equation in (17), we obtain $\beta_{1:\emptyset} = 0$. This completes the proof. □

3.4. **The uniruledness of $\overline{\Theta}^-_{g,1}$ for small g.** In the range when the general curve of genus g lies on a $K3$ surface, we can show that the divisor $\overline{\Theta}^-_{g,1}$ is uniruled. In fact, we have a more precise result which will be used when determining the Kodaira dimension of $\overline{\mathcal{S}}^-_{g,1}$:

Theorem 3.7. *For $g \leq 9$ or $g = 11$, the universal theta-characteristic $\overline{\Theta}^-_{g,1}$ is uniruled. Precisely through a general point of $\overline{\Theta}^-_{g,1}$ there passes a rational curve $\Gamma \subseteq \overline{\Theta}^-_{g,1}$, such that $\Gamma \cdot \overline{\Theta}^-_{g,1} = 0$.*

Proof. We fix a general point $[C, p, \vartheta] \in \mathcal{S}^-_{g,1}$ and we may assume $h^0(C, \vartheta) = 1$ and that $\mathrm{supp}(\vartheta)$ consists of $g-1$ distinct points $p = p_1, p_2, \ldots, p_{g-1}$. When $g \leq 11$ and $g \neq 10$, there exists a smooth $K3$ surface $S \supseteq C$. We may even assume $\mathrm{Pic}(S) \cong \mathbb{Z} \cdot C$. Note that in the embedding $S \stackrel{|C|}{\longrightarrow} \mathbf{P}^g$, the points $p_1, \ldots, p_{g-1}$ span a codimension two subspace $\Pi = \langle p_1, \ldots, p_{g-1}\rangle \subseteq \mathbf{P}^g$ such that $\Pi \cdot S = 2(p_1 + \cdots + p_{g-1})$. Following [FV3, Theorem 3.10], we consider the pencil of hyperplanes $\{H_t\}_{t \in \mathbf{P}^1}$ in $\mathbf{P}^g$ containing Π. This induces a rational curve in moduli

$$\Gamma := \Big\{\big[C_t := H_t \cap S,\ p_1,\ \ \vartheta_t = \mathcal{O}_{C_t}(p_1 + \cdots + p_{g-1})\big] : t \in \mathbf{P}^1\Big\} \subseteq \overline{\Theta}^-_{g,1}. \tag{18}$$

It follows that $\Gamma \cdot \lambda = g+1$, $\Gamma \cdot \beta_0 = g-1$ and that $\Gamma \cdot \alpha_0 = 4g+20$, see [FV3, Corollary 3.8]. The points in the intersection $\Gamma \cdot \beta_0$ correspond to those hyperplanes H_t such that the intersection $H_t \cap S$ is nodal at one of the points p_i. All the elements in the pencil $\{H_t \cap S\}_{t \in \mathbf{P}^1}$ are irreducible and have at most one singular point.

Observe that $\Pi \cap \mathbf{T}_p(S)$ is a 1-dimensional space and this tangent direction may be regarded as a point v on the exceptional divisor E_p of the blow-up $\widetilde{S} = \mathrm{Bl}_{\{p_1, \ldots, p_{g-1}\}}(S)$. Set $\epsilon\colon S' := \mathrm{Bl}_v(S') \to S$ and denote by $E'_p \subseteq S'$ the proper transform of E_p. The family of pointed spin curves in (18) is then induced by the fibration $f'\colon S' \to \mathbf{P}^1$, where

$$f'^*\big(\mathcal{O}_{\mathbf{P}^1}(1)\big) \cong \mathcal{O}_{S'}(\epsilon^*(C) - E'_p - E_{p_2} - \cdots - E_{p_{g-1}}).$$

We have $\Gamma \cdot \psi = -\big(E'_p\big)^2 = 2$. Using Theorem 3.6, we obtain $\Gamma \cdot \overline{\Theta}^-_{g,1} = \frac{g+1}{4} + 1 - \frac{4g+20}{16} = 0$. □

3.5. **The positivity of $K_{\overline{\mathcal{S}}_{g,n}}$.** In order to prove that $K^{\pm}_{\overline{\mathcal{S}}_{g,n}}$ is big, we introduce some known effective divisors on moduli spaces related to $\overline{\mathcal{S}}^{\pm}_{g,n}$. We first recall the class [F1] of the compactification $\overline{\Theta}_{\mathrm{null}}$ of the theta-null divisor $\Theta_{\mathrm{null}} := \Big\{[C, \vartheta] \in \mathcal{S}^+_g : H^0(C, \vartheta) \neq 0\Big\}$:

$$[\overline{\Theta}_{\mathrm{null}}] = \frac{1}{4}\lambda - \frac{1}{16}\alpha_0 - \frac{1}{2}\sum_{i=1}^{\lfloor \frac{g}{2} \rfloor} \beta_i \in CH^1(\overline{\mathcal{S}}^+_g). \tag{19}$$

On $\overline{\mathcal{S}}_g^-$, we have the class of the closure of the locus Z_g of odd spin curves with a non-reduced support [FV3, Theorem 0.4]:

$$[\overline{Z}_g] = (g+8)\lambda - \frac{g+2}{4}\alpha_0 - 2\beta_0 - \sum_{i=1}^{g-1} 2(g-i)\alpha_i \in CH^1(\overline{\mathcal{S}}_g^-). \tag{20}$$

Observe that Z_g is the branch divisor of the generically finite forgetful map $\Theta_{g,1}^- \to \mathcal{S}_g^-$. On $\overline{\mathcal{M}}_{g,n}$, we have the class of Logan's divisor [Log, Theorems 5.3-5.7] defined as the closure in $\overline{\mathcal{M}}_{g,g}$ of the following locus $D_g := \Big\{[C, p_1, \ldots, p_g] \in \mathcal{M}_{g.g} : h^0\big(C, \mathcal{O}_C(p_1 + \cdots + p_g)\big) \geq 2\Big\}$:

$$[\overline{D}_g] = -\lambda + \sum_{i=1}^{g} \psi_i - \sum_{i \geq 0, S} \binom{\big||S| - i\big| + 1}{2} \delta_{i:S} \in CH^1(\overline{\mathcal{M}}_{g,g}).$$

We are now in a position to prove Theorem 1.3:

Proof of Theorem 1.3 for $\overline{\mathcal{S}}_{g,n}^+$. We first treat the case $n \geq g$. We shall use the fact that the class $\sum_{i=1}^n \psi_i$ is big on $\overline{\mathcal{S}}_{g,n}^+$. For a subset $S \subseteq [n]$ with $|S| = g$, let $\pi_S \colon \overline{\mathcal{M}}_{g,n} \to \overline{\mathcal{M}}_{g,g}$ be the morphism forgetting all the marked points in S^c and we consider the symmetrized version of the pull-back of Logan's effective divisor $D_{g,n} := \frac{1}{\binom{n}{g}} \pi_S^*(D_g)$ on $\overline{\mathcal{M}}_{g,n}$. Then using standard formulas for the pull-backs of boundary classes under forgetful maps [AC], we compute

$$\big[\overline{D}_{g,n}\big] = -\lambda + \frac{g}{n} \sum_{i=1}^{n} \psi_i - \frac{g(g-3+2n)}{n(n-1)} \sum_{|S|=2} \delta_{0:S} - \cdots \in CH^1(\overline{\mathcal{M}}_{g,n}). \tag{21}$$

We choose an effective divisor D on $\overline{\mathcal{M}}_g$ with $[D] = s\lambda - \delta_{\mathrm{irr}} - \sum_{i=1}^{\lfloor \frac{g}{2} \rfloor} b_i \cdot \delta_i \in CH^1(\overline{\mathcal{M}}_g)$, with $b_i \geq 1$, and where $s = 6 + \frac{12}{g+1}$ when $g+1$ is composite (in which case D is a multiple of a Brill-Noether divisor [EH, Theorem 1]), respectively $s = \frac{17}{2}$ when $g = 4$, respectively $s = \frac{47}{6}$ when $g = 6$ (in both these cases D being a multiple of the corresponding Petri divisor, see [EH, Theorem 2]) and finally $s = 7$ when $g = 10$, in which case D is the $K3$ divisor considered in [FP]. Denoting by $\sigma \colon \overline{\mathcal{S}}_{g,n} \to \overline{\mathcal{M}}_g$ respectively by $\mu \colon \overline{\mathcal{S}}_{g,n} \to \overline{\mathcal{S}}_g$ the natural forgetful maps and recalling that $\pi \colon \overline{\mathcal{S}}_{g,n}^+ \to \overline{\mathcal{M}}_{g,n}$ is the covering map, we form the following effective combination on $\overline{\mathcal{S}}_{g,n}^+$

$$8\big[\mu^*(\overline{\Theta}_{\mathrm{null}})\big] + \frac{2n(n-1)}{g(g-3+2n)}\big[\pi^*(\overline{D}_{g,n})\big] + \frac{3}{2}\big[\sigma^*(D)\big] = \Big(\frac{3s+4}{2} - \frac{2n(n-1)}{g(g-3+2n)}\Big)\lambda +$$
$$\frac{2(n-1)}{g-3+2n} \sum_{i=1}^{n} \psi_i - 2\alpha_0 - 3\beta_0 - 2\alpha_{0:2} - \sum_{i,S} \big(\overline{\alpha}_{i:S} \cdot \alpha_{i:S} + \overline{\beta}_{i:S} \cdot \beta_{i:S}\big) \in CH^1(\overline{\mathcal{S}}_{g,n}^+),$$

where $\overline{\alpha}_{i:S}, \overline{\beta}_{i:S} \geq 2$ in all cases except when $i = 0$ and $|S| = 2$, in which case the corresponding coefficients are equal to zero. We compare this formula to the expression (15). We observe that the coefficient of $\sum_{i=1}^n \psi_i$ is smaller than one, whereas the range of (g, n) in the statement of Theorem 1.3 is choses such that the λ-coefficient is less than 13. It follows that $K_{\overline{\mathcal{S}}_{g,n}^+}$ is big. Coupled with Proposition 3.3, this shows that $\overline{\mathcal{S}}_{g,n}^+$ is of general type in this range.

For $g=8$, it is easy to see that $K_{\overline{\mathcal{S}}^+_{8,1}}$ can be expressed as a combination with positive coefficients of $[\sigma^*(D)]$, $\bigl[\mu^*(\overline{\Theta}_{\mathrm{null}})\bigr]$, ψ, the pull-back of the Weierstrass divisor under the forgetful map $\overline{\mathcal{S}}^+_{8,1}\to\overline{\mathcal{M}}_{8,1}$ and certain boundary divisors. It follows that $K_{\overline{\mathcal{S}}^+_{8,1}}$ is big.

Finally, on $\overline{\mathcal{S}}^+_{7,3}$, we consider the closure of the divisor $\mathfrak{D}_{2:2:3}$ consisting of those pointed spin curves $\bigl[C,p_1,p_2,p_3,\vartheta\bigr]\in\mathcal{S}^+_{7,3}$, such that there exists a permutation $\tau\in\mathfrak{S}_3$ with

$$h^0\bigl(C,\mathcal{O}_C(2p_{\tau(1)}+2p_{\tau(2)}+3p_{\tau(3)})\bigr)\geq 2.$$

Using [Log, Theorems 5.4-5.7], its class is equal up to a positive rational constant, to

$$[\overline{\mathfrak{D}}_{2:2:3}]=-3\lambda+12\bigl(\psi_1+\psi_2+\psi_3\bigr)-40\cdot\alpha_{0:2}-0\cdot\alpha_0-0\cdot\beta_0-\cdots\in CH^1(\overline{\mathcal{S}}^+_{7,3}).$$

By direct calculation, we observe that $K_{\overline{\mathcal{S}}^+_{7,3}}-8\bigl[\mu^*(\overline{\Theta}_{\mathrm{null}})\bigr]-\frac{1}{12}[\overline{\mathfrak{D}}_{2:2:3}]-\frac{3}{2}[\sigma^*(D)]$ is an effective combination of boundary classes, which completes the proof. $\Box$

We now move on to prove Theorem 1.3 for the moduli space of odd spin curves.

Proof of Theorem 1.3 for $\overline{\mathcal{S}}^-_{g,n}$. We first treat the case when $g\geq 4$ and $n\geq g$. Since the proof bears similarities with the one for $\overline{\mathcal{S}}^+_{g,n}$, we skip a few details. We recall that we considered the Logan divisor $D_{g,n}$ on $\overline{\mathcal{M}}_{g,n}$ and the effective divisor D on $\overline{\mathcal{M}}_g$ having minimal slope s. Denoting by $\pi_i\colon\overline{\mathcal{S}}^-_{g,n}\to\overline{\mathcal{S}}^-_{g,1}$ the morphism forgetting all marked points except the one labelled by the index i, we set $\overline{\Theta}_{g,n}:=\pi_1^*\bigl(\overline{\Theta}_{g,1}\bigr)+\cdots+\pi_n^*\bigl(\overline{\Theta}_{g,1}\bigr)$. From Theorem 3.6,

$$[\overline{\Theta}^-_{g,n}]=\frac{n}{4}\lambda+\frac{1}{2}\sum_{i=1}^n\psi_i-\frac{n}{16}\alpha_0-\beta_{0:2}-\cdots\in CH^1(\overline{\mathcal{S}}^-_{g,n}).\tag{22}$$

We form the following effective linear combination on $\overline{\mathcal{S}}^-_{g,n}$:

$$\frac{8}{n}\bigl[\overline{\Theta}_{g,n}\bigr]+\frac{2(n-4)(n-1)}{g(g-3+2n)}\bigl[\pi^*(\overline{D}_{g,n})\bigr]+\frac{3}{2}\bigl[\sigma^*(D)\bigr]=\Bigl(\frac{3s+4}{2}-\frac{2(n-4)(n-1)}{g(g-3+2n)}\Bigr)\lambda$$
$$+\frac{2(n^2+2g-n-2)}{n(g-3+2n)}\sum_{i=1}^n\psi_i-2\alpha_0-3\beta_0-2\alpha_{0:2}-\sum_{i,S}\overline{\alpha}_{i:S}\cdot\alpha_{i:S},$$

where $\overline{\alpha}_{i:S}\geq 2$, except when $i=0$ and $|S|=2$, when the corresponding coefficient equals zero. Again, observe that the coefficient of $\sum_{i=1}^n\psi_i$ is smaller than one. Then for $4\leq g\leq 7$, the λ-coefficient of this effective divisor is smaller than $\frac{13}{2}$ precisely in the range in the statement of Theorem 1.3.

Assume now $n\leq g$. We use the effective divisors provided by (20), (21) and (22). We can easily show that for every $g\in\{8,9,10,11\}$ and $n\geq h(g)$, we have that

$$K_{\overline{\mathcal{S}}^-_{g,n}}\in\mathbb{Q}_{\geq 0}\Bigl\langle\bigl[\overline{\Theta}^-_{g,n}\bigr],\ \bigl[\mu^*(\overline{Z}_g)\bigr],\ \bigl[\sigma^*(D)\bigr],\ \bigl[\pi^*(\overline{D}_{g,n})\bigr],\ \frac{1}{n}\sum_{i=1}^n\psi_i\Bigr\rangle,$$

where the coefficient of $\psi_1+\cdots+\psi_n$ in this expression is positive. It follows that $K_{\overline{\mathcal{S}}^-_{g,n}}$ is big. For the boundary cases $\overline{\mathcal{S}}^-_{8,4}$ and $\overline{\mathcal{S}}^-_{9,3}$, we have the following expressions for the respective canonical divisors: $K_{\overline{\mathcal{S}}_{8,4}^-} - 2\big[\overline{\Theta}_{8:4}^-\big] - \frac{3}{2}[\sigma^*(D)] \in \mathbb{Q}_{\geq 0}\big\langle \alpha_{i:S}\big\rangle$, respectively

$$K_{\overline{\mathcal{S}}_{9,3}^-} - 2\big[\overline{\Theta}_{9:3}^-\big] - \frac{10}{7}[\sigma^*(D)] - \frac{1}{14}\big[\mu^*(\overline{Z}_9)\big] \in \mathbb{Q}_{\geq 0}\Big\langle \alpha_{i:S} : i \geq 0, S \subseteq [n]\Big\rangle.$$

This finishes the proof. □

We now discuss the case $g = 11$.

Proof of Theorem 1.4. The same consideration as in the proof of Theorem 1.3 show that one can represent the canonical class of $\overline{\mathcal{S}}_{11,1}^-$ as follows

$$K_{\overline{\mathcal{S}}_{11,1}^-} = 2\big[\overline{\Theta}_{11:1}^-\big] + \frac{4}{3}\big[\sigma^*(D)\big] + \frac{1}{6}\big[\mu^*(\overline{Z}_{11})\big] + \sum_{i=1}^{10}\big(a_i \cdot \alpha_i + b_i \cdot \alpha_{i:\emptyset}\big), \tag{23}$$

where $a_i, b_i \geq 0$. In fact, using [FP, Proposition 10.5], there is a 19-dimensional family of effective divisors on $\overline{\mathcal{M}}_{11}$ having slope 7, therefore we obtain that $\kappa(\overline{\mathcal{S}}_{11,1}^-) \geq 19$.

It remains to show that $\overline{\mathcal{S}}_{11,1}^-$ is not of general type. To that end, we consider the family of rational curves constructed in the proof of Theorem 3.7. A general member $\Gamma \subseteq \overline{\Theta}_{11,1}^-$ of this family passes through a general point of $\overline{\Theta}_{11,1}^-$ and has the intersection numbers $\Gamma \cdot \lambda = 12$, $\Gamma \cdot \psi = 2$, $\Gamma \cdot \alpha_0 = 64$ and $\Gamma \cdot \beta_0 = 10$, while $\Gamma \cdot \alpha_{i:1} = \Gamma \cdot \alpha_{i:\emptyset} = 0$, for $i = 1, \ldots, 10$.

We observe via Theorem 3.7 and (20) that Γ has intersection number zero with *every* divisor appearing on the right hand side of (23), in particular, also $\Gamma \cdot K_{\overline{\mathcal{S}}_{11,1}^-} = 0$. Since Γ fills-up a divisor in $\overline{\mathcal{S}}_{11,1}^-$ this rules out the possibility of expressing $K_{\overline{\mathcal{S}}_{11,1}^-}$ as an effective combination of an ample and an effective class. In particular, $K_{\overline{\mathcal{S}}_{11,1}^-}$ is not big, that is, $\overline{\mathcal{S}}_{11,1}^-$ is not of general type. The case of $\overline{\mathcal{S}}_{11:[2]}^-$ is similar and we skip the details. □

3.6. **Holomorphic forms on $\overline{\mathcal{S}}_{1,n}^+$.** We now discuss the geometry of $\overline{\mathcal{S}}_{1,n}^+$ and establish Theorem 1.5. Note that one has an identification $\overline{\mathcal{S}}_{1,1}^+ \cong X_1(2) \cong \mathbf{P}^1$.

Proof of Theorem 1.5. One can use (15) in genus one, together with the relations $12\lambda = \delta_{\mathrm{irr}}$ and $12\psi_i = \delta_{\mathrm{irr}} + 12\sum_{i\in S}\delta_{0:S}$, to obtain the following formula [BF, Proposition 3]:

$$K_{\overline{\mathcal{M}}_{1,n}} = (n-11)\lambda + \sum_{a=2}^{n-1}\sum_{S\subseteq[n], |S|=a}(a-2)\delta_{0:S} + (n-3)\delta_{0:n} \in CH^1(\overline{\mathcal{M}}_{1,n}).$$

The finite cover $\pi\colon \overline{\mathcal{S}}_{1,n}^+ \to \overline{\mathcal{M}}_{1,n}$ is branched along the divisor B_0. Since we have the isomorphism $\overline{\mathcal{S}}_{1,1}^+ \cong \mathbf{P}^1$ at the level of coarse moduli spaces and both divisors A_0 and B_0 corresponds to points on $\overline{\mathcal{S}}_{1,1}^+$, it follows that $\alpha_0 = \beta_0 \in CH^1(\overline{\mathcal{S}}_{1,n}^+)$. Since $\pi^*(\delta_{\mathrm{irr}}) = \alpha_0 + 2\beta_0$ and $\pi^*(\Delta_{0:S}) = A_{0:S}$ for every subset $S \subseteq [n]$ with $|S| \geq 2$, by the Hurwitz formula we can write

$$K_{\overline{\mathcal{S}}_{1,n}^+} = (n-11)\lambda + \sum_{a=2}^{n-1}(a-2)\alpha_{0:a} + (n-3)\delta_{0:n} + \beta_0 = \frac{n-7}{4}\alpha_0 + \sum_{a=2}^{n}(a-2)\alpha_{0:a} - \alpha_{0:n}. \tag{24}$$

In particular, for $n \geq 8$, via Remark 3.4 using the forgetful morphism $\overline{\mathcal{S}}_{1,n}^+ \to \overline{\mathcal{S}}_{1,1}^+$, we obtain that $\kappa(\overline{\mathcal{S}}_{1,n}^+) = \kappa(\overline{\mathcal{S}}_{1,n}^+, K_{\overline{\mathcal{S}}_{1,n}^+}) \geq \kappa(\overline{\mathcal{S}}_{1,1}^+, \alpha_0) = \kappa(\mathbf{P}^1, \mathcal{O}_{\mathbf{P}^1}(1)) = 1$. On the other hand, as explained in [BF, Proposition 4], since the general fibre of the morphism $\overline{\mathcal{S}}_{1,n}^+ \to \overline{\mathcal{S}}_{1,1}^+$ has

Kodaira dimension zero, by the *easy addition* theorem we write $\kappa(\overline{\mathcal{S}}^+_{1,n}) \le \dim(\overline{\mathcal{S}}^+_{1,1}) = 1$, thus $\kappa(\overline{\mathcal{S}}^+_{1,n}) = 1$, for $n \ge 8$.

For $n = 7$, we obtain from (24) that $H^0(\overline{\mathcal{S}}^+_{1,7}, K_{\overline{\mathcal{S}}^+_{1,7}}) \neq 0$. Finally, $K_{\overline{\mathcal{S}}^+_{1,7}}$ admits an effective representative consisting of boundary divisors $A_{0:S}$, where $S \subseteq [7]$. For each such divisor, by varying the j-invariant of the elliptic curve corresponding to a general point, one constructs a rational curve $\Gamma_S \subseteq A_{0:S}$ passing through a general point and satisfying $\Gamma_S \cdot \alpha_{0:S} = -1$ and $\Gamma_S \cdot \alpha_{0:S'} = 0$, for every $S' \subseteq [7]$ with $S' \neq S$. It follows that $\kappa(\overline{\mathcal{S}}^+_{1,7}) = 0$. □

4. Correspondences between moduli spaces of pointed spin curves.

We now explain a structure result for the spin moduli space $\overline{\mathcal{S}}_7^-$ which makes no reference to Mukai's structure theorem [M2] for curves of genus 7 in terms of the spinorial variety $OG(5,10) \subseteq \mathbf{P}^{15}$, that is, of the birational isomorphism

$$\overline{\mathcal{M}}_7 \overset{\cong}{\dashrightarrow} \mathbf{G}(9,15)//\mathrm{Aut}\, OG(5,10).$$

This construction establishes a correspondence between $\overline{\mathcal{S}}_7^-$ and $\overline{\mathcal{S}}^+_{3,6}$.

We denote by $\mathcal{HS}_7^-$ the *Hurwitz spin* stack parametrizing triples $[C,\vartheta,A]$, consisting of an odd spin curve $[C,\vartheta] \in \mathcal{S}_7^-$ and a pencil $A \in W^1_5(C)$. The general fibre of the forgetful map $\mathcal{HS}_7^- \to \overline{\mathcal{S}}_7^-$ is a smooth curve. By Serre duality $L := \omega_C \otimes A^\vee \in W^2_7(C)$. We consider the induced plane model

$$\phi\colon C \xrightarrow{|L|} \Gamma \subseteq \mathbf{P}^2 = \mathbf{P}H^0(C,L)^\vee.$$

For a general choice of $[C,\vartheta,A]$, it is straightforward that Γ is a nodal septic curve with 8 nodes, which we denote by $z_1,\ldots,z_8 \in \mathbf{P}^2$. Setting $\vartheta := \mathcal{O}_C(p_1+\cdots+p_6)$, it then follows that there exists a smooth plane quadric curve $D \in \left|\mathcal{O}_{\mathbf{P}^2}(4)\right|$ such that

$$D\cdot\Gamma = 2(z_1+\cdots+z_8+p_1+\cdots+p_6). \tag{25}$$

Then $\eta := \mathcal{O}_D(p_1+\cdots+p_6+z_1+\cdots+z_8)\otimes\omega_D^{-3} \in \mathrm{Pic}^2(D)$ is an even theta-characteristic.

To obtain a parametrization of $\mathcal{HS}_7^-$ (and ultimately of $\overline{\mathcal{S}}_7^-$), we consider the following locally trivial $\mathbf{P}^5$-bundle over $\mathcal{S}^+_{3,6}$. Let $\mathfrak{u}\colon \mathcal{C} \to \mathcal{S}^+_{3,6}$ be the universal curve and $\mathcal{L} \in \mathrm{Pic}(\mathcal{C})$ be the universal spin bundle. We denote by $\sigma_i\colon \mathcal{S}^+_{3,6} \to \mathcal{C}$ the section corresponding to the i-th marked point and set $\Sigma_i := \mathrm{Im}(\sigma_i)$, for $i=1,\ldots,6$. Let

$$\mathcal{P}_{3,6} := \mathbf{P}\big(\mathfrak{u}_*(\mathcal{L}\otimes\omega_{\mathfrak{u}}^3(-\Sigma_1-\cdots-\Sigma_6)\big) \longrightarrow \mathcal{S}^+_{3,6}.$$

A point of $\mathcal{P}_{3,6}$ corresponds to an element $\big(D,\eta,\overline{p},z_1+\cdots+z_8\big)$, where $[D,\eta]\in\mathcal{S}^+_{3,6}$ is an even spin curve, $\overline{p}=(p_1,\ldots,p_6)$ and $z_1+\cdots+z_8$ is an effective divisor on D such that

$$p_1+\cdots+p_6+z_1+\cdots+z_8 \in \left|\omega_D^3\otimes\eta\right|.$$

Note that $2(p_1+\cdots+p_6)+2(z_1+\cdots+z_8) \in \left|\omega_D^7\right| = \left|\mathcal{O}_D(7)\right|$. From the exact sequence

$$0\longrightarrow H^0(\mathbf{P}^2,\mathcal{O}_{\mathbf{P}^2}(3)) \xrightarrow{\cdot D} H^0(\mathbf{P}^2,\mathcal{O}_{\mathbf{P}^2}(7)) \longrightarrow H^0(D,\mathcal{O}_D(7)) \longrightarrow 0,$$

we conclude there exists a plane curve $\Gamma \in \left|\mathcal{O}_{\mathbf{P}^2}(7)\right|$, with $D\cdot\Gamma = 2(p_1+\cdots+p_6+z_1+\cdots+z_8)$. We set $\mathcal{I}_{2\overline{p}+2\overline{z}} := \mathcal{I}^2_{p_1}\cap\ldots\cap\mathcal{I}^2_{p_6}\cap\mathcal{I}^2_{z_1}\cap\ldots\cap\mathcal{I}^2_{z_8} \subseteq \mathcal{O}_{\mathbf{P}^2}$. From the exact sequence

$$0\longrightarrow H^0(\mathbf{P}^2,\mathcal{O}_{\mathbf{P}^2}(3)) \longrightarrow H^0\big(\mathbf{P}^2,\mathcal{I}_{2\overline{p}+2\overline{z}}(7)\big) \longrightarrow H^0(D,\mathcal{O}_D) \longrightarrow 0, \tag{26}$$

it follows one has a 10-dimensional linear system of plane curves of degree 7 cutting out the divisor $2\overline{p}+2\overline{z}$ on D.

Definition 4.1. We denote by $\mathcal{Y}\to\mathcal{P}_{3,6}$ the parameter space of objects $\bigl(D,\eta,\overline{p},z_1+\cdots+z_8,\Gamma\bigr)$, where $\Gamma\in\bigl|\mathcal{I}_{2\overline{p}+2\overline{z}}(7)\bigr|$ is a curve such that $\mathrm{Sing}(\Gamma)=\{z_1,\ldots,z_8\}$.

We denote by $\mu\colon\mathcal{Y}\dashrightarrow\mathcal{S}_{3,6}^+$ the forgetful morphism $\bigl(D,\eta,\overline{p},z_1+\cdots+z_8,\Gamma\bigr)\mapsto[D,\eta]$.

Proposition 4.2. *The morphism $\mathcal{Y}\dashrightarrow\mathcal{P}_{3,6}$ is birationally a $\mathbf{P}^2$-bundle.*

Proof. Having fixed a general element $\bigl(D,\eta,\overline{p},z_1+\cdots+z_8\bigr)$, from (26) the linear system $\bigl|\mathcal{I}_{2\overline{p}+2\overline{z}}(7)\bigr|$ is 10-dimensional. Requiring that $z_i\in\mathrm{sing}(\Gamma)$ is a linear condition for $i=1,\ldots,8$ and via a local calculation is straightforward to see that these 8 conditions are linearly independent. It follows that $\mathcal{Y}\to\mathcal{P}_{3,6}$ is birational to a $\mathbf{P}^2$-bundle. $\Box$

Summarizing this construction, one has the following diagram,

$$\mathcal{Y}=\Bigl\{(D,\eta,\overline{p},z_1+\cdots+z_8,\Gamma):D\cdot\Gamma=2\bigl(\textstyle\sum_{i=1}^6 p_i+\sum_{j=1}^8 z_j\bigr)\Bigr\} \tag{27}$$

$$\mathcal{S}_{3,6}^+\xleftarrow{\ \mu\ }\mathcal{Y}\xrightarrow{\ \chi\ }\mathcal{HS}_7^-\longrightarrow\overline{\mathcal{S}}_7^-$$

where χ is given by $\bigl(D,\overline{p},\eta,z_1+\cdots+z_8,\ \Gamma\bigr)\mapsto\bigl(C,\mathcal{O}_C(\nu^{-1}(p_1)+\cdots+\nu^{-1}(p_6)),\omega_C(-1)\bigr)$, with $\nu\colon C\to\Gamma$ being the normalization map.

Theorem 4.3. *The Hurwitz spin stack $\mathcal{HS}_7^-$ is birational to a double projective bundle over $\mathcal{S}_{3,6}^+$.*

A variant of this construction provides a birational correspondence between $\overline{\mathcal{S}}_8^-$ and $\overline{\mathcal{S}}_{4,7}^+$.

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