REPRESENTATIONS OF COMPACT LIE GROUPS AND ELLIPTIC OPERATORS

Jochen Brüning and Ernst Heintze *)

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INTRODUCTION

Let E be a Hermitian vector bundle over a compact Riemannian manifold M and $P:C^\infty(E) \longrightarrow C^\infty(E)$ a strongly elliptic symmetric and positive linear differential operator of order 2k. Then P has eigenvalues $0 \leq \lambda_1 < \lambda_2 < \ldots$ tending to infinity and each eigenspace $P^\lambda := \{ f \in C^\infty(E) \mid Pf = \lambda f \}$ has finite dimension. Although one cannot expect precise information about the individual eigenvalues in general one has the asymptotic formula

$$N(t) := \sum_{\lambda \leq t} \dim P^\lambda \sim \text{const } t^{\frac{\dim M}{2k}}, \ t \longrightarrow \infty ,$$

where the constant is given by a certain integral over T_1^*M (the unit cotangent bundle) involving the symbol of P. This is the famous formula of Weyl [31], if P is the Laplacian acting on functions and M a bounded open set in Euclidean space, extended to compact Riemannian manifolds by Minakshisundaram and Pleijel [26]. We now refine these considerations by bringing in a compact Lie group G. In addition to the above data we assume that E is a G-vector bundle, that G acts on M by isometries and unitarily on the fibers, and that P commutes with the action of G on $C^\infty(E)$. Then the eigenspaces P^λ are unitary G-modules. Given a fixed unitary irreducible representation ϱ of G we are interested in the multiplicity $\nu_\varrho(\lambda)$ of ϱ in P^λ, and we study the asymptotic behavior as t goes to infinity of the function

$$N_\varrho(t) := \sum_{\lambda \leq t} \nu_\varrho(\lambda).$$

Of course the classical case is covered by $G = \{e\}$ since then ϱ is trivial and $\nu_\varrho(\lambda) = \dim P^\lambda$. Our result will be given in theorem 3.1 under considerably weaker assumptions than described above. We single out two special cases. In the first case (cor. 3.2) we assume that P is only transversally (strongly) elliptic in the sense of Atiyah [3] . Although the eigenspaces will then be infinite dimensional in general, each ϱ occurs only finitely many times in each P^λ. In the second case (cor. 3.4) we allow M to have a boundary and consider certain elliptic boundary value problems e.g. the Dirichlet or Neumann problem for the Laplacian acting on functions.

The main steps in the proof and the organization of the paper are as follows. By a simple reduction we may assume that ϱ is the trivial representation, hence

$$N_\varrho(t) = \sum_{\lambda \leq t} \dim (P^\lambda)^G.$$

Thus we only have to consider the space $L^2(E)^G$ of G-invariant elements in $L^2(E)$ (i.e. the subspace on which G acts like $\varrho = 1$). We observe that the selfadjoint operator generated by P allows restriction to $L^2(E)^G$. Denoting this restriction by S, N_ϱ turns out to be the counting function of Spec S. Now by a general construction $L^2(E)^G$ may be identified with $L^2(F,h)$, where F is a Hermitian vector bundle over M_o/G, M_o the union of principal orbits in M, and h is defined by h(q) := volume of the orbit over q, $q \in M_o/G$. This result can be thought of as a generalized Frobenius reciprocity theorem. As immediate consequences we give necessary and sufficient conditions for a unitary irreducible representation of G to occur in $L^2(E)$ (and thus in some P^λ) and determine its multiplicity. Using the identification $L^2(E)^G \simeq L^2(F,h)$ S corresponds to a selfadjoint operator in $L^2(F,h)$, say T, which will be studied in section 2. We show that T is generated by a strongly elliptic differential operator. Since N_ϱ is also the counting function of Spec T we have thus reduced the problem to the case $G = \{e\}$. In contrast to the classical situation however, M_o/G will not be compact in general (we have taken out the singularities of M/G corresponding to exceptional and singular orbits) and this causes the main difficulty. In section 3 we attack the asymptotics of N_ϱ via its Laplace transform which for $s > o$ is given by the trace in $L^2(F,h)$ of the operator e^{-sT}. This in turn is given as an integral over M_o/G involving the heat kernel of T. Using local results of Hörmander [21] we find the asymptotic behavior of the integrand as $s \to o$. Thus by the Lebesgue-Fatou lemma and a Tauberian theorem for the Laplace transform the asymptotic behavior of N_ϱ will follow from a pointwise estimate of the heat kernel and the fact that vol M_o/G is finite. Both results will be derived in section 4. In section 5 we give a remainder estimate for finite G of the form $O(t^{\frac{\dim M-1}{2k}} \log t)$. This will be applied in section 6 to improve the Gelfand-Gangolli-Wallach formula for the spectra of discrete uniform subgroups of semisimple Lie groups.

Our main results (for the Laplacian acting on functions) have been announced in [11] . As mentioned there our work on this problem has been stimulated by the paper of Huber [23] who proved 3.2 for M a compact Riemann surface of genus $\geqslant 2$ and $P = \Delta$ the Laplacian acting on functions. More generally, Wallach [29] proved 3.2 for operators with Laplacian type symbols and G finite. While preparing the manuscript we learned that Donelly [13] proved 3.2 for Δ acting on functions and general compact G, using quite different methods however.

1. A BASIC CONSTRUCTION

As a motivation and to introduce some of our technical devices we begin by determining how many times a given irreducible unitary representation of G occurs in $L^2(E)$. The answer being of interest in itself will give strong hints how to get more refined informations about the multiplicity of the representation in spectral subspaces of G-invariant selfadjoint operators, especially those arising from an elliptic differential operator.

Let us collect some notations. For a Riemannian manifold M we denote by d_M the distance function and by dM the volume element. G will be a compact Lie group kept fixed throughout this paper. We choose a left invariant Riemannian metric with volume element dG on G. For a G-manifold M and $p \in M$ let Gp be the G-orbit of p and G_p the isotropy group at p. $\pi_G : M \longrightarrow {}^M/_G$ will be the orbit map and M^G the set of elements left fixed by each $g \in G$. Let M_o be the union of principal orbits in M. It is an open and dense subset of M and $M - M_o$ has measure o. Also ${}^{M_o}/_G$ is a manifold which is connected if M is (cf.[9] , th. IV, 3.1, th. IV, 3.3 and prop. IV, 3.7). We consider moreover Hermitian vector bundles over Riemannian manifolds. If $E \longrightarrow M$ is such a bundle we denote by $\langle \cdot | \cdot \rangle_E$ the Hermitian structure and by $| \cdot |_E$ the corresponding norm. Let C(E) ($C_o(E)$) and $C^\infty(E)$ ($C_o^\infty(E)$) be the spaces of continuous and differentiable sections of E (respectively with compact support). $C_o(E)$ with the norm

$$f \mapsto \left(\int_M \langle f | f \rangle_E (p) \, dM(p) \right)^{\frac{1}{2}}$$

becomes a pre-Hilbert space. Its completion will be called $L^2(E)$ and we write $(\cdot | \cdot)_{L^2(E)}$ and $\| \cdot \|_{L^2(E)}$ for its scalar product and norm. Since we will have to integrate several functions over G we note that every continuous function on a compact measure space with values in a complete locally convex Hausdorff space can be integrated and that the integral commutes with continuous linear operators between such spaces ([8], Ch. III, pp. 81 and 85).

We now fix a Hermitian G-vector bundle E over the connected Riemannian manifold M, where G acts on M by isometries and preserves the Hermitian structure in the fibers. Then G acts also on the continuous sections of E by

$$U_g f(p) := g(f(g^{-1}(p))),\ f \in C(E),\ g \in G,\ p \in M,$$

thus defining a unitary representation of G in $L^2(E)$. Let ϱ be a fixed irreducible unitary representation of G. To determine the multiplicity of ϱ in $L^2(E)$ we recall the following facts about unitary representations of compact groups in Hilbert spaces. If $\mathcal{H}$ is a Hilbert space with a unitary representation of G, then for $\tau \in \hat{G}$ (the set of equivalence classes of finite dimensional irreducible unitary representations of G) we choose a representative G-module W and put

1.1) $$\mathcal{H}_\tau := \psi(\mathrm{Hom}_G(W,\mathcal{H}) \otimes W) \subset \mathcal{H},$$

where $\psi : f \otimes v \longmapsto f(v)$ is the natural homomorphism. Then ψ is in fact a G-equivariant isomorphism onto its image and $\mathcal{H}_\tau$ is the subspace of $\mathcal{H}$ on which G acts like τ, for example $\mathcal{H}_1 = \mathcal{H}^G$, 1 denoting the trivial representation. We have

1.2) $$\mathcal{H} = \sum_{\tau \in \hat{G}} \mathcal{H}_\tau$$

as sum of mutually orthogonal subspaces. This is a well known consequence of the classical Peter-Weyl theorem (and essentially equivalent to it). For convenience of the reader we give the following direct proof.

Lemma 1.1: Let G be a compact topological group and $\mathcal{H}$ a Hilbert space with a continuous unitary representation U of G. Then $\mathcal{H}$ can be orthogonally decomposed into finite dimensional G-irreducible subspaces.

Proof: By elementary representation theory and Zorn's lemma it is sufficient to exhibit a G-invariant subspace $\mathcal{H}'$ of $\mathcal{H}$ with $0 < \dim \mathcal{H}' < \infty$. To do so let $T(\mathcal{H})$ be the trace class in $\mathcal{H}$ and recall the following facts: $T(\mathcal{H})$ is a Banach space under the trace norm contained in the set of compact operators, the conjugation $*$

of operators and the trace Tr define continuous linear maps $T(\mathcal{H}) \longrightarrow T(\mathcal{H})$ and $T(\mathcal{H}) \longrightarrow \mathbb{C}$ respectively, and for each $A \in T(\mathcal{H})$ the function $G \ni g \longmapsto U_g A U_g^* \in T(\mathcal{H})$ is continuous (cf. [28], sec. 1.15). Now pick a Hermitian element $A \in T(\mathcal{H})$ with $\mathrm{Tr} A > 0$. Then

$$\tilde{A} := \frac{1}{\mathrm{vol}\ G} \int_G U_g A U_g^* \, dG(g)$$

is also a Hermitian element of $T(\mathcal{H})$ with the same trace and commuting with G. Now being compact Hermitian and nonzero $\tilde{A}$ must possess a nontrivial eigenspace which has the desired properties.

Now by (1.1) the multiplicity of τ in $\mathcal{H}$ is $\dim \mathrm{Hom}_G(W,\mathcal{H})$ (which may be infinite). By the finiteness of dim W we have

$$\mathrm{Hom}_G(W,\mathcal{H}) \simeq (\mathcal{H} \otimes W^*)^G,$$

where G acts on W^* via the conjugate representation τ^*. Therefore, choosing a representative G-module V for the representation ρ in question we find that the multiplicity of ρ in $L^2(E)$ is equal to

1.3) $$\dim \mathrm{Hom}_G(V,L^2(E)) = \dim(L^2(E) \otimes V^*)^G = \dim L^2(E \otimes V^*)^G,$$

since we have a canonical isomorphism $L^2(E) \otimes V^* \longrightarrow L^2(E \otimes V^*)$. Replacing the Hermitian G-vector bundle $E \otimes V^*$ by E we see that it is important to describe the space $L^2(E)^G$. Our basic observation will be that $L^2(E)^G$ is naturally isomorphic to a space of sections of a certain Hermitian bundle F over M_o/G. To describe F we introduce the set

$$E' := \bigcup_{p \in M_o} E_p^{G_p},$$

i.e. the union of all elements in $E|M_o$ which are invariant under the isotropy group of their base point. We then have

Lemma 1.2: E' is a G-invariant subbundle of $E|M_o$.

Proof: The G-invariance of E' is clear from $g(E_p^{G_p})=E_{g(p)}^{G_{g(p)}}$ for $g \in G$, $p \in M_o$. To prove local triviality of E' we may by the slice theorem ([9] , cor. VI, 2.4) assume that $M_o=G/H \times D$, where D is an open ball in some Euclidean space and the G-action is given by

$$g(xH,d)=(gxH,d),\ x\in G,\ d\in D.$$

In particular $G_p=H$ for $p\in D':=HxD$. But then $E'':=E'|D'=(E|D')^H$ and this is a C^∞ vector bundle because it is the image of the C^∞ vector bundle homomorphism

$$\frac{1}{\mathrm{vol}\ H}\int_H U_h\, dH(h)$$

(note that $U_h \in C^\infty \mathrm{Hom}(E|D', E|D')$). By making D smaller if necessary we may assume that E'' is trivial. Now if $(s_i)_{i=1}^k$ is a maximal set of linear independent sections of E'', such a set for E' is defined by

$$\bar{s}_i((gH,d)):=g\ s_i((H,d)),$$

$1\leq i\leq k$, $g\in G$, $d\in D$, and therefore E' is also trivial. Since M_o/G is connected the dimension of E' is constant on M_o.

Now G acts on E' with exactly one orbit type (namely H) and therefore $E'/G=:F$ is a manifold. Using the slice theorem as in the proof of 1.2 it is easy to see that F is a C^∞ vector bundle over M_o/G which inherits a Hermitian structure from E'. We provide M_o/G with the Riemannian structure that turns π_G into a Riemannian submersion and put for $q\in M_o/G$

(1.4) $$h(q):=\mathrm{vol}\ \pi_G^{-1}(q).$$

Denoting by $L^2(F,h)$ the closure of $C_o(F)$ under the norm

$$f \longmapsto \Big(\int_{M_o/G} \langle f | f \rangle_F (q)\, h(q)\, d^{\,M_o/G}(q) \Big)^{\frac{1}{2}}$$

we come to the following basic construction.

<u>Theorem 1.3:</u> There is an isometric isomorphism of Hilbert spaces

$$\Phi : L^2(E)^G \longrightarrow L^2(F,h).$$

With $\pi'_G : E' \longrightarrow F$ denoting the orbit map Φ is given by

$$\Phi f_1 \circ \pi_G(p) = \pi'_G \circ f_1(p),\ f_1 \in C_o(E|M_o)^G,\ p \in M_o\ ,$$

and its inverse Φ^{-1} by

$$\Phi^{-1} f_2(p) = f_2 \circ \pi_G(p) \cap E_p,\ f_2 \in C_o(F),\ p \in M_o .$$

<u>Proof:</u> Note first of all that $C_o(E|M_o)^G \subset C_o(E')$ so that Φ is well defined. It is obvious from the definition that $\Phi(C_o^{(\infty)}(E|M_o)^G) \subset C_o^{(\infty)}(F)$. Next it follows from the slice theorem again that $\Phi^{-1}(C_o^{(\infty)}(F)) \subset C_o^{(\infty)}(E|M_o)^G$ implying

$$\Phi(C_o^{(\infty)}(E|M_o)^G) = C_o^{(\infty)}(F).$$

Now we observe that the orthogonal projection $Q : L^2(E) \longrightarrow L^2(E)^G$ is given by

(1.5) $$Qf = \frac{1}{\mathrm{vol}\ G} \int_G U_g\, f\, dG(g).$$

Therefore, Q maps $C_o^{(\infty)}(E|M_o)$ to $C_o^{(\infty)}(E|M_o)^G$. But since $C_o(E|M_o)$ and $C_o(F)$ are dense in $L^2(E|M_o) = L^2(E)$ and $L^2(F,h)$ respectively the proof will be completed by showing that Φ is isometric on $C_o(E|M_o)^G$, which is an easy consequence of Fubini's theorem for Riemannian submersions (cf.[4], p. 16):

$$\int_{M_o/G} | \Phi f |_F^2 (q)\, h(q)\, d^{\,M_o/G}(q)$$

$$= \int_{M_o/G} \int_{\pi_G^{-1}(q)} |f|_E^2(p)\, d\, \pi_G^{-1}(q)(p)\, d\, {}^{M_o}/_G(q)$$

$$= \int_{M_o} |f|_E^2(p)\, d\, M\, (p).$$

For later reference we collect the following properties of Φ.

Corollary 1.4: 1) $\Phi(C_o^{(\infty)}(E|M_o)^G) = C_o^{(\infty)}(F)$.

2) $|\Phi f \circ \pi_G(p)|_F = |f(p)|_E$ for $f \in C_o(E|M_o)^G$ and $p \in M_o$.

3) $\Phi^{-1}(\varphi f) = \varphi \circ \pi_G \, \Phi^{-1} f$ for $f \in C_o(F)$ and $\varphi \in C({}^{M_o}/_G)$.

4) $\operatorname{supp} \Phi f = \pi_G(\operatorname{supp} f)$, $f \in C_o(E|M_o)^G$,
$\operatorname{supp} \Phi^{-1} f = \pi_G^{-1}(\operatorname{supp} f)$, $f \in C_o(F)$.

5) For $f \in C_o^\infty(F)$ and $p \in M_o$ we have
$j_k(\Phi^{-1} f)(p) = o$ if $j_k(f)(\pi_G(p)) = o$, where j_k denotes the k-jet, $k \in \mathbb{Z}_+$.

Proof: 1) - 4) are immediate consequences of 1.3. To prove 5) pick $q \in {}^{M_o}/_G$ and assume $j_k(f)(q) = o$ for some k. By definition ([27], Ch. IV) we have a representation

$$f = \sum_{i=1}^{r} \varphi_i f_i$$

with $\varphi_i \in C_o^\infty({}^{M_o}/_G)$, $f_i \in C_o^\infty(F)$ and $j_k(\varphi_i)(q) = o$, $1 \leq i \leq r$. By 3) we find

$$\Phi^{-1} f = \sum_{i=1}^{r} \varphi_i \circ \pi_G (\Phi^{-1} f_i),$$

which implies $j_k(\Phi^{-1} f)(p) = o$ for $p \in \pi_G^{-1}(q)$.

We now use this result to derive the desired information about the multiplicity of ϱ in $L^2(E)$. Applying 1.3 to $E \otimes V^*$ and denoting the resulting bundle on ${}^{M_o}/_G$ by F_ϱ we get immediately

Corollary 1.5: (Generalized Frobenius reciprocity theorem)
There are natural isomorphisms

$$\mathrm{Hom}_G(V,L^2(E)) \simeq (L^2(E)\otimes V^*)^G \simeq L^2(E\otimes V^*)^G$$
$$\simeq L^2(F_\rho,h).$$

If G acts transitively on M, i.e. M is homogeneous, 1.5 reduces by the construction of F_ρ to

1.6) $$\mathrm{Hom}_G(V,L^2(E)) \simeq \mathrm{Hom}_H(V,W)$$

with $H:=G_p$ and $W:=E_p$ for some $p\in M$. But E may be identified with $G\times_H W$, so that (1.6) gives Bott's formulation ([7], Prop. 2.1) of the classical Frobenius reciprocity theorem, which is due to Frobenius for finite groups and to Weil for compact groups. By 1.5 and (1.3) we therefore see that the multiplicity of ρ in $L^2(E)$ is given by

$$\dim L^2(F_\rho,h),$$

and this is nonzero iff $\dim \mathrm{Hom}_{G_p}(V,E_p)=\dim F_{\rho,\pi_G(p)}$ is nonzero for some (and hence for all) $p\in M_o$ iff there exists some irreducible unitary representation of G_p contained in V and E_p.

Corollary 1.6: Let E,M,ρ,V, and F_ρ be as above and put

(1.7) $$l:=\dim F_\rho=\dim \mathrm{Hom}_{G_p}(V,E_p),\ p\in M_o.$$

1) ρ occurs in $L^2(E)$ iff $l>o$.
2) If G does not act transitively on M, i.e. if $\dim M_o/G>o$, then ρ occurs with infinite multiplicity if it occurs at all.
3) If G acts transitively on M, i.e. if M is homogeneous, then ρ occurs with multiplicity l.
4) If G is finite and acts effectively on M, then every $\rho\in\hat{G}$ occurs with infinite multiplicity.
5) If each G_p acts trivially on E_p for $p\in M_o$ (e.g. if $E=M\times\mathbb{C}$ with trivial action on the fibers), then precisely the class one representations occur, i.e. those for which G_p has a nonzero fixed

vector in V.

6) If G is not finite and acts effectively on M, then $L^2(E)$ contains infinitely many elements of $\hat{G}$.

<u>Proof:</u> 1) - 5) follow at once from the above considerations.
6) Under our conditions for each principal orbit O, $L^2(E|O)$ has infinite dimension. By 3) and (1.7) it therefore contains infinitely many $\varrho_i \in \hat{G}$, $i \in \mathbb{N}$. Then if V_i is a representative G-module for ϱ_i we have by 1) that for $p \in O$

$$\dim \mathrm{Hom}_{G_p} (V_i, E_p) > 0,$$

and thus - again by 1) - that ϱ_i occurs in $L^2(E)$, $i \in \mathbb{N}$.

2. REPRESENTATIONS OF G IN SPECTRAL SUBSPACES OF SELFADJOINT OPERATORS

In addition to the data of the preceding section - the G-bundle E over M and the representation ϱ of G in V - we now bring in a selfadjoint operator R in $L^2(E)$ which commutes with G in the following sense: if $\mathcal{D}(R)$ denotes the domain of definition of R then

$$U_g(\mathcal{D}(R)) \subset \mathcal{D}(R)$$

(2.1) and

$$U_g R(x) = RU_g(x), \quad g \in G, \; x \in \mathcal{D}(R).$$

One of the most interesting cases arises when R is generated by an elliptic differential operator P acting on $C^\infty(E)$ and M is compact. Then P is essentially selfadjoint and thus R uniquely determined by P. Moreover (2.1) must only be checked on $C^\infty(E)$. However for other applications we have in mind we need a greater generality. Let

$$R = \int_{-\infty}^{+\infty} t \, d\, R_t$$

be the spectral resolution of R with $t \longmapsto R_t$ left continuous. We identify R_t with its image in $L^2(E)$. Then R_t is G-invariant for $t \in \mathbb{R}$, and we may consider the function

(2.2)
$$N(t) := \text{multiplicity of } \varrho \text{ in } R_t$$
$$= \dim \operatorname{Hom}_G(V, R_t) = \dim(R_t \otimes V^*)^G, \quad t \in \mathbb{R}.$$

It is the asymptotic behavior of this function that we want to study. N may take infinite values, but if e.g. the spectrum Spec R of R consists of positive eigenvalues with finite multiplicity only and has no finite accumulation points then

$$N(t) = \sum_{\substack{\lambda \in \operatorname{Spec} R \\ \lambda \leq t}} \nu(\lambda)$$

where $\nu(\lambda)$ denotes the multiplicity of ϱ in the eigenspace of R with eigenvalue λ. N is related to our preceding considerations by

Lemma 2.1: We have

$$\lim_{t\to\infty} N(t)=\dim \mathrm{Hom}_G(V,L^2(E))=\dim L^2(F_\varrho ,h)$$

$$= \text{multiplicity of } \varrho \text{ in } L^2(E).$$

Proof: $\mathrm{Hom}(V,L^2(E))\simeq L^2(E)\otimes V^*$ has a natural Hilbert space structure which for $t\in\mathbb{R}$ implies the orthogonal decomposition

$$L^2(E)\otimes V^* = R_t\otimes V^* \oplus \sum_{j=1}^{\infty}(R_{t+j}-R_{t+j-1})\otimes V^*.$$

Now the orthogonal projection (1.5) maps each member of the sum into itself, so we get by taking dimensions

$$\dim \mathrm{Hom}_G(V,L^2(E))=N(t)+\sum_{j=1}^{\infty} \dim \mathrm{Hom}_G(V,R_{t+j}-R_{t+j-1}).$$

From this formula the assertion is obvious.

Thus ϱ occurs in $L^2(E)$ iff it occurs in some spectral subspace R_t of R. If G acts transitively on M we see from 1.6, 3) and 2.1 that the asymptotic behavior of N is completely determined independent of R. Therefore, from now on we assume that G does not act transitively on M or, what is the same, that $\dim {}^{M_0}/_G \geq 1$. In that case we have $\lim_{t\to\infty} N(t)=\infty$ if ϱ occurs in $L^2(E)$ at all and it is reasonable to investigate the asymptotic behavior of N as $t\longrightarrow\infty$, which can be expected to involve R in an essential way. To get hold of N we would like to restrict R to $L^2(E)_\varrho$ (the subspace of $L^2(E)$ on which G acts like ϱ) and to analyze the spectral resolution of the resulting operator, because the ϱ-part of R_t is contained in $L^2(E)_\varrho$. Now the following observation will allow us again to assume that ϱ is the trivial representation. We have the G-bundle $E\otimes V^*$ (where G acts via ϱ^* on V^*) and in $L^2(E\otimes V^*)\simeq L^2(E)\otimes V^*$ the operator $\tilde{R}:=R\otimes \mathrm{id}_{V^*}$. Note that R is selfadjoint with spectral resolution $\tilde{R}_t=R_t\otimes V^*$ and commutes with G. Hence by (2.2)

$$N(t)=\dim(R_t \otimes V^*)^G=\dim \tilde{R}_t^G.$$

Thus all we have to do is to restrict R to $L^2(E)^G$, and this is possible by the following lemma.

Lemma 2.2: Put $S:=R|\mathcal{D}(R)\cap L^2(E)^G$. Then S is a well defined selfadjoint operator on $L^2(E)^G$ with $\mathcal{D}(S)=\mathcal{D}(R)^G$. If its spectral resolution is

$$S=\int_{-\infty}^{+\infty} t \, d\, S_t$$

then for $t\in\mathbb{R}$

(2.3) $$\dim S_t=N(t).$$

Thus N is the counting function of Spec S.

Proof: Since $\mathcal{D}(R)$ is a Hilbert space with the graph norm, which is bigger then the given one, we easily see that the restriction of Q (where Q is given by (1.5)) to $\mathcal{D}(R)$ maps into $\mathcal{D}(R)$. Moreover $R:\mathcal{D}(R)\longrightarrow L^2(E)$ is continuous which implies that Q commutes with R on $\mathcal{D}(R)$ by (2.1). Therefore, S is well defined and obviously selfadjoint, and its spectral resolution is given by

(2.4) $$S_t=R_t^G,\ t\in\mathbb{R}.$$

From this (2.3) is immediate by (2.2).

We now use 1.3 to replace S by a unitarily equivalent operator T acting on a Hilbert space of sections. Thus let F denote the bundle corresponding to E under the construction of 1.3, and let Φ be the isometric isomorphism $L^2(E)^G\longrightarrow L^2(F,h)$. We put

$$T:=\Phi\circ S\circ\Phi^{-1}\,|\,\Phi(\mathcal{D}(S)).$$

Then T is a selfadjoint operator in $L^2(F,h)$ with $\mathcal{D}(T)=\Phi(\mathcal{D}(S))$, and if

$$T = \int_{-\infty}^{+\infty} t \, d\, T_t$$

is its spectral resolution we have by (2.3)

2.5) $\dim T_t = N(t)$.

Of course the study of the asymptotic behavior of N will only make sense if $N(t) < \infty$ for $t \in \mathbb{R}$. In view of (2.5) this is equivalent to the following property of T:

2.6) T is bounded below and Spec T consists entirely of eigenvalues of finite multiplicity without finite accumulation points.

To check (2.6) in concrete cases the following simple remark will be very useful.

Lemma 2.3: Spec $T \subset$ Spec R, and if $\lambda \in$ Spec T is an isolated eigenvalue of R then λ is also an eigenvalue of T.

Proof:

Denoting as before by Q the orthogonal projection $L^2(E) \rightarrow L^2(E)^G$ we see that Q is a reducing subspace for $(R-z)^{-1}$ whenever $z \notin$ Spec R. Therefore Spec S $\subset$ Spec R. If λ is an isolated eigenvalue of R with eigenspace W and $Q(W)=o$ then it follows from (2.4) that S_t is constant near λ and $\lambda \notin$ Spec S which completes the proof.

Thus (2.6) is certainly satisfied if it is satisfied by R. The most interesting examples in applications are the operators arising from elliptic problems and these are known to fulfill (2.6) if e.g. M is compact. Therefore, we now discuss the case when R is generated by a differential operator P on the C^∞ sections of E which means simply that $C_o^\infty(E) \subset \mathcal{D}(R)$ and $R \mid C_o^\infty(E) = P$. This necessarily implies the following properties of P:

2.7) $$(Pf_1 \mid f_2)_{L^2(E)} = (f_1 \mid Pf_2)_{L^2(E)}, \quad f_1, f_2 \in C_o^\infty(E),$$

and

2.8) $$U_g Pf = PU_g f, \quad g \in G, \; f \in C_o^\infty(E).$$

We recall (see [27] , Ch. IV for these facts) that a differential operator P is a linear map $C^\infty(E) \longrightarrow C^\infty(E)$ satisfying for some $k \in \mathbb{Z}_+$ $j_k(Pf)(p) = o$ whenever $j_k(f)(p) = o$ for $f \in C^\infty(E)$ and $p \in M$. Here j_k denotes the k-jet map. The smallest k with this property is called the order of P. For a k^{th} order operator P the principal symbol $\sigma(P)$ of P is defined as an element of $C^\infty \mathrm{Hom}(\pi^*E, \pi^*E)$, where $\pi : T^*M \longrightarrow M$ is the natural projection. More precisely, for $p \in M$, $\xi \in T^*M_p$ and $e \in \pi^*E_\xi$ we choose $\varphi \in C^\infty(M)$ satisfying $\varphi(p)=o$ and $d\varphi_p = \xi$, and $f \in C^\infty(E)$ satisfying $f(p)=e$. Then

$$\sigma(P)(\xi)(e) = P\left(\frac{(-i)^k}{k!} \varphi^k f\right)(p),$$

and this is independent of the choices involved (we insert the factor $(-i)^k$ for convenience in later formulas). Now P is called elliptic if $\sigma(P)(\xi)$ is an isomorphism for $\xi \in T_1^*M$ and strongly elliptic, if k is even and $\sigma(P)(\xi)$ is positive definite for $\xi \in T_1^*M$. In dealing with G-manifolds we also have the weaker notion of transversal ellipticity: let $T_G^*M := \{\xi \in T^*M \mid \xi(X)=o \text{ for all } X \in TM_{\pi(\xi)}, \text{ tangent to } G\pi(\xi)\}$ and call P transversally elliptic if $\sigma(P)(\xi)$ is an isomorphism for $\xi \in T_1^*M \cap T_G^*M$.

It is natural to ask whether T is generated by a differential operator whenever R is. This is in fact so as we are going to show. It is again sufficient to deal with the trivial representation: if R is generated by a differential operator P then $\tilde{R}=R \otimes id_{V^*}$ is generated by the differential operator $\tilde{P}=P \otimes id_{V^*}$ on $C^\infty(E \otimes V^*)$, and the symbols of P and $\tilde{P}$ are related by

2.9) $$\sigma(\tilde{P}) = \sigma(P) \otimes id_{V^*}.$$

<u>Theorem 2.4:</u> If R is generated in $L^2(E)$ by a differential operator P of order k, then T is generated in $L^2(F,h)$ by a certain differential operator P' of order k. For the principal symbol of P' we have the formula

2.1o) $$\sigma(P')(\xi)(\pi_G^!(e)) = \pi_G^!(\sigma(P)(\pi_G^*\xi_p)(e))$$

for $q \in {}^{M_o}/_G$, $\xi \in (T_1^* \, {}^{M_o}/_G)_q$, $p \in \pi_G^{-1}(q)$ and $e \in E'_p$.
As consequence P' is elliptic when-ever P is transversally elliptic.

<u>Proof:</u> By 1.4, 1) we have $C_o^\infty(F) \subset \mathcal{D}(T)$. Since R is generated by P it then follows from 1.4, 4) that for $f \in C_o^\infty(F)$

2.11) $$\operatorname{supp} Tf = \operatorname{supp} \Phi(P(\Phi^{-1} f)) \subset \operatorname{supp} f.$$

Thus $T \mid C_o^\infty(F)$ possesses a linear extension P' to $C^\infty(F)$ satisfying (2.11) for $f \in C^\infty(F)$. In fact P' is uniquely determined and given by

$$P'f = \sum_{i \in \mathbb{N}} T(\varphi_i f) \;, \quad f \in C^\infty(F),$$

where $(\varphi_i)_{i \in \mathbb{N}}$ is a C^∞ partition of unity on ${}^{M_o}/_G$. Now suppose $j_k f(q) = o$ for $f \in C^\infty(F)$ and $q \in {}^{M_o}/_G$. Choose $\psi \in C_o^\infty({}^{M_o}/_G)$ with $\psi = 1$ in a neighbourhood of q and $p \in \pi_G^{-1}(q)$. Then we find from (2.11) and 1.4, 5)

$$P'f(q) = P'(\psi f)(q) = \Phi(P(\Phi^{-1}\psi f))(\pi_G(p))$$
$$= \pi'_G (P(\Phi^{-1}(\psi f))(p)) = o$$

showing that P' is a differential operator on $C^\infty(F)$ of order $\leq k$.
To prove (2.1o) let $q \in {}^{M_o}/_G$, $\xi \in (T_1^* \, {}^{M_o}/_G)_q$, and $e' \in F_q$, and choose $\varphi \in C_o^\infty({}^{M_o}/_G)$ satisfying $\varphi(q) = o$ and $d\varphi_q = \xi$, and $f \in C_o^\infty(F)$ satisfying $f(q) = e'$, Moreover pick $p \in \pi_G^{-1}(q)$ and $e \in E'_p$ with $\pi'_G(e) = e'$
Then we find with 1.4, 3)

$$\sigma(P')(\xi)\,(\pi'_G(e)) := P'\left(\frac{(-i)^k}{k!}\varphi^k f\right)(q)$$

$$= \Phi\left(P\left(\Phi^{-1}\left(\frac{(-i)^k}{k!}\varphi^k f\right)\right)\right)(\pi_G(p))$$

$$= \pi'_G\left(P\left(\frac{(-i)^k}{k!}(\varphi \circ \pi_G)^k \, \Phi^{-1} f\right)(p)\right)$$

$$= \pi'_G(\sigma(P)(\pi_G^* \xi_p)(e)).$$

The proof is complete.

As a corollary of the proof we note that E'_p is an invariant subspace of $\mathfrak{S}(P)(\xi)$ whenever $\xi \in (T^*_G M)_p$ and $p \in M_o$.

3. THE ASYMPTOTIC BEHAVIOR OF N

We now investigate the asymptotic behavior of N in a special case. We consider a set of conditions on R the validity of which will be discussed afterwards. We start with

(A) R is generated by a strongly elliptic operator P of order 2k.

Then P **automatically** satisfies (2.7) and (2.8). P is called formally positive if

(B) $(Pf|f)_{L^2(E)} \geqslant 0, \; f \in C_0^\infty(E).$

The next condition is concerned with Spec T (the operator constructed from R in section 2) is **and is almost the same as (2.6):**

(C) T is positive and Spec T consists entirely of eigenvalues of finite multiplicity without finite accumulation points.

We also want some rough control over the growth of N:

(D) N is of polynomial growth as $t \longrightarrow \infty$.

Next we bring in the spectral function e_t^R of R, $t \in \mathbb{R}$, which is the Schwartz kernel of R_t, i.e. the distributional section of $E \boxtimes E^*$ defined by

$$e_t^R(f^* \otimes f) := f^*(R_t f), \quad f \in C_0^\infty(E), \; f^* \in C_0^\infty(E^*).$$

where $\boxtimes$ denotes the external tensor product (see [2], sec. 5 for these notions). The ellipticity of P then **easily implies that** $e_t^R \in C^\infty(E \boxtimes E^*)$ (cf. [21]). Moreover it follows from the **Sobolev lemma and the local regularity results for elliptic systems** ([1] **sec.** 6) **that**

$$|\psi(e_t^R)| \leqslant a(\psi)\, t^{b(\psi)}, \quad t \in \mathbb{R},$$

for every distributional section ψ **of** $E^* \boxtimes E$ **with compact**

support and certain positive constants $a(\psi)$, $b(\psi)$. Therefore, by [8] Ch. VI, §1 n°2, prop. 7 and the reflexivity of $C^{\infty}(E \boxtimes E^*)$ the integral

(3.1) $$\Gamma_s^R := s \int_0^\infty e^{-st} e_t^R \, dt, \quad s > 0,$$

defines an element of $C^{\infty}(E \boxtimes E^*)$ which is called the heat kernel of R and is easily identified with the Schwartz kernel of e^{-sR}. We put n:=dim M. Then the condition on Γ_s^R is

(E) $$|\Gamma_s^R|_{E \boxtimes E^*}(p,q) \leq C_1 \, s^{-\frac{n}{2k}} \, e^{-C_2 \frac{d_M(p,q)^{\frac{2k}{2k-1}}}{s^{\frac{1}{2k-1}}}}$$

for certain $C_1 > 0$, $C_2 > 0$ and all $p,q \in M$, $s > 0$.
Finally, we need a condition on M, namely

(F) M can be isometrically and G-equivariantly imbedded into a compact Riemannian G-manifold X as an open subset.

We now present the main result of this paper.
Theorem 3.1: Suppose that $m := \dim M_o/G \geq 1$ and that (A) to (F) are satisfied. Then we have for

$$N(t) = \dim R_t^G$$

the following asymptotic behavior as $t \longrightarrow \infty$:

(3.2) $$N(t) \sim \frac{t^{\frac{m}{2k}}}{m(2\pi)^m} \int_{M_o} \frac{1}{\mathrm{vol}\, G_p} \int_{(T_1^* M_o)_p \cap T_G^* M_o} \mathrm{Tr}_{E_p'}(\sigma(P)(\xi))^{-\frac{m}{2k}} dw(\xi) \, dM(p).$$

Here dw denotes the volume element induced by Lebesgue measure.

Before proving 3.1 we give several corollaries in order to make our conditions more explicit.

Corollary 3.2: Let M be a compact Riemannian manifold, E a Hermitian G-vector bundle over M such that G acts on M by isometries and unitarily on the fibers. Let P be a transversally strongly elliptic differential operator of order $2k$ on $C^\infty(E)$ commuting with the action of G. Assume further that P is essentially selfadjoint in $L^2(E)$ (which is automatically fulfilled if P is strongly elliptic and symmetric on $C^\infty(E)$). For an irreducible unitary representation $\varrho : G \longrightarrow \mathrm{Aut}(V)$ denote by $\nu_\varrho(\lambda)$ the multiplicity of ϱ in the eigenspace of P with eigenvalue λ and put

$$N_\varrho(t) := \sum_{\lambda \leqslant t} \nu_\varrho(\lambda).$$

Then N_ϱ is finitely-valued and for $t \longrightarrow \infty$

$$N_\varrho(t) \sim \frac{t^{\frac{m}{2k}}}{m(2\pi)^m} \int_{M_o} \frac{1}{\mathrm{vol}\ G_p} \int_{(T_1^* M_o)_p \cap T_G^* M_o} \mathrm{Tr}_{(E_p \otimes V^*)^{G_p}} \left(\sigma(P)(\xi)^{-\frac{m}{2k}} \otimes \mathrm{id}_{V^*}\right) \times dw(\xi)\, dM(p). \tag{3.3}$$

Here m and dw are as in 3.1.

Proof: By (2.9) it is clearly sufficient to prove the corollary for the trivial representation. Assume first of all that P is strongly elliptic. It is well known that P is essentially selfadjoint and that $R := P^*$ satisfies condition (2.6) (see [27] Ch. XI, Th. 14). Moreover R commutes with G. By adding some multiple of the identity to P we may assume that P is formally positive and hence that R is positive. This is no loss of generality since the asymptotic behavior of N will not be changed. Thus conditions (A) and (B) are satisfied and (C) follows from the definition of T and 2.3. Now we clearly have

$$N(t) \leqslant \dim R_t =: N_R(t), \quad t \in \mathbb{R}.$$

Therefore, condition (D) holds since N_R is of polynomial growth (a well known fact which follows e.g. from formula (3.12) below, condition (E) and the Tauberian theorem in [12], p. 517). Finally, the estimate (E) is proved in [17], theorem 1.4.3. Thus in this case 3.2 follows from 3.1 with E replaced by $E \otimes V^*$ and P replaced by $P \otimes id_{V^*}$.

Now consider more generally a transversally strongly elliptic operator P which moreover is assumed to be essentially selfadjoint. Then $R := P^*$ is selfadjoint and commutes with G. We will reduce this case to the previous one by constructing a differential operator P' of order 2 such that $P'' := P + CP'^k$ is strongly elliptic for some constant C, commutes with G, and leads to the same function N as P (compare [3] sec. 2). To do so we choose on the Lie algebra $\mathfrak{g}$ of G a basis $(X_i)^l_{i=1}$ orthonormal with respect to a Ad G-invariant scalar product. We denote by $\hat{X}_i$ the corresponding Killing vector fields on M and define first order differential operators on $C^\infty(E)$ by

$$\tilde{X}_i f(p) := \frac{d}{dt}\Big|_{t=o} U_{\exp t X_i} f(p), \quad f \in C^\infty(E), \ p \in M,$$

$1 \leq i \leq l$. It is then easily checked that the second order differential operator

$$P' := -\sum_{i=1}^{l} \tilde{X}_i^2$$

is formally positive on $C^\infty(E)$ and commutes with G, and that its symbol is given by

3.4) $$\sigma(P')(\xi) = \Big(\sum_{i=1}^{l} \xi(\hat{X}_i(\pi(\xi)))^2\Big) \, id_{E_{\pi(\xi)}}.$$

Now since $T_1^*M \cap T_G^*M$ is a closed subset of the compact set T_1^*M and P is transversally strongly elliptic, we can find constants $C_3, C_4 > o$ such that for $\xi \in T_1^*M$

$$\sigma(P)(\xi) + C_3\sigma(P')^k(\xi) \geq C_4 \, |\xi|^{2k} \, \mathrm{id}_{E_{\pi(\xi)}} .$$

Thus $P'':=P+C_3P'^k$ is a strongly elliptic differential operator of order 2k on $C^\infty(E)$ and satisfies (2.7) and (2.8). Now let $R'':=P''^*$ and denote by S, T, S'', T'' the operators constructed form R and R'' as in section 2. We want to show that S=S''. This implies that the multiplicities of the trivial representation in R_t and R''_t are the same for $t \in \mathbb{R}$. Therefore, we can apply the part of the theorem already proved to P'' and since $\sigma(P'')(\xi)=\sigma(P)(\xi)$ for $\xi \in T^*_G M$ by (3.4) we are done.

Now a differential operator of order 2k on $C^\infty(E)$ extends by continuity to the Sobolev space $H^{2k}(E)$ (see [27] Ch. IX for the definition). It follows from this fact that

$$\mathcal{D}(R'')=H^{2k}(E) \subset \mathcal{D}(R)$$

and thus (with Q given by (1.5))

$$\mathcal{D}(S'')=Q\,\mathcal{D}(R'') \subset Q\,\mathcal{D}(R)=\mathcal{D}(S).$$

Let $f \in \mathcal{D}(R)$ and choose $(f_n)_{n\in\mathbb{N}} \subset C^\infty(E)$ with $f_n \longrightarrow f$ and $Pf_n \longrightarrow Rf$ in $L^2(E)$. Clearly $P'f=o$ for $f \in C^\infty(E)^G$ and we find

$$QPf_n = PQf_n = P''Qf_n \longrightarrow QRf = RQf.$$

But R'' is the closure of P'' which implies

$$Qf \in Q\,\mathcal{D}(R'') \text{ and } R''Qf = RQf.$$

Thus $\mathcal{D}(S) = \mathcal{D}(S'')$ and S=S''. The proof is complete.

<u>Corollary 3.3:</u> Let M,E and P be as in Cor. 3.2. Then the eigenvectors of P span $L^2(E)$.
<u>Proof:</u> We recall the decomposition (1.2):

$$L^2(E) = \sum_{\tau\in\hat{G}} L^2(E)_\tau .$$

Using the G-equivariantisomorphism $L^2(E \otimes W^*)^G \otimes W \to L^2(E)_\tau$ of section 1 and noting that

$$L^2(E \otimes W^*)^G \otimes W \simeq L^2(E \otimes W^* \otimes W)^G$$

where G acts trivially on W, we see that $R|L^2(E)_\tau$ is unitarily equivalent to the operator $\tilde{S}$ constructed from $\tilde{R}:=R \otimes id_{V^*} \otimes id_V$ in $L^2(E \otimes W^* \otimes W)$ as in section 2. Now the assertion follows from (the proof of) 3.2. It should be noted that for P transversally strongly elliptic the multiplicity of the eigenspaces need not be finite.

Corollary 3.4: Let M be the interior of a compact Riemannian manifold $\overline{M}$ with boundary, $\overline{E}$ a Hermitian G-vector bundle over $\overline{M}$ such that G acts by isometries on $\overline{M}$ and unitarily on the fibers and put $E:=\overline{E}|M$. Let P be a strongly elliptic differential operator of order 2k on $C_0^\infty(\overline{E})$ commuting with the action of G. Assume further that P is formally positive and let R be a positive selfadjoint extension of P in $L^2(\overline{E}) = L^2(E)$ arising from an elliptic boundary value problem and commuting with G. For an irreducible unitary representation $\varrho : G \longrightarrow \mathrm{Aut}(V)$ denote by $\nu_\varrho(\lambda)$ the multiplicity of ϱ in the eigenspace of P with eigenvalue λ and put

$$N_\varrho(t) := \sum_{\lambda \leq t} \nu_\varrho(\lambda).$$

Then N_ϱ is finitely-valued and has the asymptotic behavior described in (3.3) as $t \to \infty$.

Proof: We have to check conditions (A) to (F). (A) and (B) are fulfilled by assumption and (C) is proved in [20] sec 10.6 (see also [27] Appendix I). (E) is proved in [17], theorem 1.4.3 and formula (2.4.22). To prove (F) we imbed $\overline{M}$ into its double X. Then X is a compact G-manifold inducing the given G-action on M. Extending the metric of $\overline{M}$ to X somehow and averaging over G gives a G-invariant metric on X which induces the given one on M. Thus 3.1 applies and the corollary is proved.

We remark that the assumptions of 3.4 are always satisfied if $\overline{E}:=\overline{M} \times \mathbb{C}$ is the trivial bundle, $P:=-\Delta$ the Laplacian acting on functions, and R is generated by Dirichlet or Neumann boundary conditions. Because of its importance we finally single out the special form of 3.1 for Laplacian type symbols.

Corollary 3.5: Suppose that the assumptions of 3.1 hold and that in addition

$$\sigma(P)(\xi) = |\xi|^2 \, id_{E_{\pi(\xi)}} , \quad \xi \in T^*_G M.$$

Then

(3.5) $$N(t) \sim t^{\frac{m}{2k}} \frac{\omega_m}{(2\pi)^m} \dim E' \operatorname{vol} {}^{M_o}/_{G},$$

where ω_m denotes the volume of the m-dimensional unit ball.

The proof of 3.1 requires several lemmas and will occupy the next two sections. Before entering into it we will add some remarks on the formula (3.2). Note first of all that the right hand side is well defined by the strong (transversal) ellipticity of P and the remark at the end of section 2. Next we specialize to $G=\{e\}$ and M compact. Then we have $E'=E$, $h(q)=1$, $q \in {}^{M_o}/_G$, $m=n$, $T^*_G M=T^*M$ and (3.2) gives

(3.6) $$N(t) \sim \frac{t^{\frac{n}{2k}}}{n(2\pi)^n} \int_M \int_{(T^*_1 M)_p} Tr_E\,(\sigma(P)(\xi))^{-\frac{n}{2k}} \, d\omega(\xi) \, dM(p).$$

This formula is due to Weyl for bounded open sets in $\mathbb{R}^n$ and the Laplacian on functions [31] and to Minakshisundaram and Pleijel for Riemannian manifolds and the Laplacian on functions [26] . A proof in the general case follows e.g. from [17] theorem 1.6.1. The question of remainder estimates in (3.6) has been discussed by many authors in many special cases and is not yet completely settled. For nontrivial G we have two partial results in this direction, one of which applies to finite G only and will be given in section 5. The other one requires a very special G-structure on M but is then an easy consequence of known theorems.

It reads as follows.

Theorem 3.6: Suppose that the assumptions of 3.1 are satisfied and that in addition $M=M_o$ and M is compact. Denoting by H the remainder in (3.2) we have as $t \longrightarrow \infty$

(3.7) $$H(t) = O(t^{\frac{m-1}{2k}}),$$

if all eigenvalues of $\sigma(P)(\xi)$ are distinct for every $\xi \in T^*M - \{o\}$ or if $\sigma(P)(\xi) = |\xi|^2 id_{E_{\pi^*(\xi)}}$ for $\xi \in T^*M$, and

(3.8) $$H(t) = O(t^{\frac{m-1}{2k} + \frac{\varepsilon}{2k}})$$

for every $\varepsilon > \frac{1}{2}$ in general.

Proof: By assumption ${}^{M_o}/_G$ is a compact Riemannian manifold and by 2.4 and (2.1o) P' is a formally positive elliptic operator on $C^\infty(F)$. Therefore, P' is essentially selfadjoint and T must be its closure in $L^2(F,h)$. Let us denote by $e_t := e_t^T$ the spectral function of T. Since T satisfies (C) one readily verifies the identity

(3.9) $$N(t) = \dim T_t = \int_{{}^{M_o}/_G} Tr_F \; \delta^* e_t(q)\, h(q)\, d\,{}^{M_o}/_G(q).$$

Here δ denotes the diagonal map. Now the asymptotic behavior of $Tr_F\, \delta^*\, e_t$ has been investigated in [21] . Hörmander's results imply

(3.1o) $$Tr_F\, \delta^* e_t(q) \sim \frac{t^{\frac{m}{2k}}}{h(q)\; m(2\pi)^m} \int_{(T_1^*\, {}^{M_o}/_G)_q} Tr_F(\sigma(P')(\xi))^{-\frac{m}{2k}}\, dw(\xi)$$

for $q \in {}^{M_o}/_G$. Now for $p \in \pi_G^{-1}(q)$ $\pi_G' : E_p' \longrightarrow F_q$ is an isometric isomorphism. By 2.4 we therefore have for $\xi \in (T_1^*\, {}^{M_o}/_G)_q$

$$Tr_F(\sigma(P')(\xi))^{-\frac{m}{2k}} = Tr_{E'}(\sigma(P)(\pi_G^* \xi_p))^{-\frac{m}{2k}} .$$

Noting that $\pi_G^*:(T_1^* {}^{M_o}/_G)_q \longrightarrow (T_1^*M)_p \cap T_G^*M$ is an isometric isomorphism too we get

$$(3.11) \qquad \mathrm{Tr}_F\, \delta^* e_t(q) \sim \frac{t^{\frac{m}{2k}}}{h(q)\; m(2\pi)^m} \int_{(T_1^*M)_p \cap T_G^*M} \mathrm{Tr}_{E'}(\sigma(P)(\xi))^{-\frac{m}{2k}} dw(\xi).$$

Therefore, the Fubini theorem for Riemannian submersions implies

$$N(t) \sim \frac{t^{\frac{m}{2k}}}{m(2\pi)^m} \int_M \frac{1}{h \circ \pi_G(p)} \int_{(T_1^*M)_p \cap T_G^*M} \mathrm{Tr}_{E'}(\sigma(P)(\xi))^{-\frac{m}{2k}}\, dw(\xi)\; dM(p).$$

But the remainder in (3.11) can be estimated uniformly on ${}^{M_o}/_G$ yielding by integration remainder estimates for N. The estimate (3.8) is established in [21] for the general case and (3.7) in [22] and [19] respectively.

3.1 was also known before in special cases when G is nontrivial. For M a compact Riemann surface of genus $\geqslant 2$, $P=-\Delta$ on functions and G a subgroup of the isometry group of M (thus necessarily finite) 3.1 was proved by Huber [23]. For $M=\Gamma \backslash G_1$, where G_1 is a semisimple Lie group and Γ a cocompact discrete subgroup, P the Casimir operator on functions, and $G = K$ a maximal compact subgroup of G_1 3.1 is due to Gelfand [15], Gangolli [14], and Wallach [29] ; Wallach also proves 3.1 for M compact, $P=-\Delta$ on functions, and G finite. This will be discussed in more detail in section 6.

We now start with the proof of 3.1. Since T is a positive selfadjoint extension of a formally positive and strongly elliptic operator P' in $L^2(F,h)$, the spectral function e_t of T is well defined and satisfies (3.11) by [21]. However the asymptotic might not be uniform on ${}^{M_o}/_G$, so we cannot just integrate over ${}^{M_o}/_G$. Instead we want to consider the Laplace transform of N. Since N satisfies condition (D) we see that for $s > o$ the Laplace transform of N is in fact well defined. Introducing for $s > o$ the heat kernel Γ_s of T as above, we get from (3.1), (3.9), and Fubini's theorem

(3.12) $$s \int_0^\infty e^{-st} N(t)\, dt = \int_{M_o/G} Tr_F\, \delta^* \Gamma_s(q)\, h(q)\, d\, {}^{M_o}/_G(q).$$

Now (3.1), (3.11), and Abelian theorems for the Laplace transform ([12] , p. 456) lead to

$$Tr_F\ \delta^* \Gamma_s(q) \sim s^{-\frac{m}{2k}} \frac{\Gamma(\frac{m}{2k}+1)}{h(q)\ m(2\pi)^m} \int_{(T_1^* M_o/G)_q} Tr_F(\sigma(P')(\xi))^{-\frac{m}{2k}}\, dw(\xi)$$

for $q \in {}^{M_o}/_G$ as $s \searrow o$.

Suppose then that we can find a constant $C_5 > o$ such that

3.13) $$\left| s^{\frac{m}{2k}}\, Tr_F\ \delta^* \Gamma_s(q) \right| \leqslant \frac{C_5}{h(q)}$$

for all $s > o$ and $q \in {}^{M_o}/_G$. As will be proved in 4.4 below vol ${}^{M_o}/_G < \infty$. Thus the Lebesgue - Fatou lemma implies the asymptotic behavior of (3.12) as $s \searrow o$. From this fact in turn we deduce theorem 3.1 by (2.5) and the Tauberian theorem for the Laplace transform ([12], p. 517). Thus we are left with the proof of (3.13).

It is useful to relate Γ_s and Γ_s^R. To do so let $Q: L^2(E) \longrightarrow L^2(E)^G$ be the orthogonal projection given by (1.5), and let Q^* be the corresponding projection in $L^2(E^*)$. We then have

<u>Lemma 3.7:</u> For $s > o$

3.14) $$Tr_F\ \delta^* \Gamma_s \circ \pi_G = Tr_{E'}\ \delta^* Q \otimes Q^* (\Gamma_S^R) .$$

<u>Proof:</u> We want to establish the identity

3.15) $$Tr_F\ \delta^*\, e_t \circ \pi_G = Tr_{E'}\ \delta^*\, Q \otimes Q^* (e_t^R)$$

for $t \in \mathbb{R}$. To do so we recall that the action of G on the dual bundle E^* is defined by

$$g(e^*)(e) := e^*(g^{-1}(e)), \quad e^* \in E_p^*, \quad e \in E_{g(p)}.$$

Denoting the isomorphism $L^2(E^*)^G \longrightarrow L^2(F^*,h)$ also by Φ we then find for $f^* \in C_o^\infty(F^*)$, $f \in C_o^\infty(F)$

(3.16) $$\begin{aligned} e_t(f^* \otimes f) &= f^*(T_t f) = f^*(\Phi Q R_t Q \Phi^{-1} f) \\ &= \Phi^{-1} f^*(Q R_t Q \Phi^{-1} f) = Q^* \Phi^{-1} f^*(R_t Q \Phi^{-1} f) \\ &= e_t^R(Q^* \Phi^{-1} f^* \otimes Q \Phi^{-1} f) \\ &= Q \otimes Q^*(e_t^R)(\Phi^{-1} f^* \otimes \Phi^{-1} f) . \end{aligned}$$

Now let $p \in M_o$ and put $q := \pi_G(p)$. We choose a basis $(U_n)_{n \in \mathbb{N}}$ of neighbourhoods of q in $M_o/_G$ and a family $(g_n)_{n \in \mathbb{N}} \subset C_o^\infty(M_o/_G)$ with $\operatorname{supp} g_n \subset U_n$ and

$$\int_{M_o/_G} g_n(q)\, h(q)\, d^{M_o/_G}(q) = 1.$$

Moreover we choose $f_i^* \in C_o^\infty(F^*)$, $f_i \in C_o^\infty(F)$ such that $(f_i(q))$ forms an orthonormal basis of F_q with dual basis $(f_i^*(q))$, $1 \leq i \leq \dim F$. Then we obviously have

$$\operatorname{Tr}_F \delta^* e_t \circ \pi_G(p) = \lim_{n\to\infty} \sum_{i=1}^{\dim F} e_t(g_n f_i^* \otimes g_n f_i).$$

On the other hand we get from 1.4, 3) and (3.16)

$$e_t(g_n f_i^* \otimes g_n f_i) = Q \otimes Q^*(e_t^R)(g_n \circ \pi_G\, \Phi^{-1} f_i^* \otimes g_n \circ \pi_G \Phi^{-1} f_i).$$

Since $\Phi^{-1} f_i^*$ and $\Phi^{-1} f_i$ are G-invariant and since $(\Phi^{-1} f_i(p))$ is an orthonormal basis of E'_p with dual basis $(\Phi^{-1} f_i(p))$, (3.15) follows. Now (3.14) follows from (3.15) and (3.1) since the integral commutes with linear maps.

Now clearly

$$Q \otimes Q^*(\Gamma_s^R) = \frac{1}{(\mathrm{vol}\ G)^2} \int_{G \times G} U_g \otimes U_{g'}^* \, (\Gamma_s^R) \, d\, G \times G(g,g')$$

and therefore for $p \in M$

$$\left| \delta^* Q \otimes Q^* (\Gamma_s^R) \right|_{E \otimes E^*}(p)$$

$$\leqslant \frac{1}{(\mathrm{vol}\ G)^2} \int_{G \times G} \left| \Gamma_s^R \right|_{E \boxtimes E^*} (g^{-1}(p), g'(p)) \, d\, G \times G\, (g,g')$$

$$\leqslant \frac{1}{\mathrm{vol}\ G} \sup_{p' \in Gp} \int_G \left| \Gamma_s^R \right|_{E \boxtimes E^*} (g(p'),p') \, d\, G(g).$$

Making use of the assumption (E) we get the following inequality

(3.17)
$$\left| s^{\frac{m}{2k}} \, Tr_E \, \delta^* Q \otimes Q^* (\Gamma_s^R) \right| (p)$$

$$\leqslant C_1 \frac{s^{\frac{m-n}{2k}}}{\mathrm{vol}\ G} \sup_{p' \in Gp} \int_G e^{-C_2 \left(\frac{d_M(g(p'),p')}{s^{\frac{1}{2k}}} \right)^{\frac{2k}{2k-1}}} d\, G(g).$$

Therefore, the proof of 3.1 is completed by the following theorem which will proved in section 4.

<u>Theorem 3.8:</u> Let M be a Riemannian manifold satisfying (F), G a compact Lie group acting on M by isometries, and $\alpha > 0$. Then there exists a constant $C_6 > 0$ depending only on G, M, and α such that for all $p \in M$ and $t > 0$

(3.18)
$$\frac{\mathrm{vol}\ Gp}{\mathrm{vol}\ G} \int_G e^{-\frac{d_M^{\alpha}(g(p),p)}{t}} \, d\, G(g) \leqslant C_6 \, t^{\frac{\dim Gp}{\alpha}} .$$

4. THE VOLUME OF ORBITS AND THE PROOF OF THEOREM 3.3

In this section we supply two results completing the proof of 3.1, namely theorem 4.4 and the proof of theorem 3.8. Since our discussion will be local the following terminology is convenient. For fixed $p \in M$ we put $N := Gp$ and $H := G_p$. Denoting the orthogonal complement of T_pN in T_pM by W we have an orthogonal representation $\varrho : H \longrightarrow O(W)$. Thus we can form the G-vector bundle $E := G \times_\varrho W$ associated to the principal bundle $H \longrightarrow G \longrightarrow G/_H$ which is defined as the quotient of $G \times W$ by the equivalence relation $(g,w) \sim (gh, \varrho(h^{-1})w)$, $g \in G$, $w \in W$, $h \in H$. The G-action on E is given by $g[g',w] := [gg',w]$ and the projection $\pi : E \longrightarrow G/_H$ by $[g,w] \longmapsto g(H)$. It is then easy to see that the map $\tau : E \ni [g,w] \longmapsto g_*(w) \in TM$ defines a G-equivariant diffeomorphism $E \longrightarrow \nu N$ where νN denotes the normal bundle to N in M. Now for ε small the normal exponential map $\exp_\nu$ gives a G-equivariant diffeomorphism from $\nu^\varepsilon N := \{ \xi \in \nu N \mid |\xi| < \varepsilon \}$ onto an invariant neighbourhood of N in M which we denote by $T^\varepsilon N$. Putting $E^\varepsilon := \tau^{-1}(\nu^\varepsilon N)$ and $\varphi := \exp_\nu \circ \tau$ we therefore have a G-equivariant diffeomorphism $\varphi : E^\varepsilon \longrightarrow T^\varepsilon M$. This is the slice theorem. For our purposes we need a special left invariant metric on E.

Lemma 4.1: There is a left invariant metric on E with the following properties:

1) $\pi : E \longrightarrow G/_H$ is a Riemannian submersion.
2) For each $v \in E$ $\pi|Gv : Gv \longrightarrow G/_H$ is a Riemannian submersion.
3) The fibers $\pi^{-1}(gH)$, $g \in G$, are canonically isometric to W with the Euclidean metric.

Proof: It is sufficient to construct a connection $\mathcal{H} = (H_v)_{v \in E}$ on E with the following properties:

1) $\mathcal{H}$ is G-invariant, i.e. $H_{g(v)} = g_*(H_v)$ for $g \in G$, $v \in E$,
2) for each $v \in E$ $H_v \subset T_v\, G_v$.

In fact given such an $\mathcal{H}$ we have $T_v = H_v \oplus \operatorname{Ker} \pi_{*,v}$ and we can provide H_v with the pull back under π_* of the metric in $T_{\pi(v)}\, G/_H$. Moreover if $v = [g,w]$ we get a linear isomorphism $W \ni w \longmapsto [g,w] \in E_{\pi(v)} = \operatorname{Ker} \pi_{*,v}$ and the induced metric on $\operatorname{Ker} \pi_{*,v}$ is independent of the choice of g. Thus requiring $H_v \perp \operatorname{Ker} \pi_{*,v}$ we have a metric on E which is easily seen to possess the desired properties.

To find $\mathcal{H}$ we start with a Ad(H) - invariant decomposition $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$, where $\mathfrak{g}$ and $\mathfrak{h}$ denote the Lie algebras of G and H respectively. Then $\{ L_{g*}(\mathfrak{m}) \mid g \in G \}$ defines a H-invariant connection for the principal bundle $H \longrightarrow G \longrightarrow {}^{G}/_{H}$. Let $\mathcal{H}$ be the corresponding connection in the associated bundle E (cf. [25] p. 87/88). If $v = [g,w] \in E$ $\mathcal{H}$ can be described as follows: $H_v := \{ \frac{d}{dt}\Big|_{t=o} [c(t),w] \Big| c : (-1,1) \longrightarrow G$ horizontal curve in G with $c(o) = g \}$.

It is now obvious that $\mathcal{H} = (H_v)_{v \in E}$ satisfies 1) and 2).

Lemma 4.2: Define the nonnegative function c on E^{ε} by

(4.1) $$c(v) := |\det \varphi_* | T_v Gv| .$$

1) For $v \in E^{\varepsilon}$ we have

(4.2) $$\text{vol } G\,\varphi(v) = c(v) \text{ vol } N \text{ vol } G_{\pi(v)}v.$$

2) For ε sufficiently small there are constants $C_7, C_8 > o$ such that for $v \in E^{\varepsilon}$

(4.3) $$C_7 \text{ vol } N \text{ vol } G_{\pi(v)}v \leq \text{vol } G\,\varphi(v) \leq C_8 \text{ vol } N \text{ vol } G_{\pi(v)}v.$$

3) If all orbits in E^{ε} are nonsingular then $c \in C^{\infty}(E^{\varepsilon})$.

Proof: 1) Let $x := \varphi(v)$, then

$$\text{vol } Gx = \int_{Gv} |\det \varphi_* | T_w Gv| \; d(Gv)(w) = c(v) \text{ vol } Gv$$

$$= c(v) \text{ vol } N \text{ vol } G_{\pi(v)}v$$

by the equivariance of φ and 4.1, 2).

2) Let C' be a uniform bound for the norm of $\varphi_{*v} : T_v G v \longrightarrow T_{\varphi(v)} G\varphi(v)$, $v \in E^{\varepsilon}$, which certainly exists for ε small. Then if $(e_i)_{i=1}^{r}$ denotes an orthonormal basis for $T_v Gv$ we get

$$c(v) = |\varphi_*(e_1) \wedge \dots \wedge \varphi_*(e_r)|$$

$$\leq \prod_{i=1}^{r} |\varphi_*(e_i)| \leq (1+C')^n =: C_8$$

which implies the second inequality. The first one follows from the same argument applied to φ^{-1}.

3) That all orbits in E^{ε} are nonsingular is equivalent to dim Gv = dim N for $v \in E^{\varepsilon}$. Denote by $\mathfrak{g}$ the Lie algebra of G and by $\tilde{X}$, $\tilde{\tilde{X}}$ the Killing vector fields corresponding to $X \in \mathfrak{g}$ on E^{ε} and $T^{\varepsilon}N$ respectively. The G-equivariance of φ implies the relation

$$(4.4) \qquad \tilde{\tilde{X}}_{\varphi(v)} = \varphi_*(\tilde{X}_v).$$

Now pick $v_o \in E^\varepsilon$ and choose a complementary subspace $\mathfrak{m}$ to the Lie algebra of G_{v_o} in $\mathfrak{g}$ with basis $(X_j)_{j=1}^l$. Then $(\tilde{X}_j(v))_{j=1}^l$ is a basis for $T_v Gv$ in a neighbourhood of v_o since the dimension is constant. Thus we have in this neighbourhood of v_o from (4.4)

$$c(v) = \frac{|\tilde{\tilde{X}}_1 \wedge \ldots \wedge \tilde{\tilde{X}}_l(\varphi(v))|}{|\tilde{X}_1 \wedge \ldots \wedge \tilde{X}_l(v)|}$$

which proves the smoothness of c.

<u>Corollary 4.3:</u> Suppose that all orbits in M are nonsingular and put

$$n(p) := \left[G_p : H_p\right],$$

where $H_p \subset G_p$ is a principal isotropy group for $p \in M$. Then

$$\tilde{h}(p) := n(p) \text{ vol } Gp$$

defines a C^∞-function on M.
In particular $h \circ \pi_G \in C^\infty(M_o)$, where h was defined in (1.4).
<u>Proof:</u> By (4.2) we have in a neighbourhood of $N = Gp_o$ with $v = \varphi^{-1}(p)$

$$\text{vol } Gp = c(v) \text{ vol } N \text{ vol } G_{\pi(v)}v.$$

The absence of singular orbits implies that c is C^∞ by 4.2, 3) and that $G_{\pi(v)}v$ is finite because $\pi|Gv$ is a submersion. Therefore

$$\text{vol } G_{\pi(v)}v = \left| {}^{G_{\pi(v)}}/_{G_v} \right| = \frac{\left[G_{\pi(v)} : H\right]}{\left[G_v : H\right]}$$

for any principal isotropy group H with $H \subset G_v \subset G_{\pi(v)}$. Since $G_v = G_p$ the proof is complete.

We now proceed to show that ${}^{M_o}/_G$ has finite volume, which was essential in reducing the proof of 3.1 to the proof of 3.8.

<u>Theorem 4.4:</u> Suppose that M satisfies (F). Then

$$\mathrm{vol}\ {}^{M_o}/_G < \infty .$$

<u>Proof:</u> By the Fubini theorem and (F) we have

$$\mathrm{vol}\ {}^{M_o}/_G = \int_{M_o} \frac{1}{\mathrm{vol}\ Gp}\ dM(p)$$

$$\leq \int_{X_o} \frac{1}{\mathrm{vol}\ Gp}\ dX(p),$$

where X is the compact Riemannian G-manifold in which M is imbedded. We therefore may assume that M is compact. Now we use induction on dim M. For dim M = 1 the assertion is obviously true. By compactness of M it is sufficient to show

$$\int_{T^{\varepsilon}N} \frac{1}{\mathrm{vol}\ Gp}\ dM(p) < \infty$$

for arbitrary $p_o \in M$, $N = Gp_o$ and some $\varepsilon > o$.
In view of (4.3) and the boundedness of c(v) for $v \in E^{\varepsilon}$ with ε small all we need to estimate is

$$\int_{E^{\varepsilon}} \frac{1}{\mathrm{vol}\ G_{\pi(v)}v}\ dE(v).$$

Now since $\pi : E^{\varepsilon} \longrightarrow N$ is a Riemannian submersion with fibers isometric under the G-action, the Fubini theorem gives

$$\int_{E^{\varepsilon}} \frac{1}{\mathrm{vol}\ G_{\pi(v)}v}\ dE(v) = \mathrm{vol}\ N \int_{E^{\varepsilon} \cap E_{p_o}} \frac{1}{\mathrm{vol}\ G_{p_o}v}\ dE_{p_o}(v).$$

Recall that $E^{\varepsilon} \cap E_{p_o} =: W^{\varepsilon}$ carries the Euclidean metric inherited from $T_{p_o}M$ and that G_{p_o} acts orthogonally on W^{ε}. Therefore

$$\int_{W^\varepsilon} \frac{1}{\operatorname{vol} G_{p_o}v}\, dW(v) = \int_o^\varepsilon \int_{S(W)} \frac{1}{\operatorname{vol} G_{p_o}rw}\, dS(W)(w)\, r^{\dim V - 1} dr$$

$$= \int_o^\varepsilon r^{\dim V - 1 - \dim G_{p_o}w} dr \int_{S(W)} \frac{1}{\operatorname{vol} G_{p_o}w}\, dS(W)(w)$$

$$\leqslant \text{const} \int_{S(W)} \frac{1}{\operatorname{vol} G_{p_o}w}\, dS(W)(w),$$

where $S(W)$ denotes the unit sphere in E_{p_o}. But $\dim S(W) < \dim W \leqslant \dim M$ and the induction hypothesis applies to the last written integral thus completing the proof.

The remainder of this section is devoted to the proof of theorem 3.8 which will be based on the following three lemmas. The first deals with the case of a single orbit.

Lemma 4.5: Let N be a compact connected Riemannian manifold and $\alpha > o$. Then there is a constant $C_9 > o$ such that for all $x \in N$ and $t > o$

$$\int_N e^{-\frac{d_N^\alpha(x,y)}{t}} dN(y) \leqslant C_9\, t^{\frac{\dim N}{\alpha}}.$$

Proof: In geodesic polar coordinates (r,v) at x we have

$$\int_N e^{-\frac{d_N^\alpha(x,y)}{t}} dN(y) = \int_{S^{l-1}} \int_o^{\varrho(v)} e^{-\frac{r}{t}} \left|\det \exp_{x*}(r,v)\right| r^{l-1}\, dr\, dw(v)$$

where $l := \dim N$, $dw :=$ volume element on S^{l-1} and $\varrho(v)$ denotes the cut locus distance in direction $v \in T_xN$. Now $|\det \exp_{x*}(r,v)|$ can be uniformly bounded in terms of a lower bound for the Ricci curvature and an upper bound for the diameter of N ([5] , p. 253) which implies the lemma.

In the next lemma we reduce to the case of connected G.

Lemma 4.6: Suppose that (3.18) has been proved for fixed M and all connected groups G. Then it holds also for general G.

Proof: Let G_o be the identity component in G and put $l := [G : G_o]$, the number of components of G. For $p \in M$ we then can find $g_i = g_i(p) \in G$, $1 \leqslant i \leqslant l$, such that $G = G_o \cup \bigsqcup_{i=1}^{l} G_o g_i$ and

(4.5) $$d_M(g\, g_i(p), p) \geqslant d_M(g_i(p), p)$$

for $g \in G_o$, $1 \leqslant i \leqslant l$. This implies

(4.6) $$d_M(g\, g_i(p), p) \geqslant \frac{1}{2}\, d_M(g(p), p)$$

for $g \in G_o$, $1 \leqslant i \leqslant l$, since otherwise by (4.5) $d_M(g_i(p), p) < \frac{1}{2}\, d_M(g(p), p)$ for some $g \in G_o$ and therefore

$$d_M(g(p), p) \leqslant d_M(g\, g_i(p),\ g(p)) + d_M(g\, g_i(p), p)$$

$$= d_M(g_i(p), p) + d_M(g\, g_i(p), p) < d_M(g(p), p).$$

Now we find with $g_o := e$ by (4.6)

$$\frac{\text{vol } Gp}{\text{vol } G} \int_G e^{-\frac{d_M^{\alpha}(g(p),p)}{t}}\, dG(g)$$

$$\leqslant \frac{\text{vol } G_o p}{\text{vol } G_o} \sum_{j=o}^{l} \int_{G_o} e^{-\frac{d_M^{\alpha}(g\, g_i(p),p)}{t}}\, dG(g)$$

$$\leqslant (l+1)\, \frac{\text{vol } G_o p}{\text{vol } G_o} \int_{G_o} e^{-\frac{d_M^{\alpha}(g(p),p)}{2^{\alpha} t}}\, dG(g).$$

We now assume that G is connected and that it does not act transitively on M. For $p \in M$ we introduce as before $T^{\varepsilon}N$ and E^{ε} for ε small, $N = Gp$. We put in addition $\psi := \varphi^{-1}$ (where $\varphi : E^{\varepsilon} \longrightarrow T^{\varepsilon}N$ was the canonical G-equivariant diffeomorphism) and $\tilde{\pi} := \pi \circ \psi$. For $x, y \in T^{\varepsilon}N$ we then define

$$A(x,y) := \left\{ v \in G_{\tilde{\pi}(y)}\, \psi(y) \;\middle|\; d_E(v, \psi(x)) \text{ is minimal} \right\}.$$

This makes sense since G is compact and connected. Finally we denote in case that codim $N \geqslant 2$ by d_{sph} the Riemannian distances induced on the spheres in the (Euclidean) fibers of E. The following technical lemma makes it possible to **prove** 3.8 by induction on dim M.

Lemma 4.7: Suppose that G is connected and does not act transitively on M. Then for $\alpha > 0$, $p \in M$ and ε small there is a constant $C_{10} = C_{10}(\alpha, p, \varepsilon)$ such that for all $x, y \in T^{\varepsilon}N$ and $v \in A(x,y)$

$$d_M^{\alpha}(x,y) \geqslant C_{10} \begin{cases} d_N^{\alpha}(\tilde{\pi}(x), \tilde{\pi}(y)) & \text{if codim } N = 1, \\ d_N^{\alpha}(\tilde{\pi}(x), \tilde{\pi}(y)) + d_{sph}^{\alpha}(\psi(y), v) & \text{otherwise.} \end{cases}$$

Proof: We begin with the remark that on a connected manifold any two Riemannian metrics are equivalent on compact subsets. Furthermore, if N is an imbedded submanifold of the Riemannian manifold N' then d_N and $d_{N'}$ are equivalent on compact subsets of N. These facts give the existence of constants C', C'' such that

(4.7) $$d_E(\psi(x)), \psi(y)) \leqslant C' d_M(x,y)$$

for $x, y \in T^{\varepsilon}N$ and

$$d_E(\psi(x), \psi(y)) \geqslant C'' d_{E_q}(\psi(x), \psi(y))$$

for $x, y \in T^{\varepsilon}N$ with $\tilde{\pi}(x) = \tilde{\pi}(y) = q$ for $q \in N$, if only ε is sufficiently small. A priori C'' depends on q, but the action of G is transitive on N and maps the fibers of E isometrically onto each other implying that C'' can be chosen independent of q. Next we recall that $\pi : E \longrightarrow N$ is a Riemannian submersion which gives

$$d_E(\psi(x), \psi(y)) \geqslant d_N(\tilde{\pi}(x), \tilde{\pi}(y)) \tag{4.8}$$

for $x,y \in T^{\varepsilon}N$. From (4.7) and (4.8) the lemma follows if codim N = 1, so we assume codim $N \geqslant 2$ from now on. For $x,y \in T^{\varepsilon}N$ with $\tilde{\pi}(x) = \tilde{\pi}(y) = q$ and $|\psi(x)|_{E_q} = |\psi(y)|_{E_q}$ we have by elementary geometry

$$d_{E_p}(\psi(x), \psi(y)) \geqslant \frac{2}{\pi} d_{sph}(\psi(x), \psi(y)).$$

We now find for $x,y \in T^{\varepsilon}N$ and $v \in A(x,y)$ by the triangle inequality

$$d_E^{\alpha}(\psi(x), \psi(y)) + d_E^{\alpha}(\psi(y), v)$$

$$\leqslant d_E^{\alpha}(\psi(x), \psi(y)) + (d_E(\psi(x), \psi(y)) + d_E(\psi(x), v))^{\alpha}$$

$$\leqslant (1+2^{\alpha})\, d_E(\psi(x), \psi(y)).$$

The proof is completed by combining the above inequalities and noting that $v \in E_{\tilde{\pi}(y)}$ with $|v| = |\psi(y)|$.

We are now ready for the proof of 3.8.

Proof of Theorem 3.8: We proceed by induction on dim M. The assertion is trivial for dim M = o. So assume the theorem is proved for all Riemannian manifolds satisfying (F) with dimension smaller then dim M. Now by (F) we can further assume M to be compact (since $d_M \geqslant d_X$), and G to be connected in view of lemma 4.6. Moreover by compactness we need only prove the theorem for $p \in T^{\varepsilon}N$ where $N = G p_o$ is arbitrary and ε sufficiently small. To do so we fix $p_o \in M$ and choose a suitable ε such that all the above lemmas hold. We put $H := G_{p_o}$ and provide G/H with the left invariant metric induced from N and G with a left invariant metric turning the canonical projection $G \longrightarrow G/H$ into a Riemannian submersion. We can use the corresponding volume element dG in (3.18) since the left hand side is independent of the choice of left invariant measure on G. Thus we get from Fubini's theorem for $p \in T^{\varepsilon}N$

(4.9) $$\frac{\text{vol } Gp}{\text{vol } G} \int_G e^{-\frac{d_M^{\alpha}(g(p),p)}{t}} \, dG(g)$$

$$= \frac{\text{vol } Gp}{\text{vol } N \text{ vol } H} \int_{G/H} \int_H e^{-\frac{d_M^{\alpha}(gh(p),p)}{t}} \, dH(h) \, d^{G}/_{H}(gH).$$

By the left invariance of dG we may assume $\tilde{\pi}(p) = p_o$. We then have for $g \in G$, $h \in H$

(4.1o) $$d_N^{\alpha}(\tilde{\pi}(gh(p)),\tilde{\pi}(p)) = d_N^{\alpha}(g(p_o),p_o).$$

If M is homogeneous we have $d_M^{\alpha}(gh(p),p) = d_N^{\alpha}(g(p_o),p_o)$ and the proof is immediate from 4.5; if codim N = 1 the proof follows from 4.7, (4.1o) and 4.5 again. So let codim $N \geq 2$. We choose $v \in A(p,gh(p))$. By definition $v \in gH(\psi(p))$ implying $g^{-1}(v) = h_o(\psi(p))$ for some $h_o = h_o(v,p) \in H$. Therefore

(4.11) $$d_{sph}^{\alpha}(gh(\psi(p)),v) = d_{sph}^{\alpha}(h(\psi(p)),h_o(\psi(p)))$$

$$= d_{sph}^{\alpha}(h_o^{-1} h(\psi(p)), \psi(p)).$$

Thus we conclude from (4.1o), 4.7 and (4.11)

$$\frac{\text{vol } Gp}{\text{vol } G} \int_G e^{-\frac{d_M^{\alpha}(g(p),p)}{t}} \, dG(g)$$

$$= \frac{\text{vol } Gp}{\text{vol } N \text{ vol } H} \int_N e^{-C_{1o}\frac{d_N^{\alpha}(q,p)}{t}} \, dN(q) \quad \times$$

$$\times \int_H e^{-C_{1o} \frac{d_{sph}^{\alpha}(h(\psi(p)),\psi(p))}{t}} dH(h).$$

Now it is easily checked that the validity of (3.18) for a given metric implies the same estimate with the same constant for any metric differing only by a scalar factor. Therefore we may assume that $|\psi(p)|_E = \frac{\varepsilon}{2}$. Consequently the induction hypothesis applies to the second integral above and (4.3) completes the proof.

5. AN IMPROVED ESTIMATE OF N FOR FINITE G

In this section we present another remainder estimate for N in a special situation. Our assumptions are the following.

(α) G is a finite group and acts effectively on M.

(β) M is compact.

(γ) R is the closure in $L^2(E)$ of a strongly elliptic and formally positive operator P on $C^\infty(E)$ of order 2k satisfying (2.1).

R is then a positive selfadjoint operator in $L^2(E)$ with $\mathcal{D}(R) = H^{2k}(E)$. Finally we require the following asymptotic behavior of the spectral function of R:

(δ)
$$\operatorname{Tr}_E \delta^* e_t^R(p) = \frac{t^{\frac{n}{2k}}}{n(2\pi)^n} \int_{T_1^* M_p} \operatorname{Tr}_E(\sigma(P)(\xi))^{-\frac{n}{2k}}\, dw(\xi) + O(t^{\frac{n-1}{2k}})$$

uniformly in $p \in M$. This is known to be true in many cases (e.g. if P acts on functions or $\sigma(P)(\xi) = |\xi|^2 \operatorname{id}_{\pi^* E}$) though not in general, cf. [22] . However this time the estimate of N will not be achieved via the heat kernel but more directly using the relation (3.9) as in the proof of 3.6. This is possible since by the finiteness of G we can construct a nice fundamental domain Y for G in M. The main tool in this construction is a recent theorem of Illman [24]. The operator T is then essentially the restriction of S to sections of $E|Y$. Therefore, we have some control over its domain and can apply a suitable globalization of the method in [10] . This yields a precise estimate of $\operatorname{Tr}_F \delta^* e_t$ which can be integrated over Y to give the following result.

<u>Theorem 5.1:</u> Suppose that conditions (α) to (δ) hold. For an irreducible unitary representation $\rho : G \longrightarrow \operatorname{Aut}(V)$ denote by $\nu_\rho(\lambda)$ the multiplicity of ρ in the eigenspace of P

with eigenvalue λ and put

$$N_\ell(t) := \sum_{\lambda \leq t} \nu_\ell(\lambda).$$

Then we have with $|G|$:= order of G.

$$(5.1) \qquad N_\ell(t) = \frac{t^{\frac{n}{2k}}}{n(2\pi)^n} \int_M \frac{1}{|G|} \int_{(T_1^*M)_p} \mathrm{Tr}_{(E_p \otimes V^*)^{G_p}} (\sigma(P)(\xi)^{-\frac{n}{2k}} \otimes \mathrm{id}_{V^*}) \times$$

$$\times\, dw(\xi)\, dM(p) + O(t^{\frac{n-1}{2k}} \log t).$$

Thus we obtain an improvement of Huber's result [23] and also of results of Wallach [29] which will be discussed in the next section. It should be remarked however that (5.1) does not seem to be best possible. One expects an estimate with the log - term removed. The proof of 5.1 will require several lemmas. To begin with we note that we can assume $2k > n$ which is convenient for technical reasons. In fact if $2k \leq n$ we choose $l \in \mathbb{N}$ with $2kl > n$. Then R^l will be the closure of P^l and for the spectral resolution we have

$$R^l{}_{t^l} = R_t \quad , t \in \mathbb{R}.$$

Therefore, it suffices to prove 5.1 for R^l. Our first task will be the construction of a fundamental domain Y for G in M. To describe it we assume that M is G-equivariantly imbedded in some $\mathbb{R}^r$ and that G acts orthogonally on M (this is possible in view of the Mostow-Palais-theorem, [9] p. 111, th. 1o.1). Then by the result of [24] there is a finite equivariant rectangular simplicial G-complex K with $|K| \subset \mathbb{R}^s$ for some s and a G-equivariant map $\varphi : |K| \longrightarrow M$, such that φ is a homeomorphism $|K| \longrightarrow M$ and $\varphi \,|\, |\sigma|$ is a differentiable imbedding for each simplex $\sigma \in K$. Let K^n be the set of n-dimensional simplexes in K, n=dim M, and denote by L_σ the affine space generated by $|\sigma|$ in $\mathbb{R}^s$ for $\sigma \in K$. Moreover let U(M) be a tubular neighbourhood of M in $\mathbb{R}^r$. Then we need the following lemma.

Lemma 5.2: 1) G acts freely on K^n.
2) For $\sigma \in K$ there is a neighborhood U_σ of $|\sigma|$ in L_σ and a C^∞ imbedding $\varphi_\sigma : U_\sigma \longrightarrow U(M)$ extending $\varphi \mid |\sigma|$.
3) $\varphi(|K| - \bigcup_{\sigma \in K^n} |\sigma|)$ has measure zero in M.

Proof: 1) G acts on K^n by the definition of K and by [24], p. 201 $g(\sigma) = \sigma$ implies $g(p) = p$ for all $p \in \varphi(|\sigma|)$. But then $g = e$ since g is an isometry.
2) By definition there is a neighbourhood $\tilde{U}_\sigma$ of $|\sigma|$ in L_σ and a C^∞ extension $\tilde{\varphi}_\sigma$ of $\varphi \mid |\sigma|$ such that $\tilde{\varphi}_{\sigma *}$ has rank n on $|\sigma|$. This easily implies the assertion.
3) This is obvious since K is finite and $\varphi \mid |\sigma|$ is differentiable for each $\sigma \in K$.

We now select one simplex from each G-orbit in K^n, say $\sigma_1, \ldots, \sigma_{t_o}$ and put

$$Y := \operatorname{Int} \bigcup_{j=1}^{t_o} \varphi(\sigma_j) .$$

Then Y is an open subset of M and vol Y = $|G|^{-1}$ vol M. Letting $F := E \mid Y$ we obtain the following simple and explicit version of 1.2.

Lemma 5.3: The map

$$\Phi : L^2(E)^G \ni f \longmapsto |G|^{\frac{1}{2}} \; f \mid Y \in L^2(F)$$

is an isometric isomorphism.

Proof: We compute for $f \in C(E)^G$

$$\int_M |f(p)|_E^2 \, dM(p) = \sum_{\substack{1 \leq j \leq t_o \\ g \in G}} \int_{g(\varphi(|\sigma_j|))} |f(p)|_E^2 \, dM(p)$$

$$= |G| \sum_{j=1}^{t_o} \int_{\varphi(|\sigma_j|)} |g^{-1} f(g(p))|_E^2 \, dM(p)$$

$$= \int_Y |\Phi f(p)|_F^2 \, dY(p)$$

Since $C_o^\infty(F) \subset \Phi(L^2(E)^G)$ the proof is complete.

As before we introduce the positive selfadjoint operators $S = R \mid \mathcal{D}(R) \cap L^2(E)^G$ and $T = \Phi \circ S \circ \Phi^{-1}$ with $\mathcal{D}(T) = \Phi(\mathcal{D}(S))$. Denoting by $(T_t)_{t \in \mathbb{R}}$ the spectral resolution and by e_t, $t \in \mathbb{R}$, the spectral function of T we have again

(5.2) $$N(t) = \dim T_t = \int_Y \mathrm{Tr}_F \, \delta^* e_t(p) \, dY(p).$$

Now P is also a differential operator on $C^\infty(F)$ and it is obvious that T is a positive selfadjoint extension of P in $L^2(F)$. We propose to estimate the function

$$B_p(t) := \mathrm{Tr}_F \, \delta^* e_t(p) - \mathrm{Tr}_E \, \delta^* e_t^R(p)$$

for $p \in Y$, $t \in \mathbb{R}$, which implies an estimate with remainder term for $\mathrm{Tr}_F \, \delta^* e_t$ because of (δ). This will be done using the method of [1o] , i.e. we will apply a Tauberian theorem of the Fatou type to the Laplace transform of B_p which in turn is given by an integral involving the Green's kernels of T and R. These are by definition the Schwartz kernels of the operators $(T - z \cdot id)^{-1}$ and $(R - z \cdot id)^{-1}$ respectively for $z \notin \mathbb{R}_+$, and we will denote them by T_z and R_z. The condition $2k > n$ yields that T_z and R_z are continuous functions uniformly bounded in the space variables. To formulate the precise estimates we choose ε with $0 < \varepsilon < \frac{\pi}{2}$ and put $Z_\varepsilon := \{ z \in \mathbb{C} \mid |\arg z| < \varepsilon \}$. Furthermore for $p \in Y$ we define

$$l(p) := \mathrm{dist} \{ p, \partial Y \}.$$

<u>Lemma 5.4:</u> For $z \notin \mathbb{R}_+$ we have $T_z \in C(F \boxtimes F^*)$ and $R_z \in C(E \boxtimes E^*)$ with continuous dependence on z. There are constants C_{1o}, C_{11}, C_{12} such that for $z \notin Z_\varepsilon$, $|z| \geq C_{1o}$, and $p, q \in Y$

(5.3) $$|T_z(p,q)|_{F \boxtimes F^*} + |R_z(p,q)|_{E \boxtimes E^*} \leq C_{11} \, |z|^{\frac{n-2k}{2k}}$$

and

(5.4) $$\left|\mathrm{Tr}_F\, \delta^* T_z(p) - \mathrm{Tr}_E\, \delta^* R_z(p)\right| \leq C_{11}\, |z|^{\frac{n-2k}{2k}}\, e^{-C_{12} l(p)\, z^{\frac{1}{2k}}} .$$

Proof: Since $2k > n$ the continuity of R_z and the estimate (5.3) for R_z alone follow from (the matrix version of) 4.2, 4.7 and 4.8 in [21] and the compactness of M. Arguing as in the proof of lemma 3.7 we obtain the relation

(5.5) $$T_z = |G| \quad Q \otimes Q^*(R_z) \Big| \; Y, \; z \notin \mathbb{R}_+ .$$

This implies the continuity of T_z and (5.3). (5.4) is proved as in [1o] , Hilfsatz 3 with obvious modifications.

As in [1o] we now find the following estimate.

Lemma 5.5: There is a constant $C_{13} > o$ such that

$$\left|B_p(t)\right| \leq \frac{C_{13}}{l(p)} \; t^{\frac{n-1}{2k}}$$

for $p \in Y$, $t > o$.

Proof: We have the identity (see [1o])

$$\int_o^\infty e^{-st}\, d\, B_p(t) = -\frac{1}{2\pi i} \int_{\partial Z_\varepsilon} e^{-s z^{\frac{1}{2k}}} (\mathrm{Tr}_F\, \delta^* T_z - \mathrm{Tr}_E\, \delta^* R_z)(p)\, dz$$

where Re s > o and $z^{\frac{1}{2k}}$ is chosen positive on the positive axis. Denoting the left hand side by $\varphi_p(s)$ we then conclude from 5.4 that φ_p has an analytic extension to $\{ s \in \mathbb{C} \mid |s| \leq C' l(p) \}$ and satisfies the estimate $\varphi_p(s) \leq \frac{C''}{l(p)^n}$ for these s, where C', C'' are independent of $p \in Y$. Thus the assumption d) of the Tauberian theorem ([1o] , Hilfssatz 4) is satisfied. Assumption a) is obviously fulfilled, since B_p is the difference of two monotone functions and c) follows from the monotonicity of $\mathrm{Tr}_F\, \delta^* e_t$ and (δ) Finally, b) follows immediately from (3.15) and (δ) again. The proof of the lemma follows from the Tauberian theorem.

With these results established 5.1 is easily proved.

Proof of Theorem 5.1 It is sufficient to deal with the trivial representation. From (δ) and 5.5 we get the estimate

$$Tr_F \delta^* e_t(p) - \frac{t^{\frac{n}{2k}}}{n(2\pi)^n} \int_{(T_1^*M)_p} Tr_{E_p}(\sigma(P)(\xi))^{-\frac{n}{2k}} dw(\xi)$$
$$= O\left(\frac{t^{\frac{n-1}{2k}}}{l(p)}\right)$$

for $p \in Y$ with O independent of p. On the other hand we have from (3.15) and (δ)

$$Tr_F \delta^* e_t(p) = O(t^{\frac{n}{2k}})$$

for $p \in Y$, again with O independent of p. Now we extend the metric on M to some metric in the tubular neighbourhood U(M), which can be pulled back under φ_{σ_j} to a metric on $|\sigma_j|$ by 5. 2, 2), and this metric is equivalent to the Euclidean metric on $|\sigma_j|$, $1 \leq j \leq t_o$. Denoting by $l_j(p)$ the boundary distance of $p \in \varphi(|\sigma_j|)$ in $\varphi(|\sigma_j|)$ we clearly have

$$l(p) \geq l_j(p), \; 1 \leq j \leq t_o.$$

Now we can apply the method in [18] , §6 to derive the estimates

$$\int_{\varphi(|\sigma_j|)} \left(Tr_F \delta^* e_t(p) - \frac{t^{\frac{n}{2k}}}{n(2\pi)^n} \int_{(T_1^*M)_p} Tr_{E_p}(\sigma(P)(\xi))^{-\frac{n}{2k}} dw(\xi)\right) dM(p)$$
$$= O(t^{\frac{n-1}{2k}} \log t)$$

for $1 \leq j \leq t_o$. Because of (3.9) the theorem is proved by summing over j.

6. APPLICATION TO THE SPECTRUM OF $\Gamma \backslash G$

In this section we give an application of our results. We consider a semisimple connected Lie group G with finite center Z(G), a maximal compact subgroup K, and a discrete cocompact subgroup Γ. Then $M := \Gamma \backslash G$ is a compact manifold. If $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ is a Cartan decomposition of the Lie algebra of G with $\mathfrak{k}$ the Lie algebra of K, and if B denotes the Killing form on $\mathfrak{g}$, then a left invariant metric $\langle \cdot | \cdot \rangle$ on G is defined by the conditions $\mathfrak{k} \perp \mathfrak{p}$, $\langle X | X \rangle := B(X,X)$ for $X \in \mathfrak{p}$, and $\langle X | X \rangle := -B(X,X)$ for $X \in \mathfrak{k}$. Thus we get a Riemannian metric on M by requiring the projection $\pi_\Gamma : G \longrightarrow \Gamma \backslash G$ to be a Riemannian submersion. Since Ad k preserves the above Cartan decomposition and the Killing form for $k \in K$, we see that K acts by isometries from the right on G and on M. Next we consider the Casimir element C in the universal enveloping algebra of $\mathfrak{g}$, defined by

$$C = \sum_{i=1}^{r+s} X_i X^i,$$

where $r = \dim \mathfrak{p}$, $s = \dim \mathfrak{k}$, and $(X_i)_{i=1}^{r+s}$ is a basis of $\mathfrak{g}$ with dual basis $(X^i)_{i=1}^{r+s}$. With C we associate the second order differential operator $\tilde{C}$ on $C^\infty(G)$ defined by

$$\tilde{C} = \sum_{i=1}^{r+s} \tilde{X}_i \tilde{X}^i,$$

where $\tilde{X}f(g) := \frac{d}{dt}\Big|_{t=0} f(g \exp tX)$ for $X \in \mathfrak{g}$, $f \in C^\infty(G)$, and $g \in G$. Obviously $\tilde{C}$ is biinvariant; thus by

$$C_\Gamma f \circ \pi_\Gamma := \tilde{C}(f \circ \pi_\Gamma), \quad f \in C^\infty(M),$$

we define a second order differential operator on $C^\infty(M)$ commuting with the action of K. Choosing bases $(X_i)_{i=1}^r$ and $(Y_j)_{j=1}^s$ for $\mathfrak{p}$ and $\mathfrak{k}$ satisfying $B(X_i,X_i) = 1$, $B(Y_j,Y_j) =$

-1, $1 \leq i \leq r$, $1 \leq j \leq s$, we can write for $f \in C^\infty(M)$, $p \in M$

$$C_\Gamma f(p) = \sum_{i=1}^{r} \pi_{\Gamma *}(\tilde{X}_i)_p^2(f) - \sum_{j=1}^{s} \pi_{\Gamma *}(\tilde{Y}_j)_p^2(f).$$

This implies

(6.1) $$\sigma(C_\Gamma)(\xi) = -|\xi|^2, \quad \xi \in T_K^* M,$$

which means that $-C_\Gamma$ is strongly transversally elliptic. Moreover $-C_\Gamma$ is certainly symmetric on $C^\infty(M)$. Now we have by the unimodularity of G a unitary representation R of G in $L^2(M)$ given by

$$R_g f(\Gamma g') := f(\Gamma g' g), \quad f \in C(M), \; g, g' \in G.$$

Using the terminology of [3o], 4.4 one finds that $L^2(M)_\infty = C^\infty(M)$ and that

(6.2) $$C_\Gamma = R_\infty(C).$$

Therefore, it follows from [3o], p. 269 that $-C_\Gamma$ is essentially selfadjoint and we can apply Cor. 3.2. To do so let ρ be a unitary irreducible representation of K in V and denote by $\nu_\rho(\lambda)$ the multiplicity of ρ in the eigenspace C^λ of $-C_\Gamma$ with eigenvalue λ. With

$$m := \dim {}^{M_o}/_K = \dim {}^{G}/_K, \; l := \dim \mathrm{Hom}_{K_p}(V, \mathbb{C}), \; p \in M_o,$$

we then find by (6.1), 3.2 and 3.5

(6.3) $$N_\rho(t) = \sum_{\substack{\lambda \in \mathrm{Spec}(-C_\Gamma) \\ \lambda \leq t}} \nu_\rho(\lambda) \sim \frac{\omega_m}{(2\pi)^m} \; l \; \mathrm{vol} \; {}^{M_o}/_K \; t^{\frac{m}{2}}.$$

Putting

$$N := \{ g \in G \mid g x K = x K \quad \text{for all } x \in G \}$$

it is not hard to see that

$$\mathrm{vol}\ {}^{M_o}/_K = \frac{\mathrm{vol}\ M\ \ |\Gamma \cap N|}{\mathrm{vol}\ K}.$$

Moreover we have

$$1 = \dim V^{\Gamma \cap N}.$$

To see this we note that $K_{\Gamma g} = g^{-1}\,\Gamma_{gK}\,g$, $g \in G$.
Since Γ acts by isometries on ${}^G/_K$ and is countable, we have $\Gamma_{gK} = \Gamma \cap N$ whenever gK lies on a principal Γ-orbit. But every $g \in G$ can be written $g = g_2 g_1$ where g_1 belongs to the compact factors and hence to K, and g_2 belongs to the noncompact factors and hence commutes with N. If G has no compact factors we have $\Gamma \cap N = \Gamma \cap Z(G) =: Z(\Gamma)$ and therefore $\dim V^{Z(\Gamma)} = 0$ or $= \dim V$. Thus ϱ occurs in $L^2(M)$ iff $\varrho \,|\, Z(\Gamma) = 1$.
Now N_ϱ can be expressed in terms of representations of G and K. Obviously C_Γ commutes with the representation R of G in $L^2(M)$. Therefore, each eigenspace C^λ is G-invariant, $\lambda \in \mathrm{Spec}(-C_\Gamma)$. Using the arguments in [16], 2.3 we then find that C^λ splits into a direct sum of G-irreducible subspaces, say

$$C^\lambda = \bigoplus_{\alpha \in A_\lambda} H^{\alpha,\lambda}.$$

If we denote the representation of G in $H^{\alpha,\lambda}$ by $U^{\alpha,\lambda}$ and if $(U^{\alpha,\lambda}, H^{\alpha,\lambda})$ belongs to $\omega \in \hat{G}$ it follows from (6.2) that

$$U_\infty^{\alpha,\lambda}(C) = C_\Gamma \,\big|\, H^{\alpha,\lambda} = \lambda\ \mathrm{id}_{H^{\alpha,\lambda}}$$

and therefore

λ = value of the Casimir element on all representations of the class ω which is denoted by λ_ω .

Thus letting $n_\Gamma(\omega)$:= multiplicity of ω in $L^2(M)$ and $[\omega|K:\varrho]$:= multiplicity of ϱ in $\omega|K$ we find

$$\nu_\varrho(\lambda) = \dim \operatorname{Hom}_K(V, C^\lambda) = \sum_{\alpha \in \mathcal{A}_\lambda} \dim \operatorname{Hom}_K(V, H^{\alpha,\lambda})$$

$$= \sum_{\substack{\omega \in \hat{G} \\ \lambda_\omega = \lambda}} n_\Gamma(\omega) \; [\omega|K:\varrho]$$

and

$$N_\varrho(t) = \sum_{\substack{\lambda_\omega \leq t \\ \omega \in \hat{G}}} n_\Gamma(\omega) \; [\omega|K:\varrho],$$

In particular, $n_\Gamma(\omega) < \infty$ for $\omega \in \hat{G}$. Summing up we have proved the formula

(6.4) $$\sum_{\substack{\lambda_\omega \leq t \\ \omega \in \hat{G}}} n_\Gamma(\omega) \; [\omega|K:\varrho] \sim \frac{\omega_m}{(2\pi)^m} \dim V^{\Gamma \cap N} \; \frac{\operatorname{vol} \Gamma\backslash G \; |\Gamma \cap N|}{\operatorname{vol} K} \; t^{\frac{m}{2}} .$$

(6.4) has been conjectured by Gelfand [15], proved by Gangolli for G complex [14], and by Wallach [29] in general (up to a minor correction which is necessary if G has nontrivial compact factors). Using the existence of a torsionfree subgroup of finite index in Γ (as done in [29] to obtain (6.4)) we now proceed to estimate the remainder in (6.4) by means of theorem 5.1.

<u>Theorem 6.1:</u> Let G,Γ,K,V, and ϱ be as above. Then we have as $t \longrightarrow \infty$

$$\sum_{\substack{\omega \in \hat{G} \\ \lambda_\omega \leq t}} n_\Gamma(\omega) \; [\omega|K:\varrho] =$$

$$\frac{\omega_m}{(2\pi)^m} \dim V^{\Gamma \cap N} \; \frac{\operatorname{vol} \Gamma\backslash G \cdot |\Gamma \cap N|}{\operatorname{vol} K} \, t^{\frac{m}{2}} + O(t^{\frac{m-1}{2}} \log t).$$

<u>Proof:</u> Let Γ_o be a torsionfree normal subgroup of Γ (which

exists by [6]) with finite index and put $H := \Gamma/\Gamma_o$. Denote by $\pi_{\Gamma_o} : G \longrightarrow \Gamma_o \backslash G$ the canonical projection and define a metric on $\Gamma_o\backslash G$ as above. Moreover let C_{Γ_o} be the Casimir operator corresponding to Γ_o. As before we see that $-C_{\Gamma_o}$ is a strongly transversally elliptic differential operator of order 2 on $C^\infty(M)$ commuting with the action of K, which is symmetric and essentially selfadjoint in $L^2(M), M := \Gamma_o \backslash G$. Applying 1.3 and 2.4 to the K-vector bundle $M \times V^*$ and the selfadjoint operator $(-C_{\Gamma_o})^* \otimes id_{V^*}$ in $L^2(M \times V^*)$ we obtain by our basic construction a Hermitian vector bundle F over $^{M_o}/_K$ and a strongly elliptic operator P on $C^\infty(F)$ with principal symbol

$$\sigma(P)(\xi) = |\xi|^2 \, id_{\pi^* F_\xi} \quad , \quad \xi \in T^* \, {}^{M_o}/_K .$$

Moreover P is symmetric in $L^2(F,h)$ with domain $C^\infty(F)$ where $h(q) = \mathrm{vol}\, \pi_K^{-1}(q)$, $q \in {}^{M_o}/_K$. Now since Γ_o is torsion-free K acts freely on M and $^{M_o}/_K = {}^{M}/_K = \Gamma_o \backslash G/K$ is a compact manifold. Denoting by g_o the Riemannian structure on $^M/_K$ induced by π_K we define a new metric by $g_q := h(q) g_{o,q}$, $q \in {}^M/_K$. We then write $L^2(F) := L^2(F,h)$ and we call R the closure of P in $L^2(F)$, Now observe that the finite group H acts on M and also on $^M/_K$ by

$$\gamma\Gamma_o(\Gamma_o g\, K) := \Gamma_o \gamma g\, K.$$

This is in fact an isometric action since h and g_o are H - invariant. Then F becomes an H-vector bundle with trivial action on the fibers. In the same way H acts isometrically on M and on the bundle $M \times V^*$, and the isomorphism of 1.3 $\Phi : L^2(M \times V^*)^K \longrightarrow L^2(F)$ is easily seen to be H-equivariant since K acts from the right. This implies that P and its closure R commute with the action of H. As before we may assume that R is positive so that the conditions (α) to (δ) of theorem 5.1 are satisfied in view of (6.1). Thus denoting by S the restriction of R to $L^2(F)^H$ (see 2.2) and by N_S the counting function of Spec S we find as $t \to \infty$

(6.5) $$N_S(t) = \text{const } t^{\frac{m}{2}} + O(t^{\frac{m-1}{2}} \log t)$$

where $m := \dim {}^{M}/_{K}$. To complete the proof we turn to $L^2(\Gamma\backslash G \times V^*)$. We denote by R_1 the closure of $-C_\Gamma \otimes id_{V^*}$ in that space and by S_1 the restriction of R_1 to $L^2(\Gamma\backslash G \times V^*)^K$. Then by 2.2 we have

(6.6) $$N_\varrho = N_{S_1},$$

N_{S_1} being the counting function of Spec S_1. Introduce $\sigma : \Gamma_o\backslash G \ni \Gamma_o g \longmapsto \Gamma g \in \Gamma\backslash G$ such that $\pi_\Gamma = \sigma \circ \pi_{\Gamma_o}$. Then the map

$$\Psi : L^2(\Gamma\backslash G \times V^*)^K \ni f \longmapsto \Phi(f \circ \sigma) \in L^2(F)^H$$

is readily seen to be an isometric isomorphism. Moreover for $f \in C^\infty(\Gamma\backslash G \times V^*)^K$ we have

$$S(\Psi f) = S(\Phi(f \circ \sigma)) = \Phi((-C_{\Gamma_o}) \otimes id_{V^*}(f \circ \sigma))$$
$$= \Phi(((-C_\Gamma) \otimes id_{V^*} f) \circ \sigma) = \Psi(S_1 f).$$

Therefore, $S_1 = \Psi^{-1} \circ S \circ \Psi$ and $N_S = N_{S_1}$. The theorem follows from (6.4), (6.5) and (6.6).

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