INVARIANT EIGENFUNCTIONS OF THE LAPLACIAN AND THEIR ASYMPTOTIC DISTRIBUTION

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1. Introduction

This is a description of recent work most of which has been done in collaboration with Ernst Heintze (theorems 1, 3, 4, 7). To keep the exposition clear the results will not be stated in full generality but only for the Laplacian. Also most proofs will only be sketched. The full details can either be found in [5] or will appear elsewhere.

We consider a Riemannian manifold M with boundary ∂M, such that $M \cup \partial M$ is compact, and its Laplacian Δ which in local coordinates $x_1, \ldots, x_n$ is given by

$$\Delta = \frac{1}{\sqrt{g}} \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i} (\sqrt{g}\, g^{ij} \frac{\partial}{\partial x_j})$$

where (g_{ij}) is the metric in these coordinates, g its determinant, and (g^{ij}) its inverse. Then $-\Delta$ is positive with domain $C_o^\infty(M)$ and therefore has selfadjoint extensions; if $\partial M = \emptyset$ there is only one such extension and if $\partial M \neq \emptyset$ we consider those generated by Dirichlet or Neumann boundary conditions. Denoting any of these extensions also by $-\Delta$ it is well known that it has eigenvalues $0 \leq \lambda_1 < \lambda_2 < \ldots < \lambda_n \to \infty$ with eigenspaces Δ_λ such that $\dim \Delta_\lambda < \infty$ and

$$L^2(M) = \bigoplus_{\lambda \geq o} \Delta_\lambda .$$

1910 H. A. Lorentz conjectured that the asymptotic behavior of the eigenvalues depends only on the Riemannian volume of M. Introducing

$$N(t) := \sum_{\lambda \leq t} \dim \Delta_\lambda$$

and putting $n := \dim M$, $\omega_n :=$ volume of the unit ball in $\mathbb{R}^n$, H. Weyl [20] proved in 1911 the following celebrated formula (stated and proved for manifolds explicitely in [15]).

Theorem

$$N(t) \sim \frac{\omega_n}{(2\pi)^n} \operatorname{vol} M \, t^{n/2}.$$

We now assume that a compact Lie group G acts on M by isometries. Then G is unitarily represented in $L^2(M)$ by defining

$$U_g f(p) := f(g^{-1}(p)), \quad g \in G, \quad f \in L^2(M), \quad p \in M.$$

We are interested in the maximal number $N_1(t)$ of linearly independent G-invariant eigenfunctions of Δ with eigenvalue $\leq t$. Denoting the G-invariant elements in a G-invariant subspace H of $L^2(M)$ by H^G we have

$$N_1(t) = \sum_{\lambda \leq t} \dim \Delta_\lambda^G .$$

Thus if G is trivial N_1 reduces to the function N considered before. We think of N_1 as measuring the asymptotic distribution of the trivial representation in the splitting of $L^2(M)$ into G-irreducible subspaces. Therefore, in generalization of Weyl's theorem we analyze the asymptotic behavior of N_1 and also of its Laplace transform in what follows. In section 2 we relate the analysis of N_1 with a differential operator on a dense submanifold of the orbit space. We use this in section 3 to derive the asymptotic behavior of N_1 including remainder estimates for finite group actions. Section 4 describes an asymptotic expansion for the Laplace transform of N_1.

2. Separation of variables

If M is homogeneous then clearly $N_1(t) \equiv 1$. Therefore, we assume from now on that G does not act transitively on M. Our first task will be to show that the asymptotic analysis of invariant eigenfunctions is equivalent to the spectral asymptotics of a singular elliptic operator. This operator is obtained by eliminating those variables which are inessential in dealing with G-invariant functions i. e. by passing to the orbit space. To do so we remark first of all that the operator S in $L^2(M)^G$ defined by

$$\mathcal{D}(s) := \mathcal{D}(\Delta) \cap L^2(M)^G, \quad Sf := -\Delta f \quad \text{for} \quad f \in \mathcal{D}(S),$$

where $\mathcal{D}$ denotes the domain, is selfadjoint and positive. Moreover its spectrum consists of eigenvalues only and if S_λ denotes the eigenspace with eigenvalue λ we have $S_\lambda = \Delta_\lambda^G$ which implies

$$N_1(t) = \sum_{\lambda \le t} \dim S_\lambda .$$

These assertions are easily derived from the explicit description of the orthogonal projection $Q : L^2(M) \to L^2(M)^G$ namely

$$Qf = \frac{1}{|G|} \int_G U_g f \, dG(g), \; f \in L^2(M),$$

where dG denotes Haar measure and $|G|$ the corresponding volume of G. To relate S to an operator on the orbit space seems to be difficult since the orbit space is not a manifold in general. However, if we denote by M_0 the union of principal orbits of the G-action then it is well known that M_0 is an open dense subset of M with $\mathrm{vol}\, M_0 = \mathrm{vol}\, M$ and that M_0/G is a manifold, too ([4], Ch. IV: 3.1, 3.3, and 3.8). Also $\dim M_0/G \ge 1$ by our assumption that G does not act transitively on M. Since we are working in a L^2-framework M_0/G is indeed a good substitute for the orbit space. Thus we consider the map

$$(2.1) \qquad \Phi : C_0^\infty(M_0/G) \ni f \to f \circ \pi \in C^\infty(M)^G$$

where $\pi : M \to M/G$ is the orbit map. We want Φ to extend to an isomorphism between some Hilbert space of functions on M_0/G and $L^2(M)^G$. The Fubini theorem for Riemannian submersions ([3], p. 16) then shows that a natural candidate is the space $L^2(M_0/G, h)$ which we define as follows: we provide M_0/G with a Riemannian metric such that π becomes a Riemannian submersion. Denoting by $d^{M_0/G}$ the volume element of this metric and putting $h(q) := \mathrm{vol}\, \pi^{-1}(\{q\})$, $q \in M_0/G$, $L^2(M_0/G, h)$ is the completion of $C_0^\infty(M_0/G)$ with respect to the norm

$$f \to \Big(\int_{M_0/G} |f(q)|^2 \, h(q) \, d^{M_0/G}(q) \Big)^{1/2}.$$

It is then easy to see that Φ in fact extends to an isomorphism of Hilbert spaces $L^2(M_0/G, h) \to L^2(M)^G$ which we also denote by Φ. Thus we are lead to study the operator

$$T := \Phi^{-1} \circ S \circ \Phi$$

which is positive and selfadjoint with domain $\Phi^{-1}(\mathcal{D}(S))$ in $L^2(M_o/G)$. Its spectrum consists of eigenvalues only and if T_λ denotes the eigenspace with eigenvalue λ we have

$$(2.2) \qquad N_1(t) = \sum_{\lambda \le t} \dim T_\lambda .$$

Concerning the structure of T we have the following crucial result.

<u>Theorem 1</u> With Δ' the Laplacian on M_o/G we have for $f \in C_o^\infty(M_o/G)$

$$(2.3) \qquad Tf = -\Delta' f - \nabla \log h \, (f).$$

<u>Proof</u> Since $P := T \mid C_o^\infty(M_o/G)$ does not increase supports it follows from Peetre's theorem ([16], p. 154) that P is a differential operator. Next, a straightforward computation using (2.1) shows that P and $-\Delta'$ have the same principal symbol. Thus the operator X defined by

$$Xf := Pf + h^{-1/2}\Delta' \, (h^{1/2} f), \; f \in C_o^\infty(M_o/G),$$

is a first order differential operator. Moreover, Xf is real if f is and X is symmetric in $L^2(M_o/G, h)$ with domain $C_o^\infty(M_o/G)$. These facts imply that X is multiplication by some function which is easily found to be $h^{-1/2}\Delta' \, (h^{1/2})$. It follows

$$(2.4) \qquad \begin{aligned} Pf &= -h^{-1/2}\Delta' \, (h^{1/2} f) + h^{-1/2}\Delta' \, (h^{1/2}) \, f \\ &= -\Delta' f - \nabla \log h \, (f). \end{aligned}$$

Thus we see that the asymptotic distribution of the invariant eigenfunctions coincides with the eigenvalue distribution of a certain selfadjoint extension of the singular second order elliptic operator P on M_o/G which may also be a singular manifold. Because of these singularities we do not have a simpler problem than before in general. The only insight we gain is the local structure of T. In fact it is more likely that we obtain new information on P and T - quite contrary to the effect of the classical method of separation of variables. We try to illustrate these remarks by means of two examples.

<u>Example 1</u> Surfaces of rotation

Let M be a surface of rotation in $\mathbb{R}^3$ obtained by rotating the C^∞-curve $c : [o,L] \ni s \to (c_1(s), o, c_2(s)) \in \mathbb{R}^3$ around the x_3 achsis. Assuming that c is parametrized by arc length we find that T is unitarily equivalent to a selfadjoint extension of the operator $\tilde{P}$ in $L^2((o,L))$ with domain $C_o^\infty((o,L))$ defined by

$$\tilde{P}f(x) := -f''(x) + \frac{1}{4c_1(x)^2}\,(2c_1(x)c_1''(x) - c_1'(x)^2)\, f(x).$$

It should be noted that the potential has a rather restricted singularity since it can always be written in the form

$$\frac{a}{x^2} + \frac{b}{(1-x)^2} + c(x)$$

where $c \in C([o,L])$. By theorem 3 below we obtain in this way the eigenvalue distribution of several classical singular equations like the Legendre equation which we get for $M = S^2$.

Example 2 t'Hooft's operator

G. t'Hooft [12] introduced the following operator: let $a_1, a_2 > -1$, $x \in (0,1)$, $u \in C_o^\infty((0,1))$, and put

$$H_{a_1,a_2}u(x) := \frac{1}{2}\int_{-\infty}^{+\infty} e^{ix\xi}|\xi|\hat{u}(\xi)d\xi + \left(\frac{a_1}{x} + \frac{a_2}{1-x}\right) u(x).$$

He conjectured that the Friedrichs extension of this operator has a pure point spectrum and that the eigenvalues μ_n obey the following asymptotic law:

$$(2.5) \qquad \mu_n = \pi^2 n + (a_1 + a_2)\log n + O(1),\ n \to \infty.$$

Hildebrandt [9] proved the assertion on the nature of the spectrum and Hildebrandt and Višnjič [10] gave the following bounds for the eigenvalues:

$$(2.6) \qquad \mu_n \le \left(1 - \frac{2\varepsilon}{\pi}\right)^{-1}(n-1)\pi^2 + 2\gamma\pi^2\,\varepsilon^{-1}$$

for $0 < \varepsilon < \frac{\pi}{2}$, where $\gamma := \max\{a_1 + 1, a_2 + 1\}$, and

$$(2.7) \qquad \mu_n \ge \begin{cases} 2(n-1) + (\sqrt{a_1+1} + \sqrt{a_2+1})^2, \\[2ex] 2\pi\left[\frac{n}{2}\right] + (\sqrt{a_1} + \sqrt{a_2})^2 \text{ if } a_1 > 0,\ a_2 > 0. \end{cases}$$

Now some additional information on the eigenvalues of H_{a_1,a_2} can be gained by fitting it into our framework. Apparently the above lower estimate is not sufficient to give the first term in t'Hooft's formula (2.5). To improve the lower estimate we consider the following situation:

on the manifold (S^1, g), where g is $\frac{1}{\pi}$ times the standard metric, we have a $\mathbb{Z}_2$ action given by the reflection σ in a fixed equator. We introduce the space $L^2(M)_{-1} := \{f \in L^2(M) \mid f \circ \sigma = -f\}$ and restrict the operator $\pi(-\Delta)^{1/2}$ to $L^2(M)_{-1}$. Then a fundamental domain for the $\mathbb{Z}_2$ action can be identified with $(0,1)$ and as above we obtain a self-adjoint operator T_{-1} on $L^2((0,1))$ with domain $H_0^1((0,1))$. This is a natural object to compare H_{a_1, a_2} with since by a theorem of Seeley [17] $\pi(-\Delta)^{1/2}$ is a pseudodifferential operator on M with symbol $\pi|\xi|$. In fact, a computation using the calculus of pseudodifferential operators leads to the inequality

$$(T_{-1}u|u) + (Cu|u) \leq (H_{1,1}u|u)$$

for $u \in C_0^\infty((0,1))$, where $(\cdot|\cdot)$ denotes the $L^2(M)$ scalar product and C is some bounded operator in $L^2(M)$. The max-min principle combined with theorem 3 below then gives a good lower estimate for the eigenvalues of H_{a_1, a_2} with $a_1, a_2 \geq 1$. With a little additional work along the same lines we obtain the following result.

<u>Theorem 2</u> Let $a_1, a_2 \geq 1$. Then

$$\mu_n \geq \pi^2 n - \text{const}, \; n \in \mathbb{N}.$$

Putting $\varepsilon := n^{-1/2}$ in (2.6) this gives

$$\mu_n = \pi^2 n + O(n^{1/2}), \; n \to \infty.$$

If $a_1, a_2 \geq 0$ then still

$$\mu_n \sim \pi^2 n, \; n \to \infty.$$

To conclude this section we remark that theorem 1 holds in much greater generality. Neither we have to restrict to the trivial representation nor to the Laplacian (in example 2 already another operator and another representation occured). The general approach is given in [5] without the explicit formula (2.3), however. The first equality in (2.4) formally coincides with Helgason's formula for the radial part of the Laplacian ([8], theorem 2.11). This is in fact no coincidence since a slight modification of the above L^2 approach yields his result, too, though he is dealing with noncompact groups also.

3. The asymptotic behavior of N_1

We now propose to determine the asymptotic behavior of N_1. Since we have the relation (2.2) we try a similar procedure as in the nonsingular case and study the Laplace transform L_1 of N_1,

$$(3.1) \qquad L_1(s) = s \int_0^\infty e^{-st} N_1(t)\, dt .$$

By a well known Tauberian argument the asymptotic behavior of N_1 as $t \to \infty$ will follow from the asymptotic behavior of L_1 as $s \to o$. As one might expect $L_1(s)$ turns out to be the trace of the operator e^{-sT} which is smoothing in view of the ellipticity of T and therefore has a C^∞ kernel denoted by Γ_s^1. We have

$$(3.2) \qquad L_1(s) = \int_{M_o/G} \Gamma_s^1 (q,q)\, h(q)\, d\, {}^{M_o}/_G (q).$$

Further, the ellipticity of T allows us to apply a theorem of Hörmander ([11], theorem 5.1) which implies the pointwise asymptotic behavior of the integrand in (3.2) as $s \to o$, namely

$$\Gamma_s^1 (q,q)\, h(q) \sim \frac{\omega_m}{(2\pi)^m}\, \Gamma\left(\frac{m}{2} + 1\right) s^{-m/2},$$

where Γ denotes the gamma function and $m := \dim {}^{M_o}/_G$. From this one conjectures the following result.

<u>Theorem 3</u> As $s \to o$

$$L_1(s) \sim \frac{\omega_m}{(2\pi)^m}\, \Gamma\left(\frac{m}{2} + 1\right) \operatorname{vol} {}^{M_o}/_G \; s^{-m/2}$$

and therefore as $t \to \infty$

$$N_1(t) \sim \frac{\omega_m}{(2\pi)^m} \operatorname{vol} {}^{M_o}/_G \; t^{m/2}.$$

The result as stated has been proved by Huber [13] for certain and by Wallach [19] for all finite groups. Donnelly [6] proved it for general compact groups and in [5] the analogous result for arbitrary representations and G-invariant elliptic operators is given. To prove the theorem we want to use the Lebesgue-Fatou lemma in (3.2). Thus we have to establish the following facts:

(3.3) $\quad \operatorname{vol} M_o/G < \infty$

and

(3.4) $\quad s^{m/2}\Gamma_s^1(q,q)\, h(q) = O(1) \quad$ for $\quad s > o,\ q \in M_o/G$.

The proof of (3.3) uses the geometry of the G-action and is given in [5]. For the proof of (3.4) we link Γ_s^1 with Γ_s which is by definition the kernel of $e^{s\Delta}$ (the "heat kernel" of M). Using the definition of T one easily shows that for $p \in M_o$

(3.5) $\quad \Gamma_s^1(\pi(p), \pi(p)) = \frac{1}{|G|} \int_G \Gamma_s(g(p),p)\, dG(g).$

To obtain an estimate for Γ_s^1 we now recall the following well known inequality

$$|\Gamma_s(p,p')| \leq C\, s^{-n/2}\, e^{-C'\, d_M^2(p,p')/s}$$

for $s > o$, $p, p' \in M$ and certain constants $C, C' > o$, where d_M denotes Riemannian distance on M. Thus the proof of theorem 3 is completed by applying the following result.

<u>Theorem 4</u> There is a constant $C > o$ such that for $p \in M$ and $s > o$

$$\frac{\operatorname{vol} Gp}{|G|} \int_G e^{-d_M^2(g(p),p)/s} dG(g) \leq C\, s^{-\dim Gp/2}$$

where Gp is the G-orbit of p.

The proof of theorem 4 uses again the geometry of the G-action and proceeds by induction on dim M; it is given in [5], sec. 4.

Having generalized Weyl's result it is natural to ask for remainder estimates. To see what we can expect we collect some results in the classical case first. Putting

$$R(t) := N(t) - \frac{\omega_n}{(2\pi)^n} \operatorname{vol} M\, t^{n/2}$$

the following estimates are known.

1. Let $\partial M = \emptyset$. Then

a) $R(t) = O(t^{n-1/2})$ (Avakumovič [1]),

b) $R(t) = o(t^{n-1/2})$ if there are "not too many" closed geodesics on M of some given length (Duistermaat and Guillemin [7]),

c) $R(t) = O(\frac{t^{n-1/2}}{\log t})$ if the curvature of M is nonpositive (Bêrard [2]).

2. Let $\partial M \neq \emptyset$. Then

a) $R(t) = O(t^{n-1/2})$ (Seeley [18]),

b) it is conjectured that generically

$$(3.6) \qquad R(t) = \mp \frac{\omega_{n-1}}{2^{n+1}\pi^{n-1}} \operatorname{vol} \partial M \, t^{n-1/2} + o(t^{n-1/2})$$

where we have - or + according to Dirichlet or Neumann boundary conditions. This has been proved in special cases where the eigenfunctions can be obtained by separation of variables. Also we have been told that R. Melrose proved (3.6) for certain manifolds with a geodesically concave boundary.

We see that though the estimate 1a) is sharp for spheres it can be improved depending on properties of the geodesic flow on M. On the other hand there is no such improvement for 2a) as can be seen from the asymptotic expansion of the trace of the heat kernel given in [14]:

$$(3.7) \qquad L(s) := \operatorname{tr} e^{s\Delta} = s\int_0^\infty e^{-st} N(t)\, dt = (4\pi s)^{-n/2}$$

$$(\operatorname{vol} M \mp \frac{\sqrt{\pi}}{2} \operatorname{vol} \partial M \, s^{1/2} + O(s)).$$

This expansion also dictates the first term in (3.6). However, (3.6) is false for the hemisphere and therefore some extra condition is necessary for its validity. Turning to G-manifolds we should expect $O(t^{n-1/2})$ as best possible remainder estimate for N_1 since $M_0/_G$ will have boundary points in $M/_G$ in general. Up to now we can present such a result only in case the group G is finite.

<u>Theorem 5</u> Let G be finite. Then we have as $t \to \infty$

$$N_1(t) = \frac{\omega_n}{(2\pi)^n} \frac{\operatorname{vol} M}{|G|} t^{n/2} + O(t^{n-1/2}).$$

This theorem improves on the remainder estimate given in [5], sec. 5 in case of the Laplacian but it does not extend to operators on vector bundles. In the proof we use the spectral function e_t of $-\Delta$ which is by definition the kernel of the orthogonal projection in $L^2(M)$ onto the space $\bigoplus_{\lambda \le t} \Delta_\lambda$. We have

$$N(t) = \int_M e_t(p,p)\, dM\,(p)$$

and (compare (3.5))

$$(3.8) \qquad N_1(t) = \frac{1}{|G|} \sum_{g \in G} \int_M e_t\,(g(p),p)\, dM\,(p).$$

To evaluate this formula we use again Hörmander's work [11]. He has shown that for every coordinate system $\varphi\colon U \to M$, $U \subset \mathbb{R}^n$, such that the induced measure on U coincides with Lebesgue measure, and $x,y \in U$ close to each other

$$(3.9) \qquad e_t(\varphi(x),\varphi(y)) = \frac{1}{(2\pi)^n} \int_{p(x,\xi)\le t} e^{i\psi(x,y,\xi)}\, d\xi + O\,(t^{n-1/2})$$

where $p(x,\xi)$ denotes the principal symbol of $-\Delta$ in the φ-coordinates and ψ is a suitably chosen phase function. Moreover, in compact subsets of the complement of the diagonal in $M \times M$ we have uniformly

$$(3.10) \qquad e_t(p,q) = O\,(t^{n-1/2}).$$

Thus the integration in (3.8) can be reduced to tubular neighborhoods of the fixed point set M^g of g which is a disjoint union of totally geodesic compact submanifolds of M. The result then follows by choosing suitable coordinates and applying the method of stationary phase. The same argument shows that an estimate similar to (3.6) above can be obtained once we can improve on (3.9) and (3.10).

<u>Theorem 6</u> Suppose that (3.9) and (3.10) hold with O's replaced by o's in both cases and that G is finite. Then

$$N_1(t) = \frac{\omega_n}{(2\pi)^n} \frac{\mathrm{vol}\, M}{|G|}\, t^{n/2} + \frac{\omega_{n-1}}{2^n \pi^{n-1}} \sum_{\substack{g \in G \\ N \text{ conn.comp.of } M^g \\ \text{codim } N=1}} \frac{\mathrm{vol}\, N}{|G|}\, t^{n-1/2} + o\,(t^{n-1/2}).$$

It is not clear at the moment which geometric properties of M imply the assumptions of theorem 6. However, the above mentioned paper of Duistermaat and Guillemin seems to indicate a connection with the geodesic flow on M.

For an application of the results of this section in the representation theory of semisimple Lie groups we refer the reader to [5], sec. 6.

4. The asymptotic expansion of L_1

As indicated in (3.7) the function L has an asymptotic expansion in powers of s as $s \to o$ ([15], [14]):

$$L(s) \sim (4\pi s)^{-n/2} \sum_{j=o}^{\infty} a_j \, s^{j/2} .$$

The coefficients a_j are interesting geometric invariants of M completely determined by the spectrum of Δ (cf. [14] for the computation of the first a_j's). This leads us to ask for an asymptotic expansion of the function L_1 defined by (3.1) since its coefficients can be expected to contain interesting geometric information on the orbit space. The solution of this problem naturally falls into two parts, namely to prove the existence of such an expansion and to calculate the coefficients, at least in principle. Donnelly has solved the whole problem for finite groups and the first part in case the G-action has no singular orbits in [6]. The following theorem gives an existence proof for connected G.

Theorem 7 Let G be connected. Then there is an asymptotic expansion as $s \to o$

$$L_1(s) \sim (4\pi s)^{-m/2} \left(a_o + \sum_{i=o}^{k} \sum_{j=1}^{\infty} a_{ij} \, (\log s)^i \, s^{j/2} \right).$$

Here k is a certain nonnegative integer and $m = \dim {}^{M_o}/_G$ as before.

The proof of the theorem is based on the formula

$$L_1(s) = \frac{1}{|G|} \int_G \int_M \Gamma_s(g(p),p) \, dM\,(p) \, dG\,(g)$$

resulting from (3.2), (3.5), and the Fubini theorem. We observe that the G-integrand is invariant under conjugation. By the Weyl integration formula we can therefore replace the G-integral by an integral over a maximal torus T of G. Now using the properties of torus actions (in particular the fact that there are only finitely many fixed point sets), and the asymptotic expansion of Γ_s in a neighborhood of the diagonal (cf. [15]) the above integral is reduced to a finite sum of integrals in "normal form". These integrals can then be shown to possess asymptotic expansions of the above type. The argument proceeds by induction on the number of fixed point sets of the T-action and requires a precise knowledge of the function d_M^2. However, the proof is rather involved and requires many technical details which cannot be given here. Also the formulas for the coefficients are very complicated; so far we have been unable to determine more than just the first one:

$$a_o = \mathrm{vol}\ {}^{M_o}/_G ,$$

thus establishing a new proof of theorem 3 for connected G. In particular it is not clear at the moment whether the logarithmic terms do really occur. If we had $a_{11} \neq o$ for some M and G, this would mean that theorem 5 does not generalize to arbitrary G.

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