manuscripta math. 39, 125 - 146 (1982)

manuscripta mathematica

ON THE EIGENVALUE PROBLEM OF 'T HOOFT

Jochen Brüning

We determine the asymptotic distribution of the eigenvalues in 't Hoofts eigenvalue problem in two-dimensional quantum chromodynamics. We formulate the problem as an eigenvalue problem for a singular pseudodifferential operator and use systematically its basic invariance properties.

1. Introduction

In 1974 't Hooft [7] proposed a model for the constitution of mesons as bound states of a quark-antiquark pair. He derived the eigenvalue equation.

$$(1)\quad \lambda u(x) = -\,\text{p.v.}\int_0^1 \frac{u(y)}{(x-y)^2}\,dy + \left(\frac{a}{x} + \frac{b}{1-x}\right) u(x),\quad 0<x<1,$$

where $a,b > -1$ and u satisfies the boundary conditions

$$(2)\quad u(0) = u(1) = 0.$$

Moreover, the principal value of the integral is given by

$$\lim_{\varepsilon\to 0} \frac{1}{2}\int_0^1 u(y)\left((x-y+i\varepsilon)^{-2} + (x-y-i\varepsilon)^{-2}\right) dy.$$

't Hooft conjectured that (1) and (2) define an operator with pure point spectrum and that the eigenvalues λ_n have the following asymp-

0025-2611/82/0039/0125/$04.40

totic behavior:

(3) $\lambda_n = \pi^2 n + (a+b) \log n + O(1),\ n\to\infty$.

In a series of papers [4], [5], [6] Hildebrandt investigated 't Hoofts eigenvalue problem. He proved the assertion on the spectrum (this was also done independently in [3]) and gave lower and upper estimates for the eigenvalues. He also estimated the number of nodal domains of the eigenfunctions and showed that the eigenfunctions do satisfy (2). This latter result was extended by Lewy [8] to a full asymptotic expansion of the eigenfunctions at o and 1. It is the purpose of this note to establish the first order asymptotics expressed in (3). We shall prove the following.

THEOREM *Let* $a, b \geq -1$. *Then the symmetric operator in* $L^2([o,1])$ *with domain* $C_o^\infty((o,1))$ *defined by* (1) *is bounded below by* o. *Its Friedrichs extension* Q *has a pure point spectrum and the operator* e^{-sQ} *belongs to the trace class,* $s > o$. *Moreover, for its trace we have*

$$\operatorname{tr} e^{-sQ} = \frac{1}{\pi^2 s} + \frac{a+b}{\pi^2} \log s + O(1),\ s\to o .$$

By standard methods we deduce from this the asymptotic behavior of the eigenvalues.

COROLLARY *The eigenvalues* $(\lambda_n)_{n\in\mathbb{N}}$ *of* Q *satisfy*

$$\lambda_n \sim \pi^2 n ,\ n\to\infty .$$

If an asymptotic relation of the form

$$\lambda_n = \pi^2 n + \alpha \log n + o(\log n),\ n\to\infty ,$$

holds then we must have $\alpha = a + b$.

Our proof starts in showing that (1) defines actually a boundary value problem for a singular elliptic pseudodifferential operator of order 1. This operator has rather bad regularity properties (cf. Lemma 2 below) preventing the application of standard techniques. We observe, however, that the operator $\bar{Q}$ arising from (1) with b = o and domain $C_o^\infty(\mathbb{R}_*)$, $\mathbb{R}_* := (o,\infty)$, has a remarkable invariance property under the natural action of $\mathbb{R}_*$ on $L^2(\mathbb{R}_+)$, $\mathbb{R}_+ := [o,\infty)$ (Proposition 3). This allows localization away from the singular point and we can apply the calculus of (smooth) pseudodifferential operators to study the operator $e^{-s\bar{Q}}$. Finally, using the regularity result we show that $e^{-s\bar{Q}}$ is a good approximation to e^{-sQ} in a neighborhood of o, and by a simple reflection argument we can also deal with the singular point 1.

2. Regularity properties

We now rewrite the operator (1) as a pseudodifferential operator. To do so we put

$$Pu(x) := \frac{1}{2} \int_{-\infty}^{+\infty} e^{ix\xi} |\xi| \hat{u}(\xi)\, d\xi \ , \ u \in C_o^\infty(\mathbb{R}) \ , \ x \in \mathbb{R},$$

$\hat{u}$ denoting the Fourier transform as usual. P is a pseudodifferential operator of order 1 with symbol $\pi|\xi|$ and extends continuously to an operator $H^s(\mathbb{R}) \to H^{s-1}(\mathbb{R})$ for every real s. For the definition and basic properties of the Sobolev spaces H^s we refer to [9] Chapter 1. In particular we have

$$(4) \quad \| Pu \|_o^2 + \pi^2 \| u \|_o^2 = \frac{\pi}{2} \| u \|_1^2 \ , \ u \in H^1(\mathbb{R}).$$

PROPOSITION 1 *We have*

$$(5) \quad Pu(x) = -\frac{1}{2} \lim_{\varepsilon \to o} \int_{-\infty}^{+\infty} u(y)((x-y+i\varepsilon)^{-2} + (x-y-i\varepsilon)^{-2})\, dy$$

for $u \in C_o^\infty(\mathbb{R})$, $x \in \mathbb{R}$.

PROOF For $\varepsilon \neq o$ we get from Fourier's inversion formula

$$\int_{-\infty}^{+\infty} u(y)((x-y+i\varepsilon)^{-2} + (x-y-i\varepsilon)^{-2})\, dy$$

$$= \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{ix\xi}\, \hat{u}(\xi) \int_{-\infty}^{+\infty} e^{iz\xi}((z-i\varepsilon)^{-2} + (z+i\varepsilon)^{-2})\, dz\, d\xi$$

and from the residue theorem

$$\int_{-\infty}^{+\infty} \frac{e^{iz\xi}}{(z-i\varepsilon)^2}\, dz = \begin{cases} -2\pi|\xi|\, e^{-\varepsilon\xi} & , \varepsilon\xi \geq 0, \\ 0 , & \varepsilon\xi < 0. \end{cases}$$

This implies

$$\int_{-\infty}^{+\infty} u(y)((x-y+i\varepsilon)^{-2} + (x-y-i\varepsilon)^{-2})\, dy$$

$$= - \int_{-\infty}^{+\infty} e^{ix\xi}\, |\xi|\, e^{-\varepsilon|\xi|}\, \hat{u}(\xi)\, d\xi$$

hence the result.

Thus we are led to investigate the operator

$$Ku(x) := Pu(x) + \left(\frac{a}{x} + \frac{b}{1-x}\right) u(x),\ u \in C_0^\infty((0,1)),\ x \in (0,1),$$

where a, b ≥ -1. K defines a symmetric operator in $L^2([0,1])$ with domain $C_0^\infty((0,1))$. We will also consider the operator

$$\bar{K}u(x) := Pu(x) + \frac{a}{x} u(x)\ ,\ u \in C_0^\infty(\mathbb{R}_*)\ ,\ x \in \mathbb{R}_*\ ,$$

which is symmetric in $L^2(\mathbb{R}_+)$ with domain $C_0^\infty(\mathbb{R}_*)$. Now using (5), the symmetry of P, and the identity

$$-2\, \mathrm{Re}\, u(y)\, \overline{u(x)} = |u(x) - u(y)|^2 - |u(x)|^2 - |u(y)|^2$$

we get for $u \in C_o^\infty((o,1))$

$$(Ku|u) = \frac{1}{2} \int_0^1\int_0^1 \left|\frac{u(x)-u(y)}{x-y}\right|^2 dx\, dy + \int_0^1 \left(\frac{1+a}{x} + \frac{1+b}{1-x}\right) |u(x)|^2 dx$$

and similarily for $u \in C_o^\infty(\mathbb{R}_*)$

$$(\bar{K}u|u) = \frac{1}{2} \int_0^\infty\int_0^\infty \left|\frac{u(x)-u(y)}{x-y}\right|^2 dx\, dy + \int_0^\infty \frac{(1+a)}{x} |u(x)|^2 dx .$$

Consequently, K and $\bar{K}$ are positive operators and their Friedrich extensions exist; we will denote them by Q and $\bar{Q}$ respectively. Moreover, from the definition ([12], p. 317 f) and from [10], p. 81 we infer that

(6) $\mathcal{D}(Q) \subset H^{1/2}((o,1))$, $\mathcal{D}(\bar{Q}) \subset H^{1/2}(\mathbb{R}_*)$,

$\mathcal{D}$ denoting the domain. By the well known compactness properties of $H^{1/2}((o,1))$ ([9] Theorem 16.1) we obtain the following result already proved in [3] and [4].

PROPOSITION 2 *The spectrum of* Q *consists only of eigenvalues* $o \le \lambda_1 < \lambda_2 < .. < \lambda_n \to \infty$ *each having finite multiplicity.*

P possesses a useful factorization. To describe it we introduce the Hilbert transform

$$Hu(x) := \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{ix\xi} \operatorname{sgn} \xi \, \hat{u}(\xi)\, d\xi \, , \ u \in C_o^\infty(\mathbb{R}) \, , \ x \in \mathbb{R} ,$$

which is a pseudodifferential operator of order o with symbol sgn ξ, thus extends to a continuous linear map $H^s(\mathbb{R}) \to H^s(\mathbb{R})$ for $s \in \mathbb{R}$. Writing

$$D := - i\pi \frac{d}{dx}$$

we arrive at the factorization

(7) $Pu = HDu = DHu \, , \, u \in C_o^\infty(\mathbb{R})$.

Note that H is unitary and that $H^2 = \mathrm{id}$. We shall need the following property of commutators.

LEMMA 1 *Let* $r \geq o$, $o \leq s \leq \frac{1}{2}$, *and* $r+s > \frac{1}{2}$. *Let* $\psi \in H^r(\mathbb{R}) \cap L^\infty(\mathbb{R})$ *and denote by* M_ψ *the corresponding multiplication operator. Then the commutator* $HM_\psi - M_\psi H : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ *restricts to a continuous map* $H^s(\mathbb{R}) \to H^{s'}(\mathbb{R})$ *for every* $s' < r+s - \frac{1}{2}$.

PROOF We have for f, $g \in L^2(\mathbb{R})$ the identity $\widehat{fg} = (2\pi)^{-1} \hat{f} * \hat{g}$, $*$ denoting convolution as usual. Hence we have for $u \in H^s(\mathbb{R})$, $v \in C_o^\infty(\mathbb{R})$

$$((HM_\psi - M_\psi H)\, u|\bar{v}) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \hat{v}(\xi)\, (\mathrm{sgn}\, \xi\, \widehat{\psi u}(\xi) - \widehat{\psi H u}(\xi))\, d\xi$$

$$= \frac{1}{(2\pi)^2} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \hat{v}(\xi)\, \hat{u}(\eta)\, \hat{\psi}(\xi-\eta)(\mathrm{sgn}\,\xi - \mathrm{sgn}\,\eta)\, d\eta\, d\xi \ .$$

On the support of the integrand we have $|\xi-\eta| = |\xi| + |\eta|$. Thus we get for $o < 2\delta < r+s - \frac{1}{2}$ by Cauchy - Schwarz

$$|((HM_\psi - M_\psi H)\, u|\bar{v})|$$

$$\leq \frac{1}{(2\pi)^2} \int_{-\infty}^{+\infty} \frac{(1+|\eta|^2)^{s/2}\, |\hat{u}(\eta)|}{(1+|\eta|^2)^{1/4+\delta}} \int_{-\infty}^{+\infty} (1+|\xi-\eta|^2)^{r/2} |\hat{\psi}(\xi-\eta)| \cdot \frac{|\hat{v}(\xi)|}{(1+|\xi|^2)^{(r+s)/2-1/4-\delta}}\, d\xi\, d\eta$$

$$\leq C_\delta\, \|\psi\|_r\, \|u\|_s\, \|v\|_{-(r+s-1/2)+2\delta} \ .$$

Using the Riesz representation theorem together with the fact that the pseudodifferential operator given by

$$\Lambda_r u(x) := \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{ix\xi} (1+|\xi|^2)^{r/2}\, \hat{u}(\xi)\, d\xi \ , \ u \in C_o^\infty(\mathbb{R}) \ , \ x \in \mathbb{R} \ ,$$

extends to a continuous linear map $H^s(\mathbb{R}) \to H^{s-r}(\mathbb{R})$ we conclude that

$(HM_\psi - M_\psi H)\, u \in H^{r+s-1/2-2\delta}(\mathbb{R})$ and that

$$\| (HM_\psi - M_\psi H)\, u \|_{r+s-1/2-2\delta} \le C_\delta \|\psi\|_r \|u\|_s .$$

The factorization (7) enables us to extend K to a linear map sending $L^2([o,1])$ into $\mathcal{D}'((o,1))$, the space of distributions on $(o,1)$. Denoting by $E_1 : L^2([o,1]) \to L^2(\mathbb{R})$ the extension by o and by $R_1 : \mathcal{D}'(\mathbb{R}) \to \mathcal{D}'((o,1))$ the restriction we put for $u \in L^2([o,1])$

$$(8) \quad Ku := R_1 DHE_1 u + \left(\frac{a}{x} + \frac{b}{1-x}\right) u$$

where differentiation and multiplication with C^∞-functions is carried out in the distribution sense. It is easily checked that the adjoint operator K^* in $L^2([o,1])$ of $K|C_o^\infty((o,1))$ is given by

$$K^* = K|K^{-1}(L^2([o,1])) .$$

Similarily we extend $\bar{K}$ to a linear map $L^2(\mathbb{R}_+) \to \mathcal{D}'(\mathbb{R}_*)$ by

$$\bar{K}u := RDHEu + \frac{a}{x}u, \quad u \in L^2(\mathbb{R}_+) ,$$

where $E : L^2(\mathbb{R}_+) \to L^2(\mathbb{R})$ denotes extension by o and $R: \mathcal{D}'(\mathbb{R}) \to \mathcal{D}'(\mathbb{R}_*)$ the restriction. We also find

$$\bar{K}^* = \bar{K}|\bar{K}^{-1}(L^2(\mathbb{R}_+)) .$$

The basic estimate of the next section will be derived from the following regularity result.

LEMMA 2 *Let* $o \le s < \frac{1}{2}$.

1) *Suppose* $u, f \in H^s((o,1))$ *and* $Ku = f$. *Then*

$$x(1-x)\, E_1 u \in H^{s'}(\mathbb{R}) \quad \textit{for every } s' < 1+s$$

and

$$u \in H^{1+s}_{loc}((o,1)) .$$

2) *Suppose* $u, f \in H^s(\mathbb{R}_*)$ *and* $\bar{K}u = f$. *Then*

$$xEu \in H^{s'}_{loc}(\mathbb{R}) \text{ for every } s' < 1+s$$

and

$$u \in H^{1+s}_{loc}(\mathbb{R}_*).$$

PROOF We give the proof of 2) only since 1) is proved analogously. Given $\varphi \in C_o^\infty(\mathbb{R})$ with supp $\varphi \subset (-A,A)$ we put

$$\varphi_+(x) := \begin{cases} x\varphi(x), & x > o, \\ o, & x \le o . \end{cases}$$

By assumption we have

$$\text{(9)} \quad RD\varphi_+HEu = R\varphi_+DHEu + R(D\varphi_+)\,HEu$$
$$= \varphi_+f - a\mathring{\varphi}u + R(D\varphi_+)\,HEu .$$

Now $\varphi_+ \in H^r(\mathbb{R})$ for every $r < \frac{3}{2}$ if $o \in$ supp φ and $\varphi_+ \in C_o^\infty(\mathbb{R})$ otherwise. Moreover, if $g \in C^1(\mathbb{R}_*)$ with $\sup_{x>o} (|g(x)| + |g'(x)|) < \infty$ then $gv \in H^s(\mathbb{R}_*)$ whenever $v \in H^s(\mathbb{R}_*)$ and $o \le s \le \frac{1}{2}$.

This implies $\varphi_+f,\ a\varphi u \in H^s((o,A))$. From the continuity of H and [9] Theorem 11.4 (for $\mathbb{R}_*$) we derive $RHEu \in H^s(\mathbb{R}_*)$ hence also $R(D\varphi_+)\,HEu \in H^s((o,A))$ and $R\varphi_+HEu \in H^s((o,A))$. But then we get from (9) and [9] Theorem 9.7 that $R\varphi_+HEu \in H^{1+s}((o,A))$. Let us write

$$\omega_1 := R\varphi_+HEu =: x\,\omega_2.$$

Then $\omega_2 \in H^s((o,A))$ and ω_1 is absolutely continuous on $[o,A]$ ([10] Chap. 2, Théorème 2.2) with $\omega_1' \in H^s((o,A))$. Consequently

$$\omega_2(x) = \frac{\omega_1(o)}{x} + \frac{1}{x}\int_o^x \omega_1'(t)\,dt,\ x \in [o,A] .$$

By Hardy's inequality ([10] Chap. 2, Lemme 5.1) $\frac{\omega_1(o)}{x} \in L^2([o,A])$ hence $\omega_1(o) = o$. This implies that $DE\omega_1 = ED\omega_1 = -i\pi\, E\,\omega_1'$ $\in H^s(\mathbb{R})$ by [9] Theorem 11.4 again hence as above

$$E\omega_1 = \varphi_+ HEu \in H^{1+s}(\mathbb{R}).$$

Finally, the identity

$$\varphi_+ Eu = H^2 \varphi_+ Eu = H(HM_{\varphi_+} - M_{\varphi_+} H)\ Eu + H\varphi_+ HEu$$

combined with Lemma 1 leads to the conclusion

$$\varphi_+ Eu \in H^{s'}(\mathbb{R}) \text{ for every } s' < 1+s$$

if $o \in \operatorname{supp} \varphi$ and

$$\varphi_+ Eu \in H^{1+s}(\mathbb{R})$$

otherwise. The proof is complete.

We expect that the above assertions can be improved to yield $s' = s+1$. The restriction $s < \frac{1}{2}$ however seems to be natural since $u = \text{const}$ is a solution of the equation $Ku = o$ when $a = b = -1$.

3. The kernel of $e^{-s\bar{Q}}$

According to the spectral theorem $e^{-s\bar{Q}}$ is a bounded operator for $s \geq o$. Moreover, the abstract Cauchy problem

$$(10) \quad (\bar{Q} + \frac{\partial}{\partial s})\ u(s) = o,\ s > o,\ \lim_{s \to o} u(s) = u_o$$

has a unique solution $u \in C^1(\mathbb{R}_*, L^2(\mathbb{R}_+)) \cap C^o(\mathbb{R}_+, L^2(\mathbb{R}_+))$ with $u(s) \in \mathcal{D}(\bar{Q})$ for $s > o$, namely

$$u(s) = e^{-s\bar{Q}}\ u_o.$$

Now fix $x > o$ and pick $\varphi \in C_o^\infty(\mathbb{R}_*)$ with $\varphi(x) = 1$. By (10) and Lemma 2,2 we have $\varphi e^{-s\bar{Q}} u \in H^1(\mathbb{R})$ for every $s > o$ and $u \in L^2(\mathbb{R}_+)$. Since $P\varphi = D(H\varphi - \varphi H) + (D\varphi)\ H + \varphi P$ it follows from (1) and Lemma 1 that

$$\| \varphi e^{-s\bar{Q}} u \|_1 \leq C(\| P\varphi e^{-s\bar{Q}} u \|_o + \| \varphi e^{-s\bar{Q}} u \|_o)$$

$$\le C_x(\|\varphi\bar{Q}e^{-s\bar{Q}}u\|_o + \|u\|_o)$$

hence by the spectral theorem

$$\|\varphi e^{-s\bar{Q}}u\|_1 \le C_x \frac{s+1}{s} \|u\|_o .$$

By the version of Sobolev's inequality given in [1], Lemma 13.2 (applied to some interval containing supp φ) we thus conclude that

$$|e^{-s\bar{Q}}u(x)| \le C_x(\frac{1+s}{s})^{1/2} \|u\|_o , \quad s, x > o.$$

But this implies by the Riesz representation theorem that the distribution kernel of $e^{-s\bar{Q}}$, to be denoted by $\bar{\Gamma}_s$, is a function satisfying

$$(11) \quad \int_o^\infty \bar{\Gamma}_s(x,y)^2 dy \le C_x(\frac{1+s}{s}) , \quad s, x > o.$$

The importance of $\bar{\Gamma}_s$ for our problem stems from the following invariance property under the natural isometric action of $\mathbb{R}_*$ on $L^2(\mathbb{R}_+)$. This is defined by

$$U_\alpha f(x) := \alpha^{1/2} f(\alpha x), \ \alpha > o, \ x > o, \ f \in L^2(\mathbb{R}_+).$$

PROPOSITION 3 1) *For* $\alpha > o$ *we have* $U_\alpha(\mathcal{D}(\bar{Q})) \subset \mathcal{D}(\bar{Q})$ *and*

$$\bar{Q}U_\alpha = \alpha U_\alpha \bar{Q} .$$

2) *For* $\alpha, s > o$ *we have*

$$U_\alpha e^{-s\alpha\bar{Q}} = e^{-s\bar{Q}} U_\alpha$$

or equivalently

$$(12) \quad \bar{\Gamma}_s(x,y) = \alpha \bar{\Gamma}_{\alpha s}(\alpha x, \alpha y) , \quad x,y > o .$$

PROOF 1) Certainly $U_\alpha(C_o^\infty(\mathbb{R}_*)) \subset C_o^\infty(\mathbb{R}_*)$. One easily checks that

$$(13) \quad \bar{K}U_\alpha u = \alpha U_\alpha \bar{K}u , \quad u \in C_o^\infty(\mathbb{R}_*),$$

hence

$$(\bar{K}U_\alpha u|U_\alpha u) = \alpha(U_\alpha \bar{K}u|U_\alpha u) = \alpha(\bar{K}u|u) .$$

By the definition of the Friedrichs extension it remains to show that $U_\alpha(\mathcal{D}(\bar{K}^*)) \subset \mathcal{D}(\bar{K}^*)$. For $v \in \mathcal{D}(\bar{K}^*)$ there is $v^* \in L^2(\mathbb{R}_+)$ such that

$$(v|\bar{K}u) = (v^*|u) \text{ for all } u \in C_o^\infty(\mathbb{R}_*) .$$

It follows for $\alpha > o$ from (13) that

$$(U_\alpha v|\bar{K}u) = (v|U_{1/\alpha} \bar{K}u) = \alpha(v|\bar{K}U_{1/\alpha} u)$$

$$= \alpha(v^*|U_{1/\alpha} u) = (\alpha U_\alpha v^*|u)$$

i.e. $U_\alpha v \in \mathcal{D}(\bar{K}^*)$ and $\bar{K}^* U_\alpha v = \alpha U_\alpha \bar{K}^* v$.

2) We put for $u \in L_2(\mathbb{R}_+)$ and $\alpha > o$

$$v_\alpha(s) := U_\alpha e^{-s\alpha\bar{Q}}u, \quad s \geq o.$$

From (13) and the spectral theorem we find

$$(\bar{Q} + \frac{\partial}{\partial s}) v_\alpha(s) = o, \quad s > o, \quad \lim_{s \to o} v_\alpha(s) = U_\alpha u ,$$

thus $U_\alpha e^{-s\alpha\bar{Q}} u = e^{-s\bar{Q}} U_\alpha u$. (12) is an obvious consequence.

We can now improve on the estimate (11) for the L^2-norm of $\bar{\Gamma}_s$.

<u>LEMMA 3</u> *For every* $\varepsilon > o$ *there is a constant* C_ε *such that for* $o < x, s \leq 1$

$$(14) \quad \int_o^\infty \bar{\Gamma}_s(x,y)^2 \, dy \leq \frac{C_\varepsilon}{x^\varepsilon s} .$$

<u>PROOF</u> We first assume $s \leq x$. Choosing $\alpha = \frac{1}{x}$ in (12) we get from (11)

$$\int_0^\infty \bar{\Gamma}_s(x,y)^2\, dy = \frac{1}{x^2} \int_0^\infty \bar{\Gamma}_{s/x}(1, \tfrac{y}{x})^2\, dy$$

$$= \frac{1}{x} \int_0^\infty \bar{\Gamma}_{s/x}(1,y)^2\, dy \le \frac{C}{s} \le \frac{C}{x^\varepsilon s} \quad .$$

Now assume $x \le s$. We choose $\alpha = \frac{1}{s}$ in (12) to find

$$(15) \quad \int_0^\infty \bar{\Gamma}_s(x,y)^2\, dy = \frac{1}{s} \int_0^\infty \bar{\Gamma}_1(\tfrac{x}{s},y)^2\, dy \; .$$

It follows from the spectral theorem and (6) that $\bar{\Gamma}_1 u$ and $\frac{\partial}{\partial s}\Big|_{s=1} \bar{\Gamma}_s u$ are elements of $H^{1/2}(\mathbb{R}_*)$ for every $u \in L^2(\mathbb{R}_+)$.

Hence we obtain from (10) and Lemma 2,2 that

$$v := xE\bar{\Gamma}_1 u \in H^{s'}_{loc}(\mathbb{R}) \text{ for every } s' < \frac{3}{2} \; .$$

As in the proof of that result we see that v is absolutely continuous and $v(o) = o$ thus

$$\bar{\Gamma}_1 u(x) = \frac{1}{x} \int_o^x v'(t)\, dt, \; x > o \; .$$

But $v' \in H^r_{loc}(\mathbb{R}_*)$ for every $r < \frac{1}{2}$ so we can apply [9] Theorem 11.2 together with the Cauchy - Schwarz inequality. This yields

$$|\bar{\Gamma}_1 u(x)| \le \frac{1}{x}\Big(\int_o^x t^{2r}\, dt\Big)^{1/2} \Big(\int_o^x \frac{v'(t)^2}{t^{2r}}\, dt\Big)^{1/2} \le C_{r,u}\, x^{r-1/2} \; .$$

In other words: the family of bounded linear functionals on $L^2(\mathbb{R}_+)$ given by $u \mapsto x^\varepsilon \bar{\Gamma}_1 u(x)$, $o < x \le 1$, is pointwise bounded for every $\varepsilon > o$. Therefore, we conclude from the resonance theorem ([12] p. 69) that

$$\int_o^\infty \bar{\Gamma}_1(x,y)^2\, dy \le \frac{C_\varepsilon}{x^\varepsilon} \text{ for every } \varepsilon > o \; , \; o < x \le 1,$$

hence by (15) and the assumption $x \le s \le 1$

$$\int_0^\infty \bar{\Gamma}_s(x,y)^2 \, dy \le \frac{C_\varepsilon s^\varepsilon}{sx^\varepsilon} \le \frac{C_\varepsilon}{sx^\varepsilon} \quad .$$

The proof is complete.

Following the method of Seeley [11] we now proceed to construct a pointwise approximation to $\bar{\Gamma}_s(x,y)$ locally away from o. Actually we only need the result for $x = y = 1$, $o < s < 1$.

LEMMA 4 *For* $o < s < 1$ *we have*

$$|\bar{\Gamma}_s(1,1) - \frac{1}{s\pi^2} + \frac{a}{\pi^2}| \le Cs^{1/4} \quad .$$

PROOF Introducing the curve $c : \mathbb{R} \ni t \mapsto t\, e^{i\frac{\pi}{4}\operatorname{sgn} t} - 1 \in \mathbb{C}$ we obtain from the spectral theorem and Cauchy's integral formula

$$(16) \quad e^{-s\bar{Q}} = \frac{1}{2\pi i} \int_c e^{-s\zeta}(\bar{Q}-\zeta)^{-1} \, d\zeta \; .$$

Obviously

$$\| (\bar{Q}-\zeta)^{-1} u \|_o \le \frac{C}{|\zeta|} \| u \|_o \; , \; \zeta \in c(\mathbb{R}) \; , \; u \in L^2(\mathbb{R}_+) \; ,$$

so the integral (16) is convergent as a Bochner integral. To construct a good approximation to $(\bar{Q}-\zeta)^{-1}$ we put

$$a_1(x,\xi) := \pi|\xi| \; , \; a_o(x,\xi) := \frac{a}{x} \; , \; x > o \; , \; \xi \in \mathbb{R} \; ,$$

and as in [11] p. 290

$$b_{-1}(x,\xi,\zeta) := \frac{1}{\pi|\xi| - \zeta} \quad ,$$

$$b_{-1-m}(a_1-\zeta) + \sum_{\substack{j+k+\ell=m \\ j<m}} \left(\frac{\partial}{\partial\xi}\right)^k b_{-1-j} \left(-i\frac{\partial}{\partial x}\right)^k \frac{a_{1-\ell}}{k!} = o \; ,$$

$m \ge 1$, $x > o$, $\xi \in \mathbb{R}$, $\zeta \notin \mathbb{R}_+$, in particular

$$b_{-2}(x,\xi,\zeta) = - \frac{a}{x(\pi|\xi| - \zeta)^2} ,$$

$$b_{-3}(x,\xi,\zeta) = \frac{ia\pi \operatorname{sgn} \xi + a^2}{x^2(\pi|\xi| - \zeta)^3} .$$

Now we choose $\varphi \in C_0^\infty(\mathbb{R}_*)$ with $\varphi = 1$ in a neigborhood of 1 and $\psi \in C_0^\infty(\mathbb{R}_*)$ with $\varphi\,\psi = \varphi$ and put for $f \in C_0^\infty(\mathbb{R}_*)$

$$F_\zeta f(x) := \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{ix\xi} \sum_{j=1}^{3} b_{-j}(x,\xi,\zeta)\, \hat{f}(\xi)\, d\xi,\ x > 0,\ \zeta \notin \mathbb{R}_+ .$$

Denoting by Z the component of $\mathbb{C} \setminus c(\mathbb{R})$ not containing $\mathbb{R}_+$ we get from [11] Lemma 2

$$\| \varphi F_\zeta \psi f \|_0 \le \frac{C}{|\zeta|} \| f \|_0 ,\ \zeta \in Z ,$$

hence

$$E_s := \frac{1}{2\pi i} \int_c e^{-s\zeta} \varphi F_\zeta \psi\, d\zeta$$

is well defined as a Bochner integral. It follows from [11] Lemma 2 that

$$\| (M_\varphi - \varphi F_\zeta \psi(\bar{Q}-\zeta))\, f \|_2 \le \frac{C}{|\zeta|} \| f \|_0 ,\ \zeta \in Z .$$

Thus the operator function

$$G_\zeta := (M_\varphi - \varphi F_\zeta \psi(\bar{Q}-\zeta))(\bar{Q}-\zeta)^{-1}$$

satisfies

(17) $$\| G_\zeta f \|_2 \le \frac{C}{|\zeta|^2} \| f \|_0 ,\ \zeta \in Z ,\ f \in L^2(\mathbb{R}_+) ,$$

and is holomorphic in Z. Using Cauchy's theorem we then find for $f \in C_0^\infty(\mathbb{R}_*)$

$$(\varphi e^{-s\bar{Q}} - E_s)\, f = \frac{1}{2\pi i} \int_C e^{-s\zeta}(\varphi(\bar{Q}-\zeta)^{-1} - \varphi F_\zeta \psi)\, f\, d\zeta$$

$$= \frac{1}{2\pi i} \int_C (e^{-s\zeta}-1)\, G_\zeta\, f\, d\zeta \ .$$

Together with (17) this implies the estimate

$$\| (\varphi e^{-s\bar{Q}} - E_s)\, f\|_2 \le C\, s^{1/2} \| f\|_o \ , \ f \in L^2(\mathbb{R}_+) \ , \ o < s < 1.$$

We now apply [2] Theorem 2.1 to the operator $(\varphi e^{-s\bar{Q}} - E_s) \mid L^2([o,2])$. This shows that the distribution kernel of $E_s | L^2([o,2])$ is a function $E_s(x,y)$ and that

(18) $|\varphi(x)\, \bar{\Gamma}_s(x,y) - E_s(x,y)| \le Cs^{1/4}, \ o < x,y < 2, \ o < s \le 1$.

It remains to compute $E_s(x,y)$. For f, g $\in C_o^\infty((o,2))$ we have

$$(E_s f|g) = \frac{1}{2\pi} \int_o^\infty \bar{g}(x)\varphi(x) \int_{-\infty}^{+\infty} e^{ix\xi}\, \widehat{\psi f}(\xi)$$

$$\frac{1}{2\pi i} \sum_{j=1}^{3} \int_C e^{-s\zeta}\, b_j(x,\xi,\zeta)\, d\zeta\, d\xi\, dx$$

since the integral converges absolutely. Hence by Cauchy's integral formula

$$(E_s f|g) = :$$

$$\frac{1}{2\pi} \int_o^\infty \int_o^\infty \bar{g}(x) f(y) \varphi(x) \psi(y) \int_{-\infty}^{+\infty} e^{i(x-y)\xi - s\pi|\xi|} \cdot$$

$$\cdot \sum_{j=o}^{2} \frac{s^j}{j!}\, c_j(x,\xi)\, d\xi\, dy\, dx$$

where

$$c_o(x,\xi) = 1, \ c_1(x,\xi) = -\frac{a}{x} \ , \ c_2(x,\xi) = \frac{ia\pi \operatorname{sgn} \xi + a^2}{x^2} \ .$$

Therefore

$$E_s(x,x) = \frac{\varphi(x)}{\pi} \int_0^\infty e^{-s\pi\xi} \left(1 - \frac{a}{x} s + \frac{a^2}{2x^2} s^2\right) d\xi$$

$$= \varphi(x) \left(\frac{1}{\pi^2 s} - \frac{a}{\pi^2 x} + \frac{a^2}{2\pi^2 x^2} s \right) .$$

From this identity and (18) the Lemma follows.

4. Asymptotic behavior of the eigenvalues

Our study of the eigenvalue distribution of Q is based on properties of the Laplace transform

$$L(s) := \int_0^\infty e^{-st} \, dN(t), \; s > 0,$$

where N(t) denotes the number of eigenvalues of Q not bigger than t (counted with multiplicity). Formally, L(s) is the trace of the operator e^{-sQ}, i. e. if Γ_s denotes the distribution kernel of e^{-sQ} we have

$$L(s) = \int_0^1 \Gamma_s(x,x) \, dx$$

provided the integral exists. Using arguments analogous to those given in section 3 we see that Γ_s is actually a function and

$$\int_0^1 \Gamma_s(x,y)^2 \, dy < \infty , \; 0 < x < 1 .$$

Now we use our preceding results to show that $\bar{\Gamma}_s$ is a good approximation to Γ_s away from 1.

LEMMA 5 *Let* $0 < s < 1$ *and* $0 < x < 2/3$. *For every* $\varepsilon > 0$ *there is a constant* C_ε *such that*

$$(19) \quad \int_0^{2/3} (\Gamma_s(x,y) - \bar{\Gamma}_s(x,y))^2 \, dy \le \frac{C_\varepsilon}{x^\varepsilon} s .$$

PROOF We choose $\varphi \in C_o^\infty((-1,1))$ with $\varphi|(-2/3, 2/3) = 1$ and $o \le \varphi \le 1$. Recall the operators of extension by zero, $E : L^2(\mathbb{R}_+) \to L^2(\mathbb{R})$, $E_1 : L^2([o,1]) \to L^2(\mathbb{R})$, and introduce in addition $E_1^+ : L^2([o,1]) \to L^2(\mathbb{R}_+)$. We put for $u \in L^2([o,1])$, $o < s < 1$

$$B_s u := \varphi E_1^+ e^{-sQ} u.$$

It is easily checked that there is a sequence $(v_n) \subset C_o^\infty((o,1))$ with $\lim_{n\to\infty} v_n = B_s u$ in $L^2(\mathbb{R}_+)$ and $(\bar{K}(v_n - v_m)|v_n - v_m) \to o$, $n, m \to \infty$. Moreover, by the properties of e^{-sQ} analogous to (10)

$$(\bar{K} + \frac{\partial}{\partial s}) B_s u = R\,(DH + \frac{a}{x})\, E\, \varphi E_1^+\, e^{-sQ}u + \varphi E_1^+ \frac{\partial e^{-sQ}u}{\partial s}$$

$$= RD(H\varphi - \varphi H)\, E_1 e^{-sQ}u + R(D\varphi)\, HE_1 e^{-sQ}u +$$

$$+ R\varphi DHE_1 e^{-sQ}u + R\varphi \frac{a}{x} E_1 e^{-sQ}u + \varphi E_1^+ \frac{\partial e^{-sQ}u}{\partial s}$$

$$=: \varphi E_1^+ (R_1 DHE_1 e^{-sQ}u + (\frac{a}{x} + \frac{b}{1-x})\, e^{-sQ}u + \frac{\partial e^{-sQ}u}{\partial s})$$

$$+ C_s u = C_s u .$$

In view of Lemma 1 C_s is a continuous linear operator $L^2([o,1]) \to L^2(\mathbb{R}_+)$ whose norm is bounded uniformly in s. Hence $B_s u \in \mathcal{D}(\bar{Q})$ and from Duhamel's principle we get

$$B_s u = e^{-s\bar{Q}}\varphi u + \int_o^s e^{-(s-t)\bar{Q}}\, C_t u\, dt .$$

Restricting now to $u \in L^2([o,1])$ with $\operatorname{supp} u \subset [o,2/3]$ we derive from Lemma 3 for $o < x < 2/3$, $o < s < 1$, and every $\varepsilon > o$

$$|e^{-sQ}u(x) - e^{-s\bar{Q}}u(x)| \le \frac{C_\varepsilon}{x^\varepsilon}\, \|u\|_o \int_o^s (s-t)^{-1/2}\, dt$$

$$\leq \frac{C_\varepsilon}{x^\varepsilon}\, s^{1/2}\, \|u\|_o$$

which implies (19).

Next we have to estimate Γ_s and $\bar{\Gamma}_s$ off the diagonal.

PROPOSITION 4 *For* $o < x < \frac{1}{2}$, $o < s < 1$, *and every* $\varepsilon > o$ *we have*

$$(20) \quad \int_{2/3}^{1} \Gamma_s(x,y)^2\, dy + \int_{2/3}^{\infty} \bar{\Gamma}_s(x,y)^2\, dy \leq \frac{C_\varepsilon}{x^\varepsilon}\, s\,.$$

PROOF We pick $\psi \in C_o^\infty((-2/3, 2/3))$ with $\psi|(-1/2, 1/2) = 1$.

For $u \in L^2([o,1])$ we compute

$$(Q + \frac{\partial}{\partial s})\, \psi\, e^{-sQ}u = \psi R_1 DHE_1 e^{-sQ}u + \psi\, (\frac{a}{x} + \frac{b}{1-x})\, e^{-sQ}u$$

$$+ \psi \frac{\partial}{\partial s}\, e^{-sQ}u + R_1 D(H\psi - \psi H)\, E_1 e^{-sQ}u + R_1 (D\psi)\, HE_1 e^{-sQ}u =: D_s u$$

where $D_s : L^2([o,1]) \to L^2([o,1])$ is linear and bounded uniformly in s.

Restricting to u with supp $u \subset [2/3, 1]$ we find as in the proof of Lemma 5 for $o < x < 1/2$

$$e^{-sQ}u(x) = \int_o^s e^{-(s-t)Q}\, D_t u\, dt$$

hence by Lemma 3 and Lemma 5

$$\int_{2/3}^{1} \Gamma_s(x,y)^2\, dy \leq \frac{C_\varepsilon}{x^\varepsilon}\, s\,.$$

A similar argument for $\bar{\Gamma}_s$ completes the proof.

We are now ready to establish our Theorem.

PROOF OF THE THEOREM Let $o < s < 1/2$. By the semigroup property of e^{-sQ} and the symmetry of Γ_s we find for $o < x < 1$

$$\Gamma_{2s}(x,x) = \int_0^1 \Gamma_s(x,y)^2\,dy < \infty.$$

Next we introduce the isometric involution

$$\sigma : L^2([o,1]) \ni f(x) \mapsto f(1-x) \in L^2([o,1]),$$

i.e. σ is reflection in 1/2. Denoting by Q' the operator arising from Q by interchanging a and b it is easily checked that $\sigma(\mathcal{D}(Q')) = \mathcal{D}(Q)$ and

$$Q' = \sigma\, Q\, \sigma$$

hence

$$\Gamma_s'(x,y) = \Gamma_s(1-x,\ 1-y),\ o < x,\ y < 1,$$

and therefore

$$(21)\quad \int_{1/2}^1 \Gamma_s(x,x)\,dx = \int_0^{1/2} \Gamma_s'(x,x)\,dx\ .$$

Now we have for $o < x < 1/2$

$$\Gamma_{2s}(x,x) = \int_0^{2/3} (\Gamma_s(x,y) - \bar\Gamma_s(x,y))^2\,dy + \int_0^{2/3} \bar\Gamma_s(x,y)^2\,dy$$

$$+ 2 \int_0^{2/3} (\Gamma_s(x,y) - \bar\Gamma_s(x,y))\,\bar\Gamma_s(x,y)\,dy + \int_{2/3}^1 \Gamma_s(x,y)^2\,dy$$

$$=: \int_0^\infty \bar\Gamma_s(x,y)^2\,dy + R_s(x)$$

$$= \bar\Gamma_{2s}(x,x) + R_s(x)\ .$$

Combining (19), (20), and (14) we find for $\varepsilon > o$ and $o < x < 1/2$

$$|\Gamma_s(x,x) - \bar\Gamma_s(x,x)| \le \frac{C_\varepsilon}{x^\varepsilon}\ .$$

Fixing $\varepsilon := 1/2$ it follows that

$$(22)\quad \int_0^{1/2} |\Gamma_s(x,x) - \bar{\Gamma}_s(x,x)|\, dx \le C .$$

Putting $\alpha := 1/x$ in (12) we find

$$(23)\quad \int_0^{1/2} \bar{\Gamma}_s(x,x)\, dx = \int_0^{1/2} \frac{1}{x}\, \bar{\Gamma}_{s/x}(1,1)\, dx$$

$$= \int_{2s}^{\infty} \frac{\bar{\Gamma}_t(1,1)}{t}\, dt$$

where we have substituted $t := s/x$ in the second integral.
From Lemma 4 we derive

$$(24)\quad \int_{2s}^{1} \frac{\bar{\Gamma}_t(1,1)}{t}\, dt =: \frac{1}{\pi^2} \int_{2s}^{1} \frac{dt}{t^2} - \frac{a}{\pi^2} \int_{2s}^{1} \frac{dt}{t} + w(s)$$

$$=: \frac{1}{2\pi^2 s} + \frac{a}{\pi^2} \log s + w_1(s)$$

where

$$|w_1(s)| \le C .$$

Thus we conclude from (22), (23), and (24) that

$$\left| \int_0^{1/2} \Gamma_s(x,x)\, dx - \frac{1}{2\pi^2 s} - \frac{a}{\pi^2} \log s \right| \le C .$$

Finally, taking (21) into account it follows that e^{-sQ} is a trace class operator and that for $s \to o$

$$\int_0^1 \Gamma_s(x,x)\, dx = \frac{1}{\pi^2 s} + \frac{a+b}{\pi^2} \log s + O(1)$$

completing the proof of the theorem.

PROOF OF THE COROLLARY The well known Tauberian theorem for the Laplace transform derives from

$$\int_0^\infty e^{-st}\, dN(t) \sim \frac{1}{\pi^2 s}, \quad s \to o,$$

the conclusion

$$N(t) \sim \frac{t}{\pi^2}, \quad t \to \infty .$$

For $\varepsilon > o$ and $n \in \mathbb{N}$ we obviously have

$$(25) \quad N(\lambda_n - \varepsilon) \leq n \leq N(\lambda_n) ,$$

which implies

$$\lim_{n\to\infty} \frac{n}{\lambda_n} = \frac{1}{\pi^2} .$$

Assume now an asymptotic relation of the form

$$\lambda_n = \pi^2 n + \alpha \log n + o(\log n) .$$

From this we get $\lambda_n - \lambda_{n-1} = o(\log n)$ and

$$n = \frac{\lambda_n}{\pi^2} - \frac{\alpha}{\pi^2} \log \lambda_n + o(\log \lambda_n) .$$

Together with (25) this implies

$$N(t) = \frac{t}{\pi^2} - \frac{\alpha}{\pi^2} \log t + o(\log t)$$

and by integration

$$\int_o^\infty e^{-st} \, dN(t) = \frac{1}{\pi^2 s} + \frac{\alpha}{\pi^2} \log s + o(\log s).$$

Thus we must necessarily have $\alpha = a + b$.

REFERENCES

1. AGMON, S.: Lectures on elliptic boundary value problems, Princeton: Van Nostrand 1965
2. AGMON, S., and Y. KANNAI: On the asymptotic behavior of spectral functions and resolvent kernels of elliptic operators, Israel J. Math. 5, 1-30 (1967)
3. FEDERBUSH, P., and A. TROMBA: A note on 't Hooft's Hamiltonian in two-dimensional QCD, Phys. Rev. D15, 2913 (1977)
4. HILDEBRANDT, S.: Mathematical aspects of 't Hooft's eigenvalue problem in two-dimensional quantum chromodynamics I, manuscripta math. 24, 45 - 79 (1978)
5. HILDEBRANDT, S.: Mathematical aspects of 't Hooft's eigenvalue problem in two-dimensional quantum chromodynamics II, Ark. Math. 17, 29 - 38 (1979)
6. HILDEBRANDT, S., and V. VIŠNJIĆ-TRIANTAFILLOU: Mathematical aspects of 't Hooft's eigenvalue problem in two-dimensional quantum chromodynamics III, Math. Z. 168, 223 - 240 (1979)
7. 'T HOOFT, G: A two-dimensional model for mesons, Nuclear Physics B 75, 461 - 470 (1974)
8. LEWY, H.: Expansion of solutions of 't Hooft's equation. A study in the confluence of analytic boundary conditions, manuscripta math. 26, 411 - 421 (1979)
9. LIONS,J.L., and E.MAGENES: Non-homogeneous boundary value problems and applications I, Berlin-Heidelberg-New York:Springer 1972
10. NEČAS, J.: Les mêthodes directes en thêorie des êquations elliptiques, Paris: Masson et Cie, 1967
11. SEELEY, R.T.: Complex powers of an elliptic operator, in: Proceedings of Symposia in Pure Mathematics 10, Providence: American Mathematical Society 1966
12. YOSIDA, K.: Functional analysis, 2. edition, Berlin-Heidelberg-New York: Springer 1968

Jochen Brüning
Fachbereich Mathematik
der Universität-Gesamthochschule
Lotharstr. 65

D-4100 Duisburg

(Received July 6, 1982)