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Analysis



# ON THE DEFICIENCY OF CERTAIN SETS OF EXPONENTIALS

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Abstract: For certain functions $f \in L^2([-1,1])$ we show that the system $(e^{iz_n t})_{n\in\mathbb{N}}$ where $(z_n)_{n\in\mathbb{N}}$ are the zeros of the Fourier transform of f or a naturally related function has finite deficiency in $L^2([-1,1])$. The deficiency is calculated.

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1. A system of oscillators can be analysed by decomposing all inputs and outpucs according to its eigenstates. If these are not known or if no satisfying mathematical model of the process in question does exist,

as for example in the case of electroencephalography, one tries to find characteristic features of the output by using one of the standard transforms. One may ask, however, for procedures which are more naturally adapted to the output and do not impose artificial frequencies on the system. It is the purpose of this note to propose one such possibility. More precisely, if $f \in L^2([-1,1])$ we try to decompose f according to the system $(e^{iz_n t})_{n\in\mathbb{N}}$ where $(z_n)_{n\in\mathbb{N}}$ are the zeros of the Fourier transform of f or some other function related to f. Obviously such a system cannot be complete in $L^2([-1,1])$. Our observation is that under simple assumptions on f the deficiency can be calculated explicitly and turns out to be finite. This means that nonharmonic Fourier analysis (which is, of course, a well established theory, see e. g. [1], [5]) could be a useful tool in many questions of the above type.

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2. Throughout Sections 2 - 5 f denotes a function of class $C^{(k)}([-1,1])$, $k \geq 0$, such that $f^{(i)}(-1) \neq 0$ for some index $i < k$ and $f^{(j)}(1) \neq 0$ for some index $j < k$, where these derivatives are interpreted in the one-sided sense. We denote by a the smallest index i such that $f^{(i)}(-1) \neq 0$ and by b the smallest index j such that $f^{(j)}(1) \neq 0$, and we set, by definition, $p := \min\{a,b\}$. Thus, p is the smallest index i such that $f^{(i)}(-1) \neq 0$ or $f^{(i)}(1) \neq 0$. According to a theorem of Pólya [2] the Fourier transform

$$F(z) := \int_{-1}^{1} e^{-izt} f(t)\,dt$$

has infinitely many distinct zeros $z_n$, $n \in \mathbb{N}$. Let $z_n$ have multiplicity $m_n$ and put

$$q := \sum_{n=1}^{\infty} (m_n - 1)$$

which might be infinite. The system of exponentials $(e^{iz_n t})_{n\in\mathbb{N}}$ spans a proper subspace of $L^2([-1,1])$ the codimension of which is called the *deficiency* of the system. Our main result reads as follows:

THEOREM *Under the above hypotheses on* f *we have* $q < \infty$ *and the deficiency of* $(e^{iz_n t})_{n\in\mathbb{N}}$ *is* $p+q+1$.

An important aspect of the Theorem is the fact that *the zeros are not counted according to their multiplicity*. In all previous results of this general type known to us (cf. [3]) the zeros are repeated, i. e. one studies the deficiency of the system

$$(t^j e^{iz_n t})_{\substack{n\in\mathbb{N} \\ 0\leq j\leq m_n-1}} .$$

Our Theorem makes use of the pure exponential terms only. It is because of this feature that the theory of nonharmonic Fourier analysis is available to study the convergence, as explained in Section 6.

3. For the proof of the Theorem it suffices to establish the following two lemmas.

LEMMA 1 *The system* (1) *has deficiency* $p+1$.

LEMMA 2 *The function* $F(z)$ *in the Theorem has only finitely many multiple zeros.*

In passing from Lemma 1 to the Theorem we omit $m_n-1$ terms in (1) for each zero of multiplicity $m_n > 1$. Hence the set of omitted terms spans a subspace of dimension q, which is finite by Lemma 2, and the Theorem follows.

Before turning to the proof of the lemmas we make two preliminary remarks. First, since $f(-t)$ is orthogonal to $e^{-izt}$ whenever $f(t)$ is orthogonal to $e^{izt}$, the completeness properties of the sets $(e^{iz_n t})$ and $(e^{-iz_n t})$ are exactly the same. It is more convenient to deal with $(e^{-iz_n t})$ in the sequel, and we shall do so. Second, if $h \in L^1([-1,1])$, an elementary theorem of Titchmarsh([4], Lemma 2.2) gives

$$(2) \qquad \left| \int_{-1}^{1} e^{-izt} h(t)dt \right| = o(e^{r|\sin\theta|})$$

uniformly in $\theta$ as $r \to \infty$, where $z = re^{i\theta}$. We use this repeatedly with various choices of h.

4. Proof of Lemma 1 By partial integration

$$(3) \qquad F(z) = e^{iz}A(z) + e^{-iz}B(z) + \frac{1}{(iz)^k} \int_{-1}^{1} e^{-izt} f^{(k)}(t)dt$$

where A(z) and B(z) are polynomials in $1/z$ of the form

$$(4) \qquad A(z) = \frac{\alpha}{z^{a+1}} + \dots, \qquad B(z) = \frac{\beta}{z^{b+1}} + \dots, \qquad \alpha\beta \neq 0,$$

where the terms not written involve only higher powers of $1/z$.

If g is orthogonal to $(t^j e^{-iz_n t})_{\substack{n\in\mathbb{N} \\ 0\leq j\leq m_n-1}}$ then

$$\int_{-1}^{1} t^j e^{-iz_n t} g(t)dt = 0, \; n \in \mathbb{N}, \; 0\leq j\leq m_n-1 ,$$

that is, the Fourier transform G of g vanishes at all zeros of F and at least of the same order. By the Hadamard factorization theorem $G(z) = F(z)H(z)$ where H if of order 1 and finite type. For fixed $\theta$, $0 < \theta < \pi$, we see from (2) and (3) that

$$|F(re^{i\theta})| \geq C \frac{e^{r \sin\theta}}{r^{b+1}} \quad \text{for } r \geq \frac{1}{C}$$

with some constant $C > o$, and by (2) again

$$G(re^{i\theta}) = o(e^{r\sin\theta}), \quad r \to \infty .$$

These inequalities combine to give $H(re^{i\theta}) = o(r^{b+1})$. An entirely similar argument with the roles of A and B interchanged gives $H(re^{i\theta}) = o(r^{a+1})$ as $r \to \infty$ and $-\pi < \theta < 0$. The two inequalities together with the Phragmén-Lindelöf theorem show first that H is a polynomial, and second that the degree of this polynomial is at most p. Hence the deficiency is at most p+1. On the other hand by partial integration

$$(iz)^j F(z) = \int_{-1}^{1} e^{-izt} f^{(j)}(t)dt, \; o \leq j \leq p .$$

Since the functions $(f^{(j)}(t))_{o\leq j\leq p}$ are obviously linearly independent this shows that the deficiency is at least p+1. This proves Lemma 1. □

Let use note that this result is consistent with Theorem 22 of [3] according to which the deficiency is $|m|$ if m is the largest integer such that $F(x)/(1+x^2)^{m/2}$ does not belong to $L^2(\mathbb{R})$. In the present case this largest integer is $m = -(1+p)$, and therefore the deficiency is 1+p.

5. Proof of Lemma 2 Since $\check{f}(t) := f(-t)$ also satisfies the assumptions of the Theorem, it suffices to show that F(z) has only finitely many multiple zeros in the upper half plane, $\mathrm{Im}\, z \geq 0$. Suppose then that $\mathrm{Im}\, z \geq 0$. If the third term on the right in (3) is denoted by I(z) we have

$$(5) \qquad F(z) = e^{iz}A(z) + e^{-iz}E(z) \text{ where } E(z) = B(z) + e^{iz}I(z).$$

For sufficiently large $|z|$ clearly

(6) $$\left|\frac{A'(z)}{A(z)}\right| \leq \frac{a+1}{|z|}$$

and a short calculation using (2) together with the hypothesis Im $z \geqq 0$ gives

$$E(z) = O(|z|^{-b-1}) \ , \ E'(z) = o(|z|^{-b-1})$$

as $z \to \infty$. Hence, for large $|z|$,

(7) $$\left|\frac{E'(z)}{E(z)}\right| \leqq 1 \ .$$

Let C be so large that (6) and (7) hold for $|z| \geqq C$, Im $z \geqq 0$ and let z be a multiple zero of F in (5) satisfying these two conditions. Thus,

$$e^{iz}A(z) + e^{-iz}E(z) = 0 \ ,$$

$$e^{iz}(A'(z) + iA(z)) + e^{-iz}(E'(z) - iE(z)) = 0 \ .$$

This is a linear homogeneous system in the two unknowns $e^{iz}$, $e^{-iz}$, which of course are not zero. Hence the coefficient determinant vanishes, giving

$$\frac{A'(z)}{A(z)} = -2i + \frac{E'(z)}{E(z)} \ .$$

By (6) and (7) this implies $|z| \leqq a+1$ which completes the proof of Lemma 2. □

6. To describe a particularly convenient situation for applications we assume f real valued and put $\tilde{f}(t) := tf(t)$. Denote by $\tilde{F}$ the Fourier transform of $\tilde{f}$ with zeros $\tilde{z}_n$. Note that $\tilde{z}_n$'s are the saddle points of F. If f satisfies our condition with p = o so does $\tilde{f}$ and if moreover $\tilde{F}$ has no multiple zeros we get from the Theorem that $(e^{i\tilde{z}_n t})_{n\in\mathbb{N}}$

has deficiency one. The orthogonal complement of the span of $(e^{i\tilde{z}_n t})_{n\in\mathbb{N}}$ obviously contains $g(t) := tf(-t)$. But

$$\int_{-1}^{1} f(t)\,\overline{g(t)}\,dt = \int_{-1}^{1} tf(t)\,f(-t)\,dt = 0$$

so we can try to represent f as a nonharmonic Fourier series

$$f(t) = \sum_{n=1}^{\infty} \alpha_n e^{i\tilde{z}_n t}. \tag{8}$$

To compute the coefficients we put

$$\overline{h_n(-t)} := \frac{i}{\tilde{F}'(\tilde{z}_n)} \int_{-1}^{t} e^{-i\tilde{z}_n(u-t)}\,\tilde{f}(u)\,du,\; t \in [-1,1],\; n \in \mathbb{N}.$$

An easy computation shows that

$$\int_{-1}^{1} e^{i\tilde{z}_m t}\,\overline{h_n}(t)\,dt = \delta_{nm},\; n, m \in \mathbb{N}\,,$$

i.e. $(h_n)_{n\in\mathbb{N}}$ is a biorthogonal system for $(e^{i\tilde{z}_n t})_{n\in\mathbb{N}}$ hence

$$\alpha_n = \int_{-1}^{1} f(t)\,\overline{h_n(t)}\,dt,\; n \in \mathbb{N}\,.$$

It remains to establish some sort of convergence for (8). To do so we assume $f(-1) + f(1) \neq 0$ and note that by (3) and (4) the zeros of $\tilde{F}$ coincide with the solutions of the equation

$$\tan z = \beta + R_{\pm}(z)\,,$$

where $\beta \in \mathbb{C}$ and $R_{\pm}(z)$ is holomorphic for $|z|$ large and $\operatorname{Im} z \gtrless 0$, and satisfies $\lim_{|z|\to\infty} |R_{\pm}(z)| = 0$. Hence outside some large circle the zeros are among the numbers

$$z_o + m\pi + \varepsilon_m$$

where $z_o \in \mathbb{C}$, $m \in \mathbb{Z}$ with $|m| \geq C$, and $|\varepsilon_m| < \frac{1}{4}$.

Using the results on Riesz bases and equiconvergence in [5] we see that the series

$$e^{-iz_o t} f(t) = \sum_{n=1}^{\infty} \alpha_n e^{i(\tilde{z}_n - z_o) t}$$

is convergent in $L^2([-1,1])$ and equiconvergent with the ordinary Fourier series of $e^{-iz_o t} f(t)$ on compact subsets of $(-1,1)$.

The analogous considerations in case $p > 0$ are more cumbersome. It should be noted, however, that $p = q = 0$ for $\tilde{f}$ is a generic condition in $C^1([-1,1])$ i.e. is satisfied in an open dense subset.

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