Diagonalization of Holomorphic Functions with Values in W^*-Algebras

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ABSTRACT

We describe an algorithm for the diagonalization of holomorphic operator functions with values in W^*-algebras.

1. INTRODUCTION

If one tries to develop a theory of holomorphic or meromorphic operator valued functions, one meets the difficulty that operator algebras have divisors of zero. To deal with it one has to restrict attention to special classes of operator functions, a very convenient one being the class of diagonalizable functions. Consequently there has been done much work to find criteria for a given meromorphic operator function to be diagonalizable; see e.g. [3] and [1]. In this note we describe an algorithm for diagonalization of functions with values in W^*-algebras which seems to be new even in the matrix case (though it is well known that every holomorphic matrix valued function is diagonalizable). This algorithm is a useful technical tool, and many of the known criteria for diagonalization (in the W^*-case) follow easily from it; other applications can be found in [2]. To be more precise, let $\mathfrak{M}$ be a W^*-algebra, $U \subset \mathbb{C}$ a neighborhood of 0, and denote by $H(U, \mathfrak{M})$ the set of holomorphic functions in U with values in $\mathfrak{M}$. Then $x \in H(U, \mathfrak{M})$ will be called *diagonalizable at* 0 if we can find a neighborhood $U' \subset U$ of 0 and $a, b \in H(U, \mathfrak{M})$ such that

$$axb(z) = \sum_{j=0}^{k} x_j z^j, \qquad z \in U', \tag{1}$$

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and the following conditions hold: x_j has a generalized inverse in $\mathfrak{M}$, i.e., there is $y_j \in \mathfrak{M}$ such that $x_j y_j x_j = x_j$, $0 \leqslant j \leqslant k$ (the set of all such elements will be denoted by $\mathrm{CR}\,\mathfrak{M}$); the families of projections $(l(x_j))_{0 \leqslant j \leqslant k}$ and $(r(x_j))_{0 \leqslant j \leqslant k}$ are mutually orthogonal, where for $x \in \mathfrak{M}$ we denote by $l(x)$ $[r(x)]$ its left [right] support; for $z \in U'$ we have $a(z) \in \mathfrak{M}^0$, $b(z) \in \mathfrak{M}^0$, where $\mathfrak{M}^0$ denotes the set of invertible elements in $\mathfrak{M}$. In this situation we call the right hand side of (1) a *diagonal representation* of x. For reference we note the following lemma on elements with generalized inverse, the simple proof of which we omit.

LEMMA. *Let* $x \in \mathrm{CR}\,\mathfrak{M} \setminus 0$.

(*a*) *There is a unique element* $x^{-1} \in \mathrm{CR}\,\mathfrak{M} \setminus 0$ *such that*

$$xx^{-1} = l(x), \; x^{-1}x = r(x).$$

(*b*) *If* $x, y \in \mathrm{CR}\,\mathfrak{M}$ *with* $r(x) \geqslant l(y) (r(x) \leqslant l(y))$, *then*

$$xy \in \mathrm{CR}\,\mathfrak{M} \text{ and } r(xy) = r(y) \; (l(xy) = l(x)).$$

Some of the results of this note are contained in [2].

2. ALGORITHM

We will now present our algorithm. Let U be a neighborhood of 0 in $\mathbb{C}$, and let $x \in H(U, \mathfrak{M})$. We write for $k \in \mathbb{Z}_+$

$$x(z) = \sum_{j=0}^{\infty} x_j z^j =: \sum_{j=0}^{k} x_j z^j + x_{k+1}(z) z^{k+1}.$$

Let us assume moreover that

$$\text{the families of projections } \left(l(x_j)\right)_{j=0}^{k}, \left(r(x_j)\right)_{j=0}^{k} \text{ are mutually orthogonal} \tag{2}$$

and that

$$l(x_j) x_{k+1}(z) = x_{k+1}(z) r(x_j) = 0 \qquad \text{for} \quad z \in U \text{ and } 0 \leqslant j \leqslant k-1. \tag{3}$$

If for a given $x \in H(U', \mathfrak{M})$ we can find a neighborhood U of 0 and $a, b \in H(U, \mathfrak{M}^0)$ such that axb satisfies (2) and (3), we shall call x *weakly diagonalizable of order k at* 0.

THEOREM 1. *Let $x \in H(U, \mathfrak{M})$ satisfy* (2) *and* (3). *If $x_k \in \mathrm{CR}\,\mathfrak{M} \setminus 0$ then x is weakly diagonalizable of order $k+1$ at* 0.

Proof. The result will be established in two steps.

Step 1. We put $p := \sum_{j=0}^{k-1} l(x_j)$, $q := \sum_{j=0}^{k-1} r(x_j)$. Then we find a neighborhood U' of 0, $U' \subset U$, and $a, b \in H(U', \mathfrak{M}^0)$ such that for $z \in U'$

$$axb(z) = \sum_{j=0}^{k} x_j z^j + y(z) z^{k+1},$$

where

$$l(y(z)) \leqslant 1_{\mathfrak{M}} - p, \qquad r(y(z)) \leqslant 1_{\mathfrak{M}} - q,$$

and

$$l(x_k) y(z)[1_{\mathfrak{M}} - q - r(x_k)] = [1_{\mathfrak{M}} - p - l(x_k)] y(z) r(x_k) = 0. \tag{4}$$

To do so we make the *Ansatz*

$$a(z) = \sum_{j=0}^{\infty} a_j(z) z^j, \qquad b(z) = \sum_{j=0}^{\infty} b_j(z) z^j, \tag{5}$$

where we require $a_0 = b_0 = 1_{\mathfrak{M}}$ and

$$r(x_i) b_j(z) = a_j(z) l(x_i) = 0, \qquad 0 \leqslant i \leqslant k-1, \quad j \in \mathbb{N}, \quad z \in U'. \tag{6}$$

Then we obtain formally

$$xb(z) = \sum_{j=0}^{k-1} x_j z^j + z^k \left(x_k + \sum_{i=1}^{\infty} [x_k b_i(z) + x_{k+1}(z) b_{i-1}(z)] z^i \right)$$

and

$$axb(z) =: \sum_{j=0}^{k} x_j z^j + z^k \sum_{i=1}^{\infty} y_i(z) z^i,$$

where we have written

$$y_i(z)=a_i(z)x_k+\sum_{\substack{l+m=i\\ m\geqslant 1}}\left[a_l(z)x_kb_m(z)+a_lx_{k+1}b_{m-1}(z)\right].$$

In particular we have

$$y_1(z)=a_1(z)x_k+x_kb_1(z)+x_{k+1}(z).$$

Obviously we can satisfy (4) for y_1 if we put

$$\begin{aligned}a_1(z):&=-\left[1_{\mathfrak{M}}-p-l(x_k)\right]x_{k+1}(z)x_k^{-1},\\ b_1(z):&=-x_k^{-1}x_{k+1}(z)\left[1_{\mathfrak{M}}-q-r(x_k)\right].\end{aligned}$$

Suppose now we have defined $a_i(z), b_i(z)$ for $1\leqslant i\leqslant i_0$ such that (4) holds for $y_i(z)$ and in addition

$$l(a_i(z))\leqslant 1_{\mathfrak{M}}-p-l(x_k),\qquad r(b_i(z))\leqslant 1_{\mathfrak{M}}-q-r(x_k).$$

Then (4) for y_{i_0+1} is satisfied if we put

$$\begin{aligned}a_{i_0+1}(z):&=-\left[1_{\mathfrak{M}}-p-l(x_k)\right]a_{i_0}x_{k+1}(z)x_k^{-1},\\ b_{i_0+1}(z):&=-x_k^{-1}x_{k+1}b_{i_0}(z)\left[1_{\mathfrak{M}}-q-r(x_k)\right].\end{aligned}$$

This leads to the definition

$$\begin{aligned}a_i(z)&=(-1)^i\left[1_{\mathfrak{M}}-p-l(x_k)\right]\left[x_{k+1}(z)x_i^{-1}\right]^i,\\ b_i(z)&=(-1)^i\left[x_k^{-1}x_{k+1}(z)\right]^i\left[1_{\mathfrak{M}}-q-r(x_k)\right]\end{aligned}\tag{7}$$

for $i\in\mathbb{N}$. Then (6) is satisfied and (4) holds for every y_i. From (7) we also conclude that the series (5) converge uniformly in some neighborhood $U'\subset U$ of 0 and that $a(z), b(z)\in\mathfrak{M}^0$ for $z\in U'$. Finally it follows from Dunford's theorem [4, p. 128] that a and b are holomorphic in U', which completes the argument of step 1.

Step 2. We write

$$\begin{aligned}y(z)&=l(x_k)y(z)r(x_k)+\left[1_{\mathfrak{M}}-p-l(x_k)\right]y(z)\left[1_{\mathfrak{M}}-q-r(x_k)\right]\\ &=:y'(z)+y''(z).\end{aligned}$$

We find a neighborhood $U'' \subset U'$ of 0 and $c \in H(U'', \mathfrak{M}^0)$ such that

$$axbc(z) = \sum_{j=0}^{k} x_j z^j + y''(z) z^{k+1}$$

for $z \in U''$, which proves the assertion of the theorem. Again we make an *Ansatz*

$$c(z) = \sum_{j=0}^{\infty} c_j(z) z^j$$

and require $c_0 = 1_{\mathfrak{M}}$ and

$$l(c_j(z)) \leqslant r(x_k), \qquad j \in \mathbb{N}.$$

We obtain formally

$$axbc(z) = \sum_{j=0}^{k} x_j z^j + y''(z) z^{k+1}$$
$$+ \sum_{l=1}^{\infty} [x_k c_l(z) + y'(z) c_{l-1}(z)] z^l.$$

Thus the definition

$$c_l(z) := (-1)^l [x_k^{-1} y'(z)]^l$$

leads to $c \in H(U'', \mathfrak{M}^0)$ for some neighborhood $U'' \subset U'$ of 0, and c meets all our requirements. ■

3. JUSTIFICATION

We have to show that the method described above yields a diagonalization for every diagonalizable function.

THEOREM 2. *Let U be a neighborhood of 0 in $\mathbb{C}$, and let $x \in H(U, \mathfrak{M})$ be diagonalizable at 0. Then a diagonalization of x can be constructed by finitely many applications of the procedure given in Theorem 1.*

Proof. By assumption there are a neighborhood $U_1 \subset U$ of 0 and $a, b \in H(U_1, \mathfrak{M}^0)$ such that

$$axb(z) = \sum_{j=0}^{k} x_j z^j$$

where $x_j \in \mathrm{CR}\,\mathfrak{M}$, $0 \leq j \leq k$, and $(l(x_j))_{j=0}^k, (r(x_j))_{j=0}^k$ are families of mutually orthogonal projections. We may assume $x_0 \neq 0$, and we have $x(0) = a(0)^{-1}x_0 b(0)^{-1} \in \mathrm{CR}\,\mathfrak{M}$. Thus we can apply the algorithm of Theorem 1 at least once. This means that we can find a neighborhood $U_2 \subset U_1$ of 0 and $c, d \in H(U_2, \mathfrak{M}^0)$ such that

$$c(z)\left(\sum_{j=0}^{k} x_j z^j\right)d(z) = \sum_{i=0}^{l} y_i z^i + y_{l+1}(z) z^{l+1},$$

where $y_i \in \mathrm{CR}\,\mathfrak{M}$, $0 \leq i \leq l-1$; $y_l \in \mathfrak{M}$; $y_{l+1} \in H(U_2, \mathfrak{M})$; $(l(y_i))_{i=0}^l, (r(y_i))_{i=0}^l$ are families of mutually orthogonal projections; and finally $l(y_i)y_{l+1}(z) = y_{l+1}(z)r(y_i) = 0$ for $0 \leq i \leq l-1$. Writing

$$c(z) = \sum_{j=0}^{\infty} c_j z^j, \qquad d(z) = \sum_{j=0}^{\infty} d_j z^j,$$

we have

$$y_j = \sum_{\substack{r+s+t=j \\ r,s,t \geq 0}} c_r x_s d_t, \qquad 0 \leq j \leq \bar{k} := \min\{k, l\}. \tag{8}$$

Since $c_0, d_0 \in \mathfrak{M}^0$, it follows easily by induction on j that for $r+s \leq \bar{k}$, $s+t \leq \bar{k}$,

$$l(c_r x_s) \leq \sum_{i=0}^{r+s} l(y_i), \qquad r(x_s d_t) \leq \sum_{i=0}^{s+t} r(y_i); \tag{9}$$

therefore

$$y_j = l(y_j) c_0 x_j d_0 r(y_j), \qquad 0 \leq j \leq \bar{k}. \tag{10}$$

We now want to show that

$$l\big(c_0 l(x_j)\big) = l(y_j), \quad r\big(r(x_j) d_0\big) = r(y_j), \qquad 0 \leq j \leq \bar{k}. \tag{11}$$

By (10) this implies

$$y_j = c_0 x_j d_0, \qquad 0 \leqslant j \leqslant \bar{k}, \tag{12}$$

which obviously implies the theorem.

We prove the first set of equalities in (11), the proof of the second one being analogous. To do so we put

$$p_j := \sum_{i=0}^{j} l(x_i), \quad q_j := l(c_0 p_j), \qquad 0 \leqslant j \leqslant \bar{k},$$

and note that it is sufficient to prove

$$q_j = \sum_{i=0}^{j} l(y_i) \quad \text{and} \quad r(q_j c_0) = p_j. \tag{13}$$

In fact from (13) we get $(1_{\mathfrak{M}} - q_j)c_0 l(x_j) = c_0 l(x_j) - q_j c_0 p_j l(x_j) = 0$ and $q_{j-1} c_0 l(x_j) = q_{j-1} c_0 p_{j-1} l(x_j) = 0$; thus

$$\begin{aligned} l(y_j) c_0 l(x_j) &= c_0 l(x_j) - (1_{\mathfrak{M}} - q_j) c_0 l(x_j) - q_{j-1} c_0 l(x_j) \\ &= c_0 l(x_j), \end{aligned}$$

i.e., $l(c_0 l(x_j)) \leqslant l(y_j)$. This together with (10) gives

$$\begin{aligned} l(c_0 l(x_j)) &\geqslant l(c_0 l(x_j) x_j d_0 r(y_j)) = l(l(y_j) c_0 x_j d_0 r(y_j)) \\ &= l(y_j), \end{aligned}$$

and hence (11). To prove the first equality in (13) we observe that by the Lemma and (9) we have

$$q_j = l\left(\sum_{i=0}^{j} c_0 x_j\right) \leqslant \sum_{i=0}^{j} l(y_i).$$

Obviously $q_0 = l(c_0 x_0) = l(c_0 x_0 d_0) = l(y_0)$. From (8) and (9) again we get

$$y_j = \sum_{s+t=j} c_0 x_s d_t r(y_j) = \sum_{s+t=j} c_0 p_j x_s d_t r(y_j), \tag{14}$$

implying

$$l(y_j) \leqslant q_j \quad \text{and} \quad \sum_{i=0}^{j} l(y_i) \leqslant q_j,$$

since $q_j \geqslant q_{j-1}$. Interchanging the roles of the y_i's and x_i's in the above argument, we find for $0 \leqslant j \leqslant k$

$$\sum_{i=0}^{j} l(x_i) = l\left(c_0^{-1} \sum_{i=0}^{j} l(y_i)\right) = r\left(\sum_{i=0}^{j} l(y_i)c_0\right),$$

which is the second equality in (13). The proof is complete. ∎

To conclude we single out the uniqueness property of diagonalizations expressed in (12).

Corollary. *Let $U \subset \mathbb{C}$ be a neighborhood of* 0, *and let $x \in H(U, \mathfrak{M})$ be diagonalizable at* 0. *Suppose there exist two diagonal representations of x at* 0,

$$axb(z) = \sum_{j=0}^{k} x_j z^j, \qquad a'xb'(z) = \sum_{j=0}^{k} x'_j z^j.$$

Then we can find $c, d \in \mathfrak{M}^0$ such that

$$x'_j = c x_j d, \qquad 0 \leqslant j \leqslant k.$$

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