Arch. Math., Vol. 42, 253–259 (1984) 0003-889X/84/4203-0253 $ 3.90/0

On the asymptotic expansion of some integrals

By

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1. Consider the integral

(1) $$I(s) := \int_0^1 g(x/s)\, f(x)\, dx$$

where $g \in \mathscr{S}(\mathbb{R})$, the Schwartz space, and $s > 0$. If $f \in C^\infty(\mathbb{R})$ then it is easy to calculate the asymptotic expansion of $I(s)$ as $s \to 0$:

$$I(s) \sim \sum_{j \geqq 0} s^{j+1} \frac{f^{(j)}(0)}{j!} \int_0^\infty g(x)\, x^j\, dx.$$

Asymptotic expansions for integrals of this type are needed in many places (e.g. in the theory of special functions) and consequently there is a vast literature on this subject (a rather comprehensive treatment can be found in [6]). One is mainly concerned with weakening the assumptions on f i.e. one replaces the Taylor expansion of f at 0 by more general asymptotic expansions involving for example real powers of x and integer powers of $\log x$.

The multidimensional versions of (1) i.e. integrals of the type

$$J(s) := \int_{[0,1]^n} g(\varphi(x)/s)\, f(x)\, dx, \qquad x = (x_1, \ldots, x_n),\ s > 0,$$

do not seem to have attracted a similar interest. If φ is assumed to be real analytic there is a systematic treatment, however, based on the coarea formula and the resolution of singularities (cf. [3], Ch. III, [4], [5]). Thus one reduces essentially to the case

(2) $$\varphi(x) = x_1^{\alpha_1} \ldots x_n^{\alpha_n},\ \alpha_i \in \mathbb{Z}_+, \qquad 1 \leqq i \leqq n.$$

Integrals of this type play an important role in the asymptotic expansion of the trace of the equivariant heat kernel [2] which lead us to look for an elementary real variable method for their asymptotic expansion. The method to be presented in this note turned out to work in much greater generality, namely we can allow certain singularities for f and, more importantly, the α_i in (2) *can be chosen arbitrarily in* $\mathbb{R}^+ := [0, \infty)$. Related integrals have been treated recently by Barlet [1] in the analysis of complex spaces. As a special case we obtain a short and simple proof of his result (Corollary to Theorem 2).

2. Before presenting our result we have to prepare some notation. For $x \in \mathbb{R}^n$ and $\alpha \in \mathbb{R}^n_+$ we will write

$$x^\alpha := x_1^{\alpha_1} \dots x_n^{\alpha_n}$$

and if $x_i > 0$, $1 \leqq i \leqq n$,

$$\log^\alpha x := \log^{\alpha_1} x_1 \dots \log^{\alpha_n} x_n.$$

For $f \in C^\infty(\mathbb{R}^n)$, g in the Schwartz space $\mathscr{S}(\mathbb{R}^n)$, $s > 0$, $\alpha, \beta \in \mathbb{R}^n_+$ and $\gamma \in \mathbb{Z}^n_+$ the integral

$$I^{\alpha\beta\gamma}(s,f) := \int_{[0,1]^n} g(x^\alpha/s)\, x^\beta \log^\gamma x\, f(x)\, dx$$

is well defined. To avoid trivial cases we will require

$$\alpha_i > 0, \quad 1 \leqq i \leqq n.$$

Let us write $K_n := [0,1]^n$. Also, since we are not going to vary the parameters α, β, γ, we will suppress the dependence of I on these variables. We want to derive an asymptotic expansion for I as $s \to 0$ but we are also interested in the dependence of the coefficients on f. Since I defines a distribution with support in K_n, $I(s,.) \in \mathscr{E}'(K_n)$ for fixed $s > 0$, it is reasonable to look for an asymptotic expansion of I in $\mathscr{E}'(K_n)$. Thus for a sequence $(\nu_n)_{n\in\mathbb{N}} \subset \mathbb{R}$ with $\lim_{n\to\infty} \nu_n = \infty$, $m \in \mathbb{Z}_+$, and distributions $I_{jk} \in \mathscr{E}'(K_n)$ we will write

$$I(s,f) \sim \sum_{\substack{j \geqq 1 \\ 0 \leqq k \leqq m}} s^{\nu_j} \log^k s\, I_{jk}(f)$$

as $s \to 0$ in $\mathscr{E}'(K_n)$ if the following is true: there are sequences $(\mu_l)_{l\in\mathbb{N}} \subset \mathbb{R}$ with $\lim_{l\to\infty} \mu_l = \infty$ and $(\lambda_l)_{l\in\mathbb{N}} \subset \mathbb{Z}_+$ such that for $N \in \mathbb{N}$ and $0 < s < 1$

$$(3) \qquad \Big|I(s,f) - \sum_{\substack{\nu_j < N \\ 0 \leqq k \leqq m}} s^{\nu_j} \log^k s\, I_{jk}(f)\Big| \leqq C_N s^{\mu_N} \sup_{\substack{x \in K_n \\ \eta \in \mathbb{Z}^n_+,\, |\eta| \leqq \lambda_N}} |\partial_x^\eta f(x)|$$

with some positive constant C_N. Here we have used the multiindex notation

$$|\eta| = \eta_1 + \dots + \eta_n \quad \text{and} \quad \partial_x^\eta f(x) = \frac{\partial^{|\eta|} f}{\partial x_1^{\eta_1} \dots \partial x^{\eta_n}}(x)$$

for $\eta \in \mathbb{Z}^n_+$.

We can now formulate our result.

Theorem 1. *Let $\alpha, \beta \in \mathbb{R}^n_+$ with $\alpha_i > 0$, $1 \leqq i \leqq n$, and $\gamma \in \mathbb{Z}^n_+$. For $g \in \mathscr{S}(\mathbb{R}^n)$ and $f \in C^\infty(\mathbb{R}^n)$ we have*

$$\int_{K_n} g(x^\alpha/s)\, x^\beta \log^\gamma x\, f(x)\, dx \sim \sum_{\substack{j \geqq 0 \\ 0 \leqq k \leqq |\gamma| + n - 1 \\ 1 \leqq l \leqq n}} s^{(\beta_l + j + 1)/\alpha_l} \log^k s\, I_{jkl}(f)$$

as $s \to 0$ asymptotically in $\mathscr{E}'(K_n)$. The I_{jkl} are distributions with support in the set $\{x \in K_n \mid x^\alpha = 0\}$.

The proof of Theorem 1 will be done by induction on n and occupy Sections 3 through 7.

3. We start with the case $n = 1$. Using the substitution $u := s^{-1/\alpha} x$ and Taylor's formula we find

$$\begin{aligned} I(s,f) &= \int_0^1 g(x^\alpha/s)\, x^\beta \log^\gamma x\, f(x)\, dx \\ &= s^{(\beta+1)/\alpha} \sum_{j=0}^{\gamma} \binom{\gamma}{j} \alpha^{-j} \log^j s \int_0^{s^{-1/\alpha}} g(u^\alpha)\, u^\beta \log^{\gamma-j} u\, f(s^{1/\alpha} u)\, du \\ &\sim \sum_{\substack{i \geqq 0 \\ 0 \leqq j \leqq \gamma}} s^{(\beta+1+i)/\alpha} \log^j s\, \frac{f^{(i)}(0)}{i!} \binom{\gamma}{j} \alpha^{-j} \int_0^\infty g(u^\alpha)\, u^{\beta+j} \log^{\gamma-j} u\, du \end{aligned}$$

as $s \to 0$ in $\mathscr{E}'([0,1])$.

4. Now suppose the theorem is true for all dimensions m, $1 \leqq m \leqq n-1$. Before treating the general case we deal with the special case that f is independent of x_n. Using the notation $x =: (x', x_n)$ for $x \in \mathbb{R}^n$ and assuming $0 < s \leqq 1$ we write

$$\begin{aligned} I(s,f) &= \left(\int_0^{s^{1/\alpha_n}} + \int_{s^{1/\alpha_n}}^1 \right) \int_{K_{n-1}} g(x^\alpha/s)\, x^\beta \log^\gamma x\, f(x')\, dx'\, dx_n \\ &=: I_1(s,f) + I_2(s,f). \end{aligned}$$

The substitution $y_n := s^{-1/\alpha_n} x_n$, $y' := x'$ yields

$$I_1(s,f) = \sum_{j=0}^{\gamma_n} s^{(\beta_n+1)/\alpha_n} \log^k s\, \alpha_n^{-k} \binom{\gamma_n}{k} \int_{K_n} g(y^\alpha)\, y^\beta \log^{\gamma'} y' \log^{\gamma_n - k} y_n\, f(y')\, dy.$$

Substituting $x_n =: s^{1/\alpha_n} u^{-1}$ in I_2 we find

$$\begin{aligned} I_2(s,f) = s^{(\beta_n+1)/\alpha_n} &\sum_{j=0}^{\gamma_n} \binom{\gamma_n}{j} (-1)^{\gamma_n - j} \alpha_n^{-j} \log^j s \\ &\cdot \int_{s^{1/\alpha_n}}^1 \log^{\gamma_n - j} u\; u^{-\beta_n - 2}\, I'(u^{\alpha_n}, f)\, du \end{aligned}$$

where

$$I'(t,f) := \int_{K_{n-1}} g(x'^{\alpha'}/t)\, x'^{\beta'} \log^{\gamma'} x'\, f(x')\, dx', \qquad 0 < t \leqq 1.$$

Using the induction hypothesis on I' we write with N sufficiently large

$$(4) \qquad I''(u^{\alpha_n}, f) := \sum_{\substack{0 \leqq j \leqq N \\ 0 \leqq k \leqq |\gamma'| + n - 2 \\ 1 \leqq l \leqq n-1}} u^{(\beta_l + j + 1)\alpha_n/\alpha_l} \log^k u^{\alpha_n}\, I_{jkl}(f)$$

and

$$\int_{s^{1/\alpha_n}}^{1} \log^{\gamma_n - j} u\, u^{-\beta_n - 2}\, I'(u^{\alpha_n}, f)\, du$$

$$= \left(\int_0^1 - \int_0^{s^{1/\alpha_n}}\right) \log^{\gamma_n - j} u\, u^{-\beta_n - 2} (I' - I'')(u^{\alpha_n}, f)\, du$$

$$+ \sum_{\substack{0 \leqq j \leqq N \\ 0 \leqq k \leqq |\gamma'| + n - 2 \\ 1 \leqq l \leqq n-1}} \alpha_n^k I_{jkl}(f) \int_{s^{1/\alpha_n}}^{1} \log^{\gamma_n - j + k} u\, u^{-\beta_n - 2 + (\beta_l + j + 1)\alpha_n/\alpha_l}\, du$$

$$=: (J_1 + J_2 + J_3)(s, f).$$

Certainly, J_1 is a distribution in $\mathscr{E}'(K_n)$ independent of s. The elementary formula, valid for $k \in \mathbb{Z}_+$, $\alpha \in \mathbb{R}$, and $u > 0$,

$$\text{(5)} \qquad \int \log^k u\, u^\alpha\, du = \begin{cases} u^{\alpha+1} \sum\limits_{l=0}^{k} c_{\alpha k l} \log^l u, & \alpha \neq -1, \\ \dfrac{\log^{k+1} u}{k+1}, & \alpha = -1, \end{cases}$$

shows that $s^{(\beta_n+1)/\alpha_n} J_3(s, f)$ has a (finite) asymptotic expansion of the desired form. Finally, if N is large enough $(I' - I'')(u^{\alpha_n}, f)$ has an asymptotic expansion of the form (4) by hypothesis but involving positive powers of u only. Using (5) again and an obvious remainder estimate we conclude that $J_3(s, f)$ has an asymptotic expansion of the announced form, too.

5. We now describe a technique that will enable us to reduce the general case to the special case already settled. We introduce a new variable $\varepsilon \in (0, 1)$ and put $\sigma := s/\varepsilon^{\alpha_n}$. Writing $I(s, f) = I(\varepsilon^{\alpha_n}\sigma, f)$ we shall show that

$$\text{(6)} \qquad I(\varepsilon^{\alpha_n}\sigma, f) \sim \sum_{\substack{j, j' \geqq 0 \\ 0 \leqq k + k' \leqq |\gamma| + n - 1 \\ 1 \leqq l, l' \leqq n}} \varepsilon^{(\beta_l + j + 1)\alpha_n/\alpha_{l'}} \sigma^{(\beta_{l'} + j' + 1)/\alpha_{l'}} \log^k \varepsilon \log^{k'} \sigma\, I_{jj'kk'll'}(f)$$

asymptotically in $\mathscr{E}'(K_n)$ as $\varepsilon^2 + \sigma^2 \to 0$. The two-variable expansion is defined similar to the one-variable case the only difference being that in the remainder estimate (3) we replace s by $(\varepsilon^2 + \sigma^2)^{1/2}$. It is easy to see that the coefficients $I_{jj'kk'll'}(f)$ are uniquely determined. Now we substitute in (6) $\varepsilon^{\alpha_n} = s^{1/2} = \sigma$, $0 < s < 1$. Then we obtain an expansion for $I(s, f)$ in $\mathscr{E}'(K_n)$ as $s \to 0$ involving the functions $\log^k s\, s^{\lambda_{jj'll'}}$ where $0 \leqq k \leqq |\gamma| + n - 1$ and $\lambda_{jj'll'} := (\beta_{l'} + j + 1)/_{2\alpha_l} + (\beta_{l'} + k' + 1)/_{2\alpha_{l'}}$. Resubstituting $s := \varepsilon^{\alpha_n}\sigma$ we obtain a second two-variable expansion for $I(\varepsilon^{\alpha_n}\sigma, f)$ and comparing the coefficients we see that nonzero coefficients occur only if the exponents of ε^{α_n} and σ are equal. This concludes the proof of Theorem 1.

6. To apply the technique described above we write

$$I(s, f) = \left(\int_0^\varepsilon + \int_\varepsilon^1\right) x_n^{\beta_n} \log^{\gamma_n} x_n \int_{K_{n-1}} g(x^\alpha/s)\, x'^{\beta'} \log^{\gamma'} x'\, f(x', x_n)\, dx'\, dx_n$$

$$=: I_1(\varepsilon, \sigma, f) + I_2(\varepsilon, \sigma, f).$$

Substituting $x_n =: \varepsilon y_n$, $x' =: y'$ we have with $\gamma_j := (\gamma', j)$

$$I_1(\varepsilon, \sigma, f) = \sum_{j=0}^{\gamma_n} \binom{\gamma_n}{j} \varepsilon^{\beta_n + 1} \log^{\gamma_n - j} \varepsilon \int_{K_n} g(y^\alpha/\sigma)\, y^\beta \log^{\gamma_j} y\, f(y', \varepsilon y_n)\, dy.$$

Replacing $f(y', \varepsilon y_n)$ by its Taylor series around $(y', 0)$ we certainly obtain an asymptotic expansion in $\mathscr{E}'(K_n)$ as $\varepsilon^2 + \sigma^2 \to 0$. Thus

$$I_1(\varepsilon, \sigma, f) \sim \sum_{\substack{i \geqq 0 \\ 0 \leqq j \leqq \gamma_n}} \binom{\gamma_n}{j} \frac{1}{i!} \varepsilon^{\beta_n + i + 1} \log^{\gamma_n - j} \varepsilon \int_{K_n} g(y^\alpha/s)\, y^\beta y_n^i \log^{\gamma_j} y\, \partial_{y_n}^i f(y', 0)\, dy.$$

Applying the result of Section 4 we obtain an expansion of the type (6) for $I_1(\varepsilon, \sigma, f)$.

7. On the support of the integrand in $I_2(\varepsilon, \sigma, f)$ we have $s/x^{\alpha_n} \leqq \sigma$. Therefore, we can apply the induction hypothesis to obtain the following asymptotic expansion in $\mathscr{E}'(K_n)$ as $\varepsilon^2 + \sigma^2 \to 0$

$$\text{(7)} \qquad I_2(\varepsilon, \sigma, f) \sim \sum_{\substack{j \geqq 0 \\ 0 \leqq k \leqq |\gamma'| + n - 2 \\ 0 \leqq k' \leqq k \\ 1 \leqq l \leqq n-1}} (\sigma \varepsilon^{\alpha_n})^{(\beta_l + j + 1)/\alpha_l} \log^{k'}(\sigma \varepsilon^{\alpha_n}) \binom{k}{k'} (-\alpha_n^{-1})^{k-k'} \cdot \int_\varepsilon^1 x_n^{\beta_n - (\beta_l + j + 1)\alpha_n/\alpha_l} \log^{\gamma_n + k - k'} x_n\, I_{jkl}(f(\cdot, x_n))\, dx_n.$$

Since by hypothesis the I_{jkl} are distributions with support in K_{n-1} we have

$$\partial_{x_n}^k I_{jkl}(f(\cdot, x_n)) = I_{jkl}(\partial_{x_n}^k f(\cdot, x_n)), \qquad k \geqq 0.$$

Using a sufficiently long part of the Taylor series of $I_{jkl}(f(\cdot, x_n))$ around $x_n = 0$ we obtain asymptotic expansions of the integrals in (7) as in Section 4. These contain negative powers of ε the smallest exponent being $\beta_n + 1 - (\beta_l + j + 1)\alpha_n/\alpha_l$ which implies for the whole sum an expansion of the type announced. This expansion is clearly asymptotic in $\mathscr{E}'(K_n)$ as $\varepsilon^2 + \sigma^2 \to 0$. Hence we have proved the expansion in Theorem 1 with $I_{jkl} \in \mathscr{E}'(K_n)$. However, if $x^\alpha \neq 0$ for $x \in \operatorname{supp} f \cap K_n$ then clearly $I(s, f) = O_{N, f}(s^N)$ for every $N \in \mathbb{N}$ proving that $\operatorname{supp} I_{jkl} \subset \{x \in K_n \mid x^a = 0\}$.

8. We finally indicate a variant of Theorem 1 which implies the result of Barlet mentioned above.

Theorem 2. *Let $g \in C^\infty(\mathbb{R})$ satisfy $g = 0$ in a neighborhood of 0 and $g = 1$ in a neighborhood of ∞. Let $\alpha, \beta \in \mathbb{R}^n$ with $\alpha_i > 0$, $1 \leqq i \leqq n$, and $\gamma \in \mathbb{Z}^n_+$. For $f \in C^\infty(\mathbb{R}^n)$ we have*

$$\int_{K_n} g(x^\alpha/s)\, x^\beta \log^\gamma x\, f(x)\, dx \sim J(f) + \sum_{\substack{j \geqq 0 \\ 0 \leqq k \leqq |\gamma| + n \\ 1 \leqq l \leqq n}} s^{(\beta_l + j + 1)/\alpha_l} \log^k s\, I_{jkl}(f)$$

asymptotically in $\mathscr{E}'(K_n)$ as $s \to 0$. Here $J, I_{jkl} \in \mathscr{E}'(K_n)$.

P r o o f. The proof is done by induction on the number p of negative entries in β. If $p = 0$ we write

$$\int_{K_n} g(x^\alpha/s)\, x^\beta \log^\gamma x\, f(x)\, dx = \int_{K_n} [1 + (g-1)(x^\alpha/s)]\, x^\beta \log^\gamma x\, f(x)\, dx,$$

and since $g - 1 \in \mathscr{S}(\mathbb{R})$ the result follows from Theorem 1 in this case.

In the inductive step assume $\beta_n < 0$. Introducing a sufficiently long part of the Taylor series of f with respect to x_n around $(x', 0)$ we split the integral into a sum. Then to a term in the sum we can either apply the induction hypothesis or we have an integral with f independent of x_n. The expansion of these is obtained repeating the arguments of Section 4. □

Corollary (Barlet [1], Proposition 2). *Let $\alpha, \beta \in \mathbb{R}^n$ with $\alpha_i > 0$, $1 \leqq i \leqq n$, and put $\alpha_0 = \beta_0 = 1$. Let $g \in C_0^\infty(\mathbb{R})$ satisfy $g = 1$ in a neighborhood of 0. Then for $f \in C_0^\infty(\mathbb{R}^n)$, $\operatorname{supp} f \subset K_n$ we have*

$$\int_{K_n} g(s/x_1 \ldots x_n)\, x^{-\beta} f(x^\alpha)\, dx \sim \sum_{\substack{j \geqq 0 \\ 0 \leqq k \leqq n \\ 1 \leqq l \leqq n}} s^{1 + j\alpha_l - \beta_l} \log^k s\, I_{jkl}(f) + J(f)$$

asymptotically in $\mathscr{D}'(\mathbb{R}^n)$ as $s \to 0$.

P r o o f. Put $\tilde{g}(x) := g(\frac{1}{x})$, $x \in \mathbb{R} \setminus \{0\}$. Then $\tilde{g}$ extends to a smooth function on $\mathbb{R}$ satisfying $\tilde{g} = 0$ in a neighborhood of 0 and $\tilde{g} = 1$ in a neighborhood of ∞. Substituting $y_i := x^{\alpha_i}$, $1 \leqq i \leqq n$, the integral becomes

$$\prod_{i=1}^{n} \alpha_i^{-1} \int_{K_n} \tilde{g}(y^{\bar{\alpha}}/s)\, y^{\bar{\beta}} f(y)\, dy$$

where $\bar{\alpha}_i := 1/\alpha_i$, $\bar{\beta}_i := 1/\alpha_i(1 - \alpha_i - \beta_i)$, $1 \leqq i \leqq n$. The assertion follows from Theorem 2. □

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Eingegangen am 12. 2. 1982

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