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Heat Equation Asymptotics for Singular Sturm-Liouville Operators

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1. Introduction

Let $a \in C^\infty(\mathbb{R})$ be real valued and consider the differential expression

$$\tau := -d_x^2 + \frac{a(x)}{x^2(1-x)^2} =: -d_x^2 + \tilde{a}(x) \tag{1.1}$$

in $I := (0, 1)$. If

$$a(i) \geqq -1/4, \quad i = 0, 1, \tag{1.2}$$

the Friedrichs extension T of $\tau | C_0^\infty(I)$ exists in $L^2(I)$ and has a pure point spectrum. In this paper we study the asymptotic expansion of $\mathrm{tr}\, e^{-sT}$ as $s \to 0+$. If the potential is smooth in $\bar{I}$ it is well known that

$$\mathrm{tr}\, e^{-sT} \sim (4\pi s)^{-1/2} \sum_{j \geqq 0} s^{j/2}(A_j + B_j) \tag{1.3}$$

(see e.g. [9]) where B_j is a universal polynomial in the variables $\tilde{a}^{(k)}(0)$, $\tilde{a}^{(l)}(1)$, k, $l \geqq 0$, and

$$A_{2j} = \int_I w_j(x)dx, \quad A_{2j+1} = 0,$$

where w_j is again a universal polynomial in the variables $\tilde{a}^{(k)}$, $k \geqq 0$, in particular $w_0(x) \equiv 1$. The proof of (1.3) starts with an asymptotic expansion of the kernel Γ_s of e^{-sT} on the diagonal, namely

$$\Gamma_s(x, x) \sim (4\pi s)^{-1/2} \sum_{j \geqq 0} s^j w_j(x), x \in I. \tag{1.4}$$

This determines already the coefficients A_j while the influence of the boundary is reflected in the B_j's. Now in the singular case we meet the principal difficulty that in the analogous expansion the coefficients are no longer integrable over I (see Theorem 4.3 below). However, the existence of an expansion generalizing (1.3) is suggested by the following example [2]. Let M be a compact surface of revolution homeomorphic to S^2 in $\mathbb{R}^3$, generated by a smooth curve $c(t) = (c_1(t), 0, c_2(t))$ of length 1. M carries a natural S^1 action and $-\Delta|L^2(M)^{S^1}$ is unitarily equivalent to the Friedrichs extension T in $L^2(I)$ of $\tau|C_0^\infty(I)$ where

$$\begin{aligned}\tau := -d_x^2 + \frac{2c_1(x)c_1''(x) - c_1'(x)^2}{4c_1(x)^2} \\ =: -d_x^2 + \frac{a(x)}{x^2(1-x)^2}.\end{aligned}$$

Assuming that c is parametrized with arc length we must have

$$\begin{aligned}c_1^{(2k)}(0) = c_1^{(2k)}(1) = 0, \quad k \geqq 0, \\ c_1'(0) = -c_1'(1) = 1.\end{aligned}$$

Hence $a \in C^\infty(\bar{I})$ and we can write for $i = 0, 1$

$$\frac{a(x)}{x^2(1-x)^2} = -\frac{1}{4(x-i)^2} + \frac{b_i(i-x)}{i-x}$$

as $x \to i$ where b_i is smooth and odd at i, i.e. $b_i^{(2k)}(i) = 0$, $k \geqq 0$. The existence of an asymptotic expansion of $\operatorname{tr} e^{-sT}$ is hence a special case of the result of [4] dealing with the equivariant heat kernel on Riemannian G-manifolds. In general, this expansion will contain also logarithmic terms which are, however, absent in the most natural examples and also in the case at hand, due to the special nature of S^1 actions [4, Theorem 7]. But the possibility of logarithmic terms even for one-dimensional problems is indicated by 'tHooft's operator this being the Friedrichs extension Q in $L^2(I)$ of

$$\begin{aligned}Pu(x) := 1/2 \int_{-\infty}^{+\infty} e^{ix\xi}|\xi|\hat{u}(\xi)\,d\xi + \left(\frac{a}{x} + \frac{b}{1-x}\right)u(x), \\ x \in I, \quad u \in C_0^\infty(I), \quad a, b \geqq -1.\end{aligned}$$

In fact, it has been shown in [3] that

$$\operatorname{tr} e^{-sQ} = \frac{1}{\pi^2 s} + \frac{a+b}{\pi^2}\log s + 0(1), \quad s \to 0.$$

On the basis of this evidence we conjectured that logarithmic terms in the expansion (1.3) can be produced by the "residues" of the potential at 0 and 1 i.e. by $b_i(0)$ if we write

$$\frac{a(x)}{x^2(1-x)^2} = \frac{a_i}{(i-x)^2} + \frac{b_i(i-x)}{i-x} \quad \text{as} \quad x \to i, \tag{1.5}$$

whereas they should not occur if b_0, b_1 are odd at 0. Besides this question we were interested to see how the structure of the coefficients for smooth potentials generalizes to the singular case because this might serve as a model in more complicated singular situations like orbit spaces.

We now present our result.

Main Theorem. *Consider the differential expression* τ *defined by* (1.1) *and satisfying* (1.2). *Assume that* $a_i > -1/4$ *or* $a_i = -1/4$ *and* $b_i(0)=0$, $i=0,1$. *Let* T *be the Friedrichs extension of* $\tau|C_0^\infty(I)$ *in* $L^2(I)$. *Then* e^{-sT} *is trace class for* $s>0$ *and*

$$\operatorname{tr} e^{-sT} \sim (4\pi s)^{-1/2} \sum_{j \geqq 0} s^{j/2}(A_j + B_j + C_j \log s).$$

Here

$$A_0 = 1, \quad A_{2j+1} = 0,$$

$$A_{2j} = \text{p.f.} \int_I \frac{w_{2j}(x)}{x^{2j}(1-x)^{2j}} dx, \quad j \geqq 0,$$

where w_{2j} *is a universal polynomial in the variables* $a^{(k)}$, $k \geqq 0$, *and the "partie fini" is calculated by integrating over* $(\varepsilon, 1-\varepsilon)$ *and taking the constant term in the asymptotic expansion as* $\varepsilon \to 0$. *The coefficients* B_j *and* C_j *depend universally on* $a^{(k)}(0)$, $a^{(l)}(1)$, $k, l \geqq 0$ *(the precise formulas are* (6.4), (6.5)*), in particular*

$$B_0 = C_0 = C_{2j+1} = 0, \quad j \geqq 0,$$

and using (1.5)

$$C_2 = -\frac{b_0(0) + b_1(0)}{8\sqrt{\pi}}.$$

If, however, b_0 *and* b_1 *are odd at* 0 *then* $C_j = 0$ *for all* j.

The case $a(x) = a(0)\,(1-x)^2$ can be treated rather explicitly using Bessel functions. This has done in [6] and with a different point of view in [5]. Cheeger's work provided the additional motivation to develop a method for the general case not using special functions at all. Thus we treat the given operator as a perturbation of the Euler operator just mentioned. But instead of exploiting the explicit formula for its heat kernel we use only the most elementary properties of equations with a regular singular point together with the invariance property of the Euler operator under the natural action of $\mathbb{R}_* := (0, \infty)$ on $L^2(\mathbb{R}_+)$, $\mathbb{R}_+ := [0, \infty)$, which is the guiding principle for the treatment of the perturbation series, too. To derive the structure of the coefficients we use an expansion of the trace with respect to two (artificial) variables ε and σ satisfying $s = \varepsilon^2 \sigma$. This allows us to use Taylor series near the singularities and to exhibit the "partie fini" terms.

The plan of the paper is as follows. In Sect. 2 we recall some known facts on the Friedrichs extension of singular second order operators. Section 3 collects the necessary estimates, Sect. 4 contains pointwise asymptotic expansions of several kernels, and in Sect. 5 we derive the perturbation series. Finally, the Main Theorem is proved in Sect. 6.

2. Selfadjoint Extensions

We briefly recall some well known facts about selfadjoint operators T in $L^2(I)$ generated by τ. We introduce the spaces

$W := \{f \in C^1(I) | f' \text{ is absolutely continuous on compact subsets of } I\}$,

$W_0 := \{f \in W | \operatorname{supp} f \text{ is compact in } I\}$,

$W_\tau := \{f \in W | f, \tau f \in L^2(I)\}$.

Put $T_0(\tau):=\tau|W_0$ and $T_1(\tau):=\tau|W_\tau$. Then $T_0(\tau)^*=T_1(\tau)$ implying that τ is symmetric. To describe the behavior of τ at the singular endpoints we write [cf. (1.5)]

$$\frac{a(x)}{x^2(1-x)^2}=:\frac{a_i}{(i-x)^2}+\frac{b_i}{(i-x)}+0(1), \quad x\to i, \quad i=0,1.$$

Then τ is bounded below on W_0 iff this is the case for

$$\tau_i:=-d_x^2+\frac{a_i}{x^2}+\frac{b_i}{x}$$

for $i=0$ and $i=1$ [11, Satz 2d]. But from [11, Hilfssatz 2] we see that this is the case iff the roots of the index equation

$$-\lambda(\lambda-1)+a_i=0,$$

i.e. $\lambda_\pm(a_i):=1/2\pm\sqrt{a_i+1/4}$, are both real which means that we must have $a_i\geqq -1/4$ for $i=0,1$. In view of [11, Satz 4] we can also say that all selfadjoint extensions of $T_0(\tau)$ are bounded below iff $a_i\geqq -1/4$ for $i=0,1$; in particular, the Friedrichs extension T does exist. We will assume that this is the case in the following. If we assume $\alpha:=\min\{a_0,a_1\}>-1/4$ then it follows from Hardy's inequality [7, p. 532] that

$$\|u\|_{H^1(I)}\leqq C((T_0(\tau)u|u)+\|u\|^2_{L^2(I)}), \quad u\in W_0, \tag{2.1}$$

where $H^1(I)$ denotes the usual Sobolev space, $(\cdot|\cdot)$ is the scalar product in $L^2(I)$, and C depends on α only. Of course, (2.1) holds without the restriction on a_0, a_1 with a constant C depending also on the distance of $\operatorname{supp} u$ and ∂I. The selfadjoint extensions of $T_0(\tau)$ are given by restriction of $T_1(\tau)$ to subspaces of W_τ which in turn can be described by boundary conditions at 0 and 1; according to the number of solutions of $(\tau+\sqrt{-1})\,u=0$ square integrable at 0 or 1 there are either two or no boundary conditions at 0 or 1 which is referred to as the limit circle case and the limit point case, respectively (cf. [7, Chap. XIII, Corollary 31]).

For the Friedrichs extension T the boundary conditions can be described very explicitly (cf. [11, Sect. 3]). For $i=0,1$ we can find $\mu_i\in\mathbb{R}$ and a fundamental system ω_i, ψ_i of solutions of $(\tau-\mu_i)\,u=0$ such that ω_i and ψ_i do not vanish in a pointed neighborhood of i and in addition $\lim\limits_{x\to i}\omega_i(x)\psi_i(x)^{-1}=0$. The domain of T is then given by

$$\mathscr{D}(T)=\{u\in W_\tau|\lim_{x\to i}u(x)\psi_i(x)^{-1}=0, i=0,1\}.$$

The same discussion applies mutatis mutandis to the differential expressions

$$\bar\tau_i:=-d_x^2+\frac{a_i}{x^2}$$

and their Friedrichs extensions $\bar T_i$ in $L^2(\mathbb{R}_+)$, $i=0,1$. It is easily checked that with

$$\psi_a(x):=\begin{cases}x^{\lambda_-(a)}, & a>-1/4,\\ x^{1/2}\log x, & x=-1/4,\end{cases}$$

we have

$$\mathscr{D}(\bar{T}_i)=\{u\in W_{\bar{\tau}_i}\,|\lim_{x\to 0} u(x)\psi_{a_i}(x)^{-1}=0\}.$$

The operators $\bar{T}_i$ have a remarkable invariance property under the natural unitary action of $\mathbb{R}_*$ on $L^2(\mathbb{R}_+)$, given by

$$U_\alpha f(x):=\alpha^{1/2}f(\alpha x),\qquad \alpha\in\mathbb{R}_*,\qquad f\in L^2(\mathbb{R}_+),\qquad x\in\mathbb{R}_+.$$

Namely, for $f\in W_{\bar{\tau}_i}$ we find

$$\bar{\tau}_i U_\alpha f=\alpha^2 U_\alpha \bar{\tau}_i f$$

implying $U_\alpha W_{\bar{\tau}_i}\subset W_{\bar{\tau}_i}$. Using the explicit form of the boundary condition we obtain the following important fact.

Lemma 2.1. *For* $\alpha\in\mathbb{R}_*$ *we have*

$$U_\alpha\mathscr{D}(\bar{T}_i)\subset\mathscr{D}(\bar{T}_i)$$

and

$$\bar{T}_i U_\alpha=\alpha^2 U_\alpha \bar{T}_i. \tag{2.2}$$

Near $x=i$ we will use the heat kernel $\bar{\Gamma}_s^i$ of $e^{-s\bar{T}_i}$ as a parametrix for the heat kernel Γ_s of e^{-sT} (it will be shown in Lemma 3.1 below that these kernels are honest functions). By a simple symmetry argument given in Sect. 6 we can restrict attention to the left endpoint so we write $a:=a_0$, $b:=b_0$, $\bar{T}:=\bar{T}_0$, $\bar{\Gamma}_s:=\bar{\Gamma}_s^0$ in what follows. The study of the comparison kernel $\bar{\Gamma}_s$ is greatly simplified by the invariance property (2.2) of $\bar{T}$ implying the following invariance of $\bar{\Gamma}_s$.

Lemma 2.2. *For* $x, y, s, \alpha>0$ *we have*

$$\bar{\Gamma}_s(x,y)=\alpha^2\,\bar{\Gamma}_{s\alpha^2}(\alpha x,\alpha y). \tag{2.3}$$

Proof. For $f\in L^2(\mathbb{R}_+)$ we find from (2.2)

$$(\bar{T}+\partial_s)U_\alpha e^{-s\alpha^2\bar{T}}f=0.$$

Since

$$\lim_{s\to 0} U_\alpha e^{-s\alpha^2\bar{T}}f=U_\alpha f$$

we must have $U_\alpha e^{-s\alpha^2\bar{T}}=e^{-s\bar{T}}U_\alpha$ implying the lemma. □

Finally, we need the following property of the domains $\mathscr{D}(T)$ and $\mathscr{D}(\bar{T})$.

Lemma 2.3. *Assume that*

$$a>-1/4\quad or\quad a=-1/4\quad and\quad b=0. \tag{2.4}$$

Let $\varphi\in C_0^\infty((-1,1))$ *with* $\varphi=1$ *in a neighborhood of* 0. *Then*

$$\varphi u\in\mathscr{D}(\bar{T})\quad for\quad u\in\mathscr{D}(T)$$

and

$$\varphi u\in\mathscr{D}(T)\quad for\quad u\in\mathscr{D}(\bar{T}).$$

Proof. Suppose $u \in \mathscr{D}(T)$. Then we have $\bar{\tau}\varphi u \in L^2(\mathbb{R}_+)$ by assumption hence $\varphi u \in W_{\bar{\tau}} = \mathscr{D}(T_0(\bar{\tau})^*)$. By definition of the Friedrichs extension there is a sequence $(u_n)_{n\in\mathbb{N}} \subset C_0^\infty(I)$ satisfying $u = \lim_n u_n$ in $L^2(I)$ and in addition

$$\lim_{n,m} (\tau(u_n - u_m)|u_n - u_m) = 0 .$$

An easy calculation using the assumption, (2.1), and Hardy's inequality shows that also

$$\lim_{n,m} (\bar{\tau}(\varphi u_n - \varphi u_m)|\varphi u_n - \varphi u_m) = 0$$

which means that $\varphi u = \lim_n \varphi u_n$ is in $\mathscr{D}(\bar{T})$. □

From now on we will assume that condition (2.4) is satisfied.

3. Estimates for $\bar{\Gamma}_s$ and Γ_s

Let us write

$$\frac{a(x)}{x^2(1-x)^2} =: \frac{a}{x^2} + \frac{b(x)}{x}, \qquad x \in I ,$$

and

$$\bar{\tau} := -d_x^2 + \frac{a}{x^2} .$$

As before, T and $\bar{T}$ will be the Friedrichs extensions of $T_0(\tau)$ and $T_0(\bar{\tau})$, respectively.

We start in showing that $\bar{\Gamma}_s(x, \cdot)$ and $\partial_x \bar{\Gamma}_s(x, \cdot)$ are in $L^2(\mathbb{R}_+)$ and estimate the norms.

Lemma 3.1. *For $x \geqq x_0 > 0$ and $0 < s \leqq s_0$ there is a constant $C = C(x_0, s_0)$ such that*

$$\int_0^\infty \bar{\Gamma}_s(x, z)^2 dz \leqq C s^{-1/2} , \tag{3.1}$$

$$\int_0^\infty (\partial_x \bar{\Gamma}_s)(x, z)^2 dz \leqq C s^{-1} . \tag{3.2}$$

Proof. Put $J := (1/2, 2)$. Then $e^{-s\bar{T}} u \in C^\infty(J)$ for $u \in L^2(\mathbb{R}_+)$, $s > 0$. The one-dimensional interpolation inequality [1, Theorem 3.1] gives

$$\begin{aligned} \|e^{-s\bar{T}}u\|_{H^2(J)} &\leqq C(\|\partial_x^2 e^{-s\bar{T}}u\|_{L^2(J)} + \|e^{-s\bar{T}}u\|_{L^2(J)}) \\ &\leqq C(\|\bar{T}e^{-s\bar{T}}u\|_{L^2(J)} + \|e^{-s\bar{T}}u\|_{L^2(J)}) \\ &\leqq C_{s_0} s^{-1} \|u\|_{L^2(\mathbb{R}_+)} \end{aligned}$$

for $0 < s \leqq s_0$ where the last estimate comes from the spectral theorem. Similarily

$$\|\partial_x e^{-s\bar{T}}u\|_{H^1(J)} \leqq C_{s_0} s^{-1} \|u\|_{L^2(\mathbb{R}_+)} .$$

Using now Agmon's version of the Sobolev inequality [1, Lemma 13.2] we obtain for $0<s\leqq s_0$ and $u\in L^2(\mathbb{R}_+)$

$$|e^{-s\bar{T}}u(1)|\leqq C_{s_0}s^{-1/4}\|u\|_{L^2(\mathbb{R}_+)},$$
$$|\partial_x e^{-s\bar{T}}u(1)|\leqq C_{s_0}s^{-1/2}\|u\|_{L^2(\mathbb{R}_+)}.$$

This proves (3.1) and (3.2) for $x=1$ in view of the Riesz representation theorem. From the invariance property (2.3) we obtain with $\alpha:=1/x$

$$\int_0^\infty \bar{\Gamma}_s(x,z)^2\,dz=1/x^2\int_0^\infty \bar{\Gamma}_{s/x^2}(1,z/x)^2\,dz$$
$$=1/x\int_0^\infty \bar{\Gamma}_{s/x^2}(1,z)^2\,dz$$

proving (3.1) in general. Differentiating (2.3) with respect to x we obtain the invariance property of $\partial_x\bar{\Gamma}_s(x,y)$, and (3.2) is proved analogously. □

Next we derive a uniform estimate for the L^2 norm of $\bar{\Gamma}_s$ as $x\to 0$.

Lemma 3.2. *Let* $0<x, s\leqq 2$. *Then there is a constant* $0<\nu<1$ *such that*

$$\int_0^\infty \bar{\Gamma}_s(x,z)^2\,dz\leqq Cs^{-1/2(1+\nu)}x^\nu. \qquad (3.3)$$

Proof. Consider the Friedrichs extension S of $\bar{\tau}$ in $L^2([0,1])$. Then S is semibounded and for $\varphi\in C_0^\infty((-1,1))$ with $\varphi=1$ in a neighborhood of 0 and $v\in\mathscr{D}(\bar{T})$ we have $\varphi v\in\mathscr{D}(S)$. Chose $c>0$ such that $S+c$ is invertible. It follows

$$(S+c)\varphi e^{-\bar{T}}u=\varphi\bar{T}e^{-\bar{T}}u-\varphi''e^{-\bar{T}}u-2\varphi'\partial_x e^{-\bar{T}}u$$
$$=:A_\varphi(u)$$

for $u\in L^2(\mathbb{R}_+)$.

We derive from Lemma 3.1

$$\|A_\varphi(u)\|_{L^2(I)}\leqq C_\varphi\|u\|_{L^2(\mathbb{R}_+)}.$$

Now let $\psi_0,\psi_1\in L^2(I)$ be the solutions of $(\bar{\tau}+c)\psi=0$ satisfying the boundary conditions for S at 0 and 1, respectively. It follows for $0<x\leqq 1$ (cf. [7, Chap. XIII.3, Theorem 10]) that with certain constants γ,γ'

$$\varphi e^{-\bar{T}}u(x)=\gamma\psi_1(x)\int_0^x\overline{\psi_0(y)}\,A_\varphi u(y)\,dy+\gamma'\psi_0(x)\int_x^1\overline{\psi_1(y)}\,A_\varphi u(y)\,dy.$$

But $\bar{\tau}+c$ is a differential equation with a regular singular point at 0 and has the index equation

$$0=-\lambda(\lambda-1)+a=-(\lambda^2-\lambda-a)$$

with solutions

$$\lambda_\pm(a)=1/2\pm\sqrt{1/4+a}.$$

From the general theory (see e.g. [8, p. 147, Satz 1]) we derive for $a > -1/4$ the representation

$$\psi_i(x) = : x^{\lambda_+(a)} g_i^+(x) + x^{\lambda_-(a)} g_i^-(x),$$

where $: 0 < x \leqq \delta < 1$, $g_i^\pm \in C^\infty([0, \delta])$, $i = 0, 1$. If $a = -1/4$ we have instead

$$\psi_i(x) = : x^{1/2} g_i(x) + x^{1/2} \log x\, h_i(x),$$

$0 < x \leqq \delta < 1$, g_i, $h_i \in C^\infty([0, \delta])$, $i = 0, 1$. We know from the discussion in Sect. 2 that $g_0^- = 0$ ($a > -1/4$) or $h_0 = 0$ ($a = -1/4$). Estimating $\varphi e^{-\bar{T}} u(x)$ by the Cauchy-Schwarz inequality we find with $0 < v < 1$

$$|e^{-\bar{T}} u(x)| \leqq C_\delta x^{v/2} \|u\|_{L^2(\mathbb{R}_+)}$$

for $0 < x \leqq \delta$, δ sufficiently small. As before we conclude that (3.3) is valid for $s = 1$ and $0 < x \leqq \delta$. Applying now (2.3) with $\alpha^2 = 1/s$ we get

$$\begin{aligned}\int_0^\infty \bar{\Gamma}_s(x, z)^2 dz &= s^{-1/2} \int_0^\infty \bar{\Gamma}_1(s^{-1/2} x, z)^2 dz \\ &\leqq C_\delta s^{-1/2(1+v)} x^v, \qquad s^{-1/2} x \leqq \delta.\end{aligned}$$

By Lemma 3.1, the same estimate holds for $s^{-1/2} x \geqq \delta$, $0 < x$, $s \leqq 2$. The proof is complete. □

Corollary 3.3. *We have for* $0 < x, y, s \leqq 2$ *with* $0 < v < 1$

$$|\bar{\Gamma}_s(x, y)| \leqq C s^{-1/2(1+v)} (xy)^{v/2}.$$

Proof. This follows from the semigroup property of $e^{-s\bar{T}}$. □

Now we derive similar estimates for Γ_s.

Lemma 3.4. *Choose* δ *with* $0 < \delta < 1$. *We have for* $0 < s \leqq 1$, $0 < x, y \leqq \delta$ *with* $0 < v < 1$

$$|\Gamma_s(x, y)| \leqq C_\delta s^{-1/2(1+v)} (xy)^{v/2}.$$

Proof. Pick $\varphi \in C_0^\infty((-1, 1))$ with $\varphi | [0, \delta] = 1$. By Lemma 2.3, our assumption, and (2.1) we have for $u \in L^2(I)$ $\varphi e^{-sT} u \in \mathscr{D}(\bar{T})$ and $\varphi \tilde{b} e^{-sT} u \in L^2(I)$, $\tilde{b}(x) := \dfrac{b(x)}{x}$. Moreover,

$$\|\varphi \tilde{b} e^{-sT} u\|_{L^2(I)} \leqq C s^{-1/2} \|u\|_{L^2(I)}, \tag{3.4}$$

which is certainly true for $b(0) = 0$ and follows from (2.1) otherwise. We compute

$$\begin{aligned}(\bar{T} + \partial_s) \varphi e^{-sT} u &= \varphi (T + \partial_s) e^{-sT} u \\ &\quad - \varphi \tilde{b} e^{-sT} u - \varphi'' e^{-sT} u - 2\varphi' \partial_x e^{-sT} u = : B_s u.\end{aligned}$$

From (3.4) and Lemma 3.1 we conclude

$$\|B_s u\|_{L^2(\mathbb{R}_+)} \leqq C_\delta s^{-1/2} \|u\|_{L^2(I)}. \tag{3.5}$$

Applying Duhamel's principle it follows for $0 < x \leqq \delta$

$$\varphi e^{-sT} u(x) = e^{-s\bar{T}} \varphi u(x) + \int_0^s e^{-(s-t)\bar{T}} B_t u(x)\, dt.$$

Using (3.5) and Lemma 3.2 we estimate

$$\left|\int_0^s e^{-(s-t)\bar{T}} B_t u(x)\,dt\right| \leqq \int_0^s \int_0^\infty |\bar{\Gamma}_{s-t}(x,z) B_t u(z)|\,dz\,dt$$

$$\leqq C_\delta \|u\|_{L^2(I)} \int_0^s t^{-1/2} \left(\int_0^\infty \bar{\Gamma}_{s-t}(x,z)^2\,dz\right)^{1/2} dt$$

$$\leqq C_\delta \|u\|_{L^2(I)} x^{\nu/2} \int_0^s t^{-1/2}(s-t)^{-1/4(1+\nu)}\,dt$$

$$\leqq C_\delta x^{\nu/2} \|u\|_{L^2(I)}\,.$$

With Lemma 3.2 again we conclude that

$$\int_0^1 \Gamma_s(x,z)^2\,dz \leqq C_\delta s^{-1/2(1+\nu)} x^\nu$$

for $0<x\leqq\delta$, $0<s\leqq 1$. Now the lemma follows from the semigroup property. □

We now proceed to derive estimates off the diagonal. To do so we put $C_*^\infty(\mathbb{R}) := \left\{\varphi \in C^\infty(\mathbb{R}) \,\middle|\, \sup_{x\in\mathbb{R}} \left|\partial_x^k \varphi(x)\right| < \infty \text{ for every } k \in \mathbb{Z}_+\right\}$. For $\varphi, \psi \in C_*^\infty(\mathbb{R})$ we introduce the operators

$$H_s^{\varphi,\psi} u := \varphi e^{-s\bar{T}} \psi u\,, \qquad K_s^{\varphi,\psi} u := \partial_x H_s^{\varphi,\psi} u\,,$$

where $u \in L^2(\mathbb{R}_+)$ and $s>0$, with the kernels

$$H_s^{\varphi,\psi}(x,y) := \varphi(x)\bar{\Gamma}_s(x,y)\psi(y)\,,$$
$$K_s^{\varphi,\psi}(x,y) := \partial_x(\varphi(x)\bar{\Gamma}_s(x,y)\psi(y))\,.$$

Lemma 3.5. *Let* $\varphi, \psi \in C_*^\infty(\mathbb{R})$ *satisfy*

$$\operatorname{supp}\varphi \quad \textit{and} \quad \operatorname{supp}\psi \quad \textit{have a positive distance} \tag{3.6}$$

and

$$0 \notin \operatorname{supp}\varphi' \cup \operatorname{supp}\psi'\,. \tag{3.7}$$

Then for every $N \in \mathbb{N}$ *there is a constant* C *depending only on* N, φ, *and* ψ *such that for* $0<x, y, s\leqq 2$ *with* $0<\nu<1$

$$|H_s^{\varphi,\psi}(x,y)| \leqq C s^N (xy)^{\nu/2}\,.$$

If condition (3.7) *is replaced by*

$$0 \notin \operatorname{supp}\varphi \cup \operatorname{supp}\psi' \tag{3.8}$$

then also

$$|K_s^{\varphi,\psi}(x,y)| \leqq C s^N y^{\nu/2}\,.$$

Proof. Suppose first that (3.6) and (3.8) are satisfied. Then $\varphi e^{-s\bar{T}}\psi u \in \mathscr{D}(\bar{T})$ for $u \in L^2(\mathbb{R}_+)$, $s>0$, and we have

$$\begin{aligned}(\bar{T}+\partial_s)\varphi e^{-s\bar{T}}\psi u &= \varphi(\bar{T}+\partial_s)e^{-s\bar{T}}\psi u \\ &\quad -\varphi'' e^{-s\bar{T}}\psi u - 2\varphi'\partial_x e^{-s\bar{T}}\psi u \\ &= \varphi'' e^{-s\bar{T}}\psi u - 2\partial_x(\varphi' e^{-s\bar{T}}\psi u) \\ &= H_s^{\varphi'',\psi}u - 2K_s^{\varphi',\psi}u =: J_s^{\varphi,\psi}u .\end{aligned}$$

Now $\lim_{s\to 0} H_s^{\varphi,\psi}u=0$ hence Duhamel gives

$$H_s^{\varphi,\psi}u = \int_0^s e^{-(s-t)\bar{T}} J_t^{\varphi,\psi}u\,dt . \tag{3.9}$$

Choose $\chi \in C_*^\infty(\mathbb{R})$ such that χ and ψ also satisfy (3.6), (3.8), and in addition $\chi\varphi=\varphi$. Then also

$$K_s^{\varphi,\psi}u = \int_0^s \partial_x \chi e^{-(s-t)\bar{T}} J_t^{\varphi,\psi}u\,dt . \tag{3.10}$$

If $\omega \in C_*^\infty(\mathbb{R})$ satisfies $0 \notin \operatorname{supp}\omega$ then we obtain from the analogue of (2.1) for $\bar{\tau}$

$$\|\partial_x \omega e^{-s\bar{T}}u\|_{L^2(\mathbb{R}_+)} \leqq C_\omega s^{-1/2}\|u\|_{L^2(\mathbb{R}_+)}, \tag{3.11}$$

consequently we have for the operator norms in $L^2(\mathbb{R}_+)$

$$\|J_s^{\varphi,\psi}\| \leqq \|H_s^{\varphi'',\psi}\| + 2\|K_s^{\varphi',\psi}\| \leqq C_{\varphi',\psi} s^{-1/2} . \tag{3.12}$$

Now suppose we had proved

$$\|J_s^{\varphi,\psi}\| \leqq C_{N,\varphi,\psi} s^{(N-1)/2} \tag{3.13}$$

for some $N \in \mathbb{Z}_+$ and $0<s\leqq 2$. Then we deduce from (3.9), (3.10), and (3.11)

$$\|H_s^{\varphi,\psi}\| \leqq \int_0^s \|J_t^{\varphi,\psi}\|\,dt \leqq C_{N,\varphi,\psi} s^{(N+1)/2} ,$$

$$\|K_s^{\varphi,\psi}\| \leqq C_\chi \int_0^s (s-t)^{-1/2} \|J_t^{\varphi,\psi}\|\,dt \leqq C_{N,\varphi,\psi} s^{N/2}$$

hence by (3.12)

$$\|J_s^{\varphi,\psi}\| \leqq C_{N,\varphi,\psi} s^{N/2} .$$

Since (3.12) establishes (3.13) for $N=0$ it is proved in general. Thus we obtain from (3.9), (3.1), and (3.13) for $x>0$, $0<s\leqq 2$, $u \in L^2(\mathbb{R}_+)$

$$|H_s^{\varphi,\psi}u(x)| \leqq C_{N,\varphi,\psi} s^N \|u\|_{L^2(\mathbb{R}_+)} \tag{3.14}$$

for every $N \in \mathbb{Z}_+$, similarly from (3.10), (3.2), and (3.13)

$$|K_s^{\varphi,\psi}u(x)| \leqq C_{N,\varphi,\psi} s^N \|u\|_{L^2(\mathbb{R}_+)} . \tag{3.15}$$

Note that (3.14) remains valid if we only require (3.6), (3.7) since then φ' and ψ satisfy (3.6) and (3.8). Now choose $\chi \in C_*^\infty(\mathbb{R})$ such that ψ, χ and $\varphi, 1-\chi$ also satisfy

(3.6) and (3.7). Using the semigroup property of $e^{-s\bar{T}}$ we obtain for $0<x, y, s \leqq 2$ the estimate

$$|H_s^{\varphi,\psi}(x,y)| \leqq \int_0^\infty (|\varphi(x)\bar{\Gamma}_{s/2}(x,z) H_{s/2}^{\psi,\chi}(y,z)| + |H_{s/2}^{\varphi,1-\chi}(x,z)\bar{\Gamma}_{s/2}(z,y)\psi(y)|)\,dz\,,$$

hence from (3.14) and (3.3)

$$|H_s^{\varphi,\psi}(x,y)| \leqq C_{N,\varphi,\psi}\, s^N (xy)^{\nu/2}$$

for every $N \in \mathbb{Z}_+$. A similar argument using (3.11), (3.3), (3.14), and (3.15) proves

$$|K_s^{\varphi,\psi}(x,y)| \leqq C_{N,\varphi,\psi} s^N y^{\nu/2}$$

if φ and ψ satisfy (3.6) and (3.8). □

The given proof can be repeated with $\bar{T}$ replaced by T and with $\varphi, \psi \in C_0^\infty((-1,1))$. Using everywhere the analogous estimates for Γ_s we obtain

Lemma 3.6. *Let $\varphi, \psi \in C_0^\infty((-1,1))$ satisfy (3.6) and (3.7). Then for every $N \in \mathbb{N}$ there is a constant C depending only on N, φ, and ψ such that for $0<x, y, s \leqq 1$ with $0<\nu<1$*

$$|\varphi(x)\Gamma_s(x,y)\psi(y)| \leqq C s^N (xy)^{\nu/2}\,.$$

In the next section we will also need estimates for $\bar{\Gamma}_s(x,y)$ as $x \to \infty$.

Lemma 3.7. *There is $s_0 \leqq 1$ such that for $x \geqq 2$, $0<y \leqq 1$, $0<s \leqq s_0$, and every $N \in \mathbb{N}$ with $0<\nu<1$*

$$|\bar{\Gamma}_s(x,y)| \leqq C_N s^N e^{-\frac{(x-y)^2}{144s}} y^{\nu/2}\,.$$

Proof. Denote by T_0 the closure of $-d_x^2$ with domain $C_0^\infty(\mathbb{R})$ in $L^2(\mathbb{R})$. Then e^{-sT_0} has the kernel

$$\Gamma_s^0(x,y) := (4\pi s)^{-1/2} e^{-\frac{(x-y)^2}{4s}}\,.$$

To compare $\bar{\Gamma}_s$ with Γ_s^0 away from 0 we choose $\varphi \in C^\infty(\mathbb{R})$ with $\operatorname{supp}\varphi \subset [3/4, \infty)$ and $\varphi|[5/3,\infty)=1$ and $\psi \in C_0^\infty((-4/3, 4/3))$ with $\psi|[-1,1]=1$. We put

$$\chi(x) := \begin{cases} -a/x^2, & x \in \operatorname{supp}\varphi\,, \\ 0 & \text{otherwise}\,. \end{cases}$$

For $u \in L^2(\mathbb{R}_+)$ we have $\varphi e^{-s\bar{T}}\psi u \in \mathscr{D}(T_0)$ (cf. Lemma 2.3) and

$$(T_0+\partial_s)\varphi e^{-s\bar{T}}\psi u = \varphi'' e^{-s\bar{T}}\psi u - 2\partial_x(\varphi' e^{-s\bar{T}}\psi u) + \chi\varphi e^{-s\bar{T}}\psi u =: L_s u + \chi\varphi e^{-s\bar{T}}\psi u\,,$$

hence

$$\varphi e^{-s\bar{T}}\psi u = \int_0^s e^{-(s-t)T_0}(L_t u + \chi\varphi e^{-t\bar{T}}\psi u)\,dt\,. \tag{3.16}$$

We put

$$M_s^0 := e^{-sT_0}, \qquad M_s^i := \int_0^s e^{-(s-t)T_0} \chi M_t^{i-1} dt, \qquad i \geqq 1.$$

For the kernel $M_s^i(x, y)$ we obviously have

$$|M_s^i(x, y)| \leqq C^i s^{(i-1)/2} e^{-\frac{(x-y)^2}{4s}} \tag{3.17}$$

for $x, y \in \mathbb{R}$, $s > 0$, and some positive constant C. From (3.16) we obtain by iteration the following identity for operators in $L^2(\mathbb{R}_+)$ (cf. the proof of Lemma 5.1)

$$\varphi e^{-s\bar{T}} \psi = \sum_{j=0}^{N} \int_0^s M_{s-t}^j L_t dt + \int_0^s M_{s-t}^N \chi \varphi e^{-t\bar{T}} \psi dt, \tag{3.18}$$

$N \in \mathbb{Z}_+$. From (3.17) and Lemma 3.2 we derive the kernel estimate

$$\left| \int_0^s M_{s-t}^N \chi \varphi e^{-t\bar{T}} \psi dt(x, y) \right| \leqq (C s^{1/2})^{N+1} y^{\nu/2}$$

if $x \geqq 2$ and $0 < s, y \leqq 1$. Thus the series (3.18) for the kernels is convergent if $x \geqq 2$, $0 < y \leqq 1$, and $0 < s \leqq s_0$ where $C s_0^{1/2} < 1$. Estimating the terms in the series by (3.17) and Lemma 3.5 and noting that $L_t(z, y) \neq 0$ only for $z \leqq 5/3$ the proof is completed. □

4. Some Convolution Integrals

The study of the perturbation series in the next section leads to the following definition. Let $(\beta_j)_{j \in \mathbb{N}}$ be a sequence of real numbers with $\beta_j \geqq -1$ and put

$$\begin{aligned} L_s^{1,\beta}(x, y) &:= \bar{\Gamma}_s(x, y), \\ L_s^{i+1,\beta}(x, y) &:= \int_0^s \int_0^\infty L_{s-t}^{1,\beta}(x, z) z^{\beta_i} L_t^{i,\beta}(z, y) dz dt, \qquad i \geqq 1. \end{aligned} \tag{4.1}$$

Of course, the definition of $L_s^{i,\beta}$ depends only on β_j for $1 \leqq j \leqq i-1$ i.e. $L_s^{i,\beta}$ is well defined for any $\beta \in \mathbb{Z}_+^{i-1}$ provided that the above integrals do converge.

Lemma 4.1. *The integrals in* (4.1) *are absolutely convergent and we have the estimates*

$$|L_s^{i,\beta}(x, y)| \leqq C_i s^{-1 + \frac{i}{2}(1-\nu)} (xy)^{\nu/2} (1+x)^{\gamma_i} \tag{4.2}$$

valid for $0 < s \leqq s_0$, $0 < y \leqq 1$, *and* $x > 0$. *Here* $C_i > 0$, $\gamma_i \in \mathbb{R}$, *and* $0 < \nu < 1$. *Moreover, we have the scaling property*

$$L_{\alpha^2 s}^{i,\beta}(\alpha x, \alpha y) = \alpha^{2i + |\beta|_{i-1} - 3} L_s^{i,\beta}(x, y), \tag{4.3}$$

where $\alpha > 0$ *and* $|\beta|_{i-1} := \sum_{j=1}^{i-1} \beta_j$.

Proof. We proceed by induction on i. For $i = 1$ (4.2) is satisfied in view of Corollary 3.3 and Lemma 3.7, and (4.3) reduces to (2.3). Assume now that (4.2) and

(4.3) are satisfied for some $i \geqq 1$. Choosing s_0 as in Lemma 3.7 we obtain (4.2) by a straightforward estimate based on Corollary 3.3, Lemma 3.7, and the induction hypothesis if we require $0<x, y \leqq 1$, $0<s \leqq s_0$.

Now let $x>1$, $0<y \leqq 1$, $0<s \leqq s_0$. From (2.3) we obtain with $\alpha=1/x$

$$L_s^{i+1,\beta}(x,y)=\frac{1}{x}\int_0^s\int_0^\infty \bar{\Gamma}_{(s-t)/x^2}(1,z/x)z^{\beta_i}L_t^{i,\beta}(z,y)\,dz\,dt$$

implying the absolute convergence of the integral as before. Using (4.3) it follows

$$\begin{aligned}L^{i+1,\beta}(x,y)&=x^{2i+|\beta|_{i-1}-4}\int_0^s\int_0^\infty \bar{\Gamma}_{(s-t)/x^2}(1,z/x)z^{\beta_i}L_{t/x^2}^{i,\beta}(z/x,y/x)\,dz\,dt\\&=x^{2(i+1)+|\beta|_i-3}\int_0^{\bar{s}/x^2}\int_0^\infty \bar{\Gamma}_{s/x^2-t}(1,z)z^{\beta_i}L_t^{i,\beta}(z,y/x)\,dz\,dt\\&=x^{2(i+1)+|\beta|_i-3}L_{s/x^2}^{i+1,\beta}(1,y/x)\end{aligned}\tag{4.4}$$

hence (4.2) follows from the previous estimate. The proof of (4.3) is similar to the proof of (4.4). □

We will also need a slightly more general class of functions. Let $(c_i)_{i\in\mathbb{N}}\subset C^\infty(\mathbb{R}_+)$ be a sequence of functions satisfying $c_i(x)=0(x^{\beta_i})$ as $x\to\infty$ for $\beta_i\geqq 0$. We put

$$\begin{aligned}L_s^{1,c}(x,y)&:=\bar{\Gamma}_s(x,y),\\L_s^{i+1,c}(x,y)&:=\int_0^s\int_0^\infty L_{s-t}^{1,c}(x,z)\frac{c_i(z)}{z}L_t^{i,c}(z,y)\,dz\,dt.\end{aligned}\tag{4.5}$$

It is obvious from the proof of Lemma 4.1 that $L_s^{i,c}$ is well defined and satisfies (4.2). With $c_i^\alpha(x):=c_i(\alpha x)$, $\alpha, x>0$, $i\in\mathbb{N}$, (4.3) generalizes to

$$L_{\alpha^2 s}^{i,c}(\alpha x,\alpha y)=\alpha^{i-2}L_s^{i,c^\alpha}(x,y).\tag{4.6}$$

Our next goal is an asymptotic expansion of the functions $L_s^{i,\beta}(x,y)$ in a neighborhood of (1/2, 1/2). We start with the following observation. If $u\in C_0^\infty((1/4,3/4))$ and if we write

$$L_s^{i,\beta}u(x):=\int_0^\infty L_s^{i,\beta}(x,y)u(y)\,dy$$

then

$$(\bar{T}+\partial_s)L_s^{i,\beta}u(x)=\begin{cases}0, & i=1,\\ x^{\beta_{i-1}}L_s^{i-1,\beta}u(x), & i>1,\end{cases}\tag{4.7}$$

and

$$\lim_{s\to 0}L_s^{i,\beta}u(x)=\begin{cases}u(x), & i=1,\\ 0, & i>1.\end{cases}$$

Now for $i=1$ we obtain an asymptotic expansion by the method of Minakshisundaram and Pleijel [10]. We make the ansatz

$$\bar{\Gamma}_s(x,y)\sim(4\pi s)^{-1/2}e^{-\frac{(x-y)^2}{4s}}\sum_{j\geqq 0}s^j u_j^1(x,y).\tag{4.8}$$

Applying $\bar{\tau}_x + \partial_s$ to the right hand side and putting the coefficient of s^{j-1} equal to zero we obtain the recursion formulas

$$(x-y)\partial_x u_j^1(x,y) + j u_j^1(x,y) = -\bar{\tau}_x u_{j-1}^1(x,y),$$

$j \geqq 0$, where $u_{-1}=0$. Requiring $u_j^1(x,y)$ to be smooth at $x=y$ and $u_0^1=1$ we obtain the explicit formulas

$$\begin{aligned} u_{j+1}^1(x,y) &= -(x-y)^{-j-1} \int_y^x (t-y)^j (\bar{\tau}_x u_j^1)(t,y)\,dt \\ &= -\int_0^1 t^j (\bar{\tau}_x u_j^1)(y+t(x-y),y)\,dt, \qquad j \geqq 0. \end{aligned}$$

Thus the u_j^1 are smooth implying the estimate

$$\sup_{x,y\in K} |\partial_x^n \partial_y^m u_j^1(x,y)| \leqq C_{n,m,j,K}$$

for any compact subset K of $\mathbb{R}_*$. Now pick φ, $\psi \in C_0^\infty((1/4,3/4))$ with $\psi = 1$ in a neighborhood of $1/2$ and $\varphi = 1$ in a neighborhood of $\operatorname{supp}\psi$ and put

$$\begin{aligned} H_s^N(x,y) &:= (4\pi s)^{-1/2} e^{-\frac{(x-y)^2}{4s}} \sum_{j=0}^N s^j \varphi(x) u_j^1(x,y)\psi(y), \quad N \in \mathbb{N}, \\ H_s^N u(x) &:= \int_0^\infty H_s^N(x,y) u(y)\,dy, \qquad u \in C_0^\infty((1/4,3/4)). \end{aligned} \tag{4.9}$$

Then we find for $x>0$

$$|(\bar{T}+\partial_s) H_s^N u(x)| \leqq C_N s^{N-1/2} \|u\|_{L^1}, \qquad \lim_{s\to 0} H_s^N u(x) = \psi u(x), \tag{4.10}$$

hence from Duhamel

$$H_s^N u(x) - e^{-s\bar{T}} \psi u(x) = \int_0^s e^{-(s-t)\bar{T}} (\bar{T}+\partial_t) H_t^N u\,dt(x),$$

therefore from (4.10) and Lemma 3.1

$$|H_s^N u(x) - e^{-sT} \psi u(x)| \leqq C_N s^N \|u\|_{L^1}, \qquad 1/4 \leqq x \leqq 3/4.$$

Thus

$$\sup_{x,y\in\psi^{-1}(1)} \left| \bar{\Gamma}_s(x,y) - (4\pi s)^{-1/2} e^{-\frac{(x-y)^2}{4s}} \sum_{j=0}^N s^j u_j^1(x,y) \right| \leqq C_N s^N, \tag{4.11}$$

i.e. (4.8) gives an asymptotic expansion near $(1/2, 1/2)$. Now assume that we have an asymptotic expansion

$$L_s^{i,\beta}(x,y) \sim (4\pi s)^{-1/2} e^{-\frac{(x-y)^2}{4s}} \sum_{j\geqq 0} s^{j+i-1} u_j^{i,\beta}(x,y)$$

near $(1/2, 1/2)$ for some $i \geqq 1$ i.e.

$$\sup_{x,y\in\psi^{-1}(1)} \left| L_s^{i,\beta}(x,y) - (4\pi s)^{-1/2} e^{-\frac{(x-y)^2}{4s}} \sum_{j=0}^N s^{j+i-1} u_j^{i,\beta}(x,y) \right| \leqq C_N s^N \tag{4.12}$$

for $N \in \mathbb{N}$. Then we make the corresponding ansatz for $L_s^{i+1,\beta}(x,y)$.

Applying $\bar{\tau}_x + \partial_s$ to the series and equating the coefficient of s^{j+i-1} in this expression with the one in the series for $x^{\beta_i} L_s^{i,\beta}$ [in view of (4.7)] we now obtain the recursion formulas

$$(x-y)\partial_x u_j^{i+1,\beta}(x,y) + (j+i)u_j^{i+1,\beta}(x,y) = -\bar{\tau}_x u_{j-1}^{i+1,\beta}(x,y) + x^{\beta_i} u_j^{i,\beta}(x,y), \quad j \geqq 0,$$

where $u_{-1}^{i+1,\beta} = 0$. Requiring $u_j^{i+1,\beta}(x,y)$ to be smooth at $x=y$ we obtain

$$u_j^{i+1,\beta}(x,y) = \int_0^1 t^{j+i}(-\bar{\tau}_x u_{j-1}^{i+1,\beta}(y+t(x-y),y) + (y+t(x-y))^{\beta_i} u_j^{i,\beta}(y+t(x-y),y))\,dt. \tag{4.13}$$

Thus all $u_j^{i+1,\beta}$ are smooth in $\mathbb{R}_* \times \mathbb{R}_*$. With $\tilde{\varphi}$ and $\tilde{\psi} \in C_0^\infty((1/4, 3/4))$ satisfying $\tilde{\varphi}\tilde{\psi} = \tilde{\psi}$, $\psi\tilde{\varphi} = \tilde{\varphi}$, $\tilde{\psi} = 1$ near 1/2, and with $H_s^{N,i,\beta}$ the obvious generalization of (4.9) we now obtain for $u \in C_0^\infty((1/4, 3/4))$, $x > 0$, $i \geqq 1$

$$|(\bar{T} + \partial_s) H_s^{N,i+1,\beta} u(x) - x^{\beta_i} H_s^{N,i,\beta} u(x)| \leqq C_N s^N \|u\|_{L^1},$$

$$\lim_{s\to 0} H_s^{N,i+1,\beta} u = 0,$$

hence from Duhamel as before

$$\left| H_s^{N,i+1,\beta} u(x) - \int_0^s \int_0^\infty \bar{\Gamma}_{s-t}(x,z)\, z^{\beta_i} H_t^{N,i,\beta} u(z)\, dz\, dt \right| \leqq C_N s^N \|u\|_{L^1}.$$

Using the induction hypothesis (4.12) we derive for $x, y \in \tilde{\psi}^{-1}(1)$, $0 < s \leqq s_0$

$$\int_0^s \int_0^\infty \bar{\Gamma}_{s-t}(x,z)\, z^{\beta_i} \tilde{\varphi}(z) L_t^{i,\beta}(z,y)\, dz\, dt = H_s^{N,i+1,\beta}(x,y) + O_N(s^N).$$

Using the estimates of Lemmas 4.1, 3.5, 3.7 we arrive at (4.12) with i replaced by $i+1$. Summing up we have proved the following result.

Theorem 4.2. *There are functions $u_j^{i,\beta} \in C^\infty(\mathbb{R}_* \times \mathbb{R}_*)$ such that for $i \geqq 1$*

$$L_s^{i,\beta}(x,y) \underset{s\to 0}{\sim} (4\pi s)^{-1/2} e^{-\frac{(x-y)^2}{4s}} \sum_{j\geqq 0} s^{j+i-1} u_j^{i,\beta}(x,y)$$

uniformly in a neighborhood of (1/2, 1/2).

More precisely, there is $\delta > 0$, $s_0 > 0$ such that for $0 < s \leqq s_0$ and $|x - 1/2| + |y - 1/2| \leqq \delta$ the estimate (4.12) *holds.*

We conclude this section with a pointwise asymptotic expansion for $\Gamma_s(x,y)$ away from 0. This is obtained by the same method the only difference being that we need more detailed information on the coefficients.

Theorem 4.3. *There are functions $u_j \in C^\infty(I \times I)$ such that*

$$\Gamma_s(x,y) \underset{s\to 0}{\sim} (4\pi s)^{-1/2} e^{-\frac{(x-y)^2}{4s}} \sum_{j\geqq 0} s^j u_j(x,y)$$

uniformly in compact subsets of $I \times I$. *We have*

$$u_0 = 1$$

and in general

$$u_j(x,x) = \frac{w_j(x)}{x^{2j}(1-x)^{2j}}, \quad j \geqq 0, \quad x \in I, \tag{4.14}$$

where w_j *is a polynomial in the variables* $a^{(l)}$, $l \geqq 0$, *with coefficients in* $C^\infty(\bar{I})$ *and independent of* a. *Finally, if* $0 < \varepsilon \leqq x, y \leqq 1/2$ *we have the remainder estimate*

$$\left| \Gamma_s(x,y) - (4\pi s)^{-1/2} e^{-\frac{(x-y)^2}{4s}} \sum_{j=0}^{N} s^j u_j(x,y) \right| \leqq C_N s^N \varepsilon^{-2N-2}. \tag{4.15}$$

Proof. As before we derive the recursion formulas

$$\begin{aligned} u_0(x,y) &= 1, \\ u_{j+1}(x,y) &= -\int_0^1 t^j (\tau_x u_j)(y + t(x-y), y)\, dt, \quad j \geqq 0, \end{aligned} \tag{4.16}$$

and a remainder estimate analogous to (4.11) in compact subsets of $I \times I$. If we restrict to $0 < \varepsilon \leqq x, y \leqq 1/2$ then it follows inductively from (4.16) that for $k \geqq 0$

$$|\partial_x^k u_j(x,y)| \leqq \frac{C_{j,k}}{\varepsilon^{2j+k}}.$$

Together with a suitable choice of φ and ψ this leads to the estimate (4.15).

Finally, it follows again by induction from (4.16) that

$$(\partial_x^k u_j)(x,x) = \frac{w_{j,k}(x)}{[x(1-x)]^{2j+k}}$$

with $w_{j,k}$ a polynomial in $a^{(l)}$ with coefficients in $C^\infty(\bar{I})$ and independent of a. This implies (4.14) for $k=0$. □

5. Asymptotic Representation of Γ_s

We are now going to derive an asymptotic series for Γ_s in terms of $\bar{\Gamma}_s$ near 0. To do so we choose functions $\varphi, \chi, \psi \in C_0^\infty((-1,1))$ with $\varphi = 1$ in a neighborhood of $\operatorname{supp}\chi$, $\chi = 1$ in a neighborhood of $\operatorname{supp}\psi$, and $\psi = 1$ in a neighborhood of 0. For $u \in L^2(I)$ and $s > 0$ we have $\varphi e^{-s\bar{T}}\chi u \in \mathscr{D}(T)$ by Lemma 2.3 and as in the proof of Lemma 3.4 we have

$$\begin{aligned} (T + \partial_s)\varphi e^{-s\bar{T}}\chi u &= \varphi(\bar{T} + \partial_s) e^{-s\bar{T}}\chi u \\ &\quad + \tilde{b}\varphi e^{-s\bar{T}}\chi u - \varphi'' e^{-s\bar{T}}\chi u - 2\varphi' \partial_x e^{-s\bar{T}}\chi u \\ &= \tilde{b}\varphi e^{-s\bar{T}}\chi u + \varphi'' e^{-s\bar{T}}\chi u - 2\partial_x \varphi' e^{-s\bar{T}}\chi u. \end{aligned}$$

Let us write

$$\begin{aligned} A_s u &:= -\tilde{b} e^{-s\bar{T}}\chi u, \\ B_s u &:= 2\partial_x \varphi' e^{-s\bar{T}}\chi u - \varphi'' e^{-s\bar{T}}\chi u, \end{aligned}$$

and note that A_s and B_s are bounded in $L^2(I)$ in view of (2.1) and Lemma 3.1. We get from Duhamel's principle

$$\varphi e^{-s\bar{T}}\chi u = e^{-sT}\chi u - \int_0^s e^{-(s-t)T}(\varphi A_t u + B_t u)\,dt.$$

Multiplying with ψ we obtain the operator equality

$$\begin{aligned}\psi e^{-sT}\chi &= \psi e^{-s\bar{T}}\chi + \int_0^s \psi e^{-(s-t)T}\chi A_t\,dt \\ &\quad + \int_0^s \psi e^{-(s-t)T}((1-\chi)\varphi A_t + B_t)\,dt \\ &=: \psi e^{-s\bar{T}}\chi + \int_0^s \psi e^{-(s-t)T}\chi A_t\,dt + C_s.\end{aligned} \tag{5.1}$$

We define for $s>0$

$$A_s^1 := A_s, \qquad A_s^{j+1} := \int_0^s A_{s-t}^1 A_t^j\,dt. \tag{5.2}$$

We know that A_s^1 is an integral operator with kernel

$$A_s^1(x,y) = -\frac{b(x)}{x}\bar{\Gamma}_s(x,y)\,\chi(y).$$

We get from Corollary 3.3 for $0<x,y,s\leqq 1$

$$|A_s^1(x,y)| \leqq C s^{-1/2(1+\nu)} x^{\nu/2-1} y^{\nu/2}$$

where $0<\nu<1$.

It follows inductively from (5.2) that for $0<x,y,s\leqq 1$

$$|A_s^j(x,y)| \leqq C_j s^{(j-1)\frac{1-\nu}{2}-\frac{1+\nu}{2}} x^{\nu/2-1} y^{\nu/2}. \tag{5.3}$$

We now get the following series representation.

Lemma 5.1. *For every $N\in\mathbb{N}$ we have*

$$\begin{aligned}\psi e^{-sT}\chi &= \psi e^{-s\bar{T}}\chi + \sum_{j=1}^{N-1}\int_0^s \psi e^{-(s-t)\bar{T}}\chi A_t^j\,dt \\ &\quad + \int_0^s \psi e^{-(s-t)T}\chi A_t^N\,dt + C_s + \sum_{j=1}^{N-1}\int_0^s C_{s-t}A_t^j\,dt \\ &=: \psi e^{-s\bar{T}}\chi + \sum_{j=1}^{N-1}\int_0^s \psi e^{-(s-t)\bar{T}}\chi A_t^j\,dt + D_s^N,\end{aligned}$$

where the sums are defined to be 0 for $N=1$.

Proof. Suppose the formula is true for some $N \geqq 1$ [which is the case for $N=1$ by (5.1)]. By (5.1) we obtain

$$\int_0^s \psi e^{-(s-t)T} \chi A_t^N \, dt = \int_0^s \psi e^{-(s-t)\tilde{T}} \chi A_t^N \, dt + \int_0^s C_{s-t} A_t^N \, dt + \int_0^s \int_0^{s-t} \psi e^{-(s-t-t')T} \chi A_{t'}^1 A_t^N \, dt' \, dt .$$

Substituting $\tilde{t} := t + t'$ and inverting the order of integration in the last integral the assertion follows. □

To get asymptotic results we have to show that D_s^N has a small kernel as $s \to 0$ for N large.

Lemma 5.2. *For* $0 < x, y, s \leqq 1$ *we have with* $0 < \nu < 1$

$$|D_s^N(x, y)| \leqq C_N s^{N\frac{1-\nu}{2} - \frac{1+\nu}{2}} (xy)^{\nu/2} . \tag{5.4}$$

Proof. Writing

$$D_s^{N,1}(x, y) := \int_0^s \int_0^1 \psi(x) \Gamma_{s-t}(x, z) \chi(z) A_t^N(z, y) \, dz \, dt$$

it follows easily from Lemma 3.4 und (5.3) that $D_s^{N,1}(x, y)$ satisfies the estimate (5.4). Similarily, using Lemma 3.6 and (5.3) we obtain the same estimate for

$$C_s^1(x, y) := \int_0^s \int_0^1 \psi(x) \Gamma_{s-t}(x, z) (1-\chi)(z) \varphi(z) A_t^1(z, y) \, dz \, dt .$$

In the terminology of Lemma 3.5 we have

$$B_s(x, y) = 2 K_s^{\varphi', \chi}(x, y) - H_s^{\varphi'', \chi}(x, y)$$

for the kernel of B_s. Thus in view of Lemmas 3.4 and 3.5

$$C_s^2(x, y) := \int_0^s \int_0^1 \psi(x) \Gamma_{s-t}(x, z) B_t(z, y) \, dz \, dt$$

also satisfies (5.5), hence also $C_s(x, y) = C_s^1(x, y) + C_s^2(x, y)$. But then by (5.3) it also follows for

$$\sum_{j=1}^{N-1} \int_0^s \int_0^1 C_{s-t}(x, z) A_t^j(z, y) \, dz \, dt$$

completing the proof of the lemma. □

The following result is now immediate.

Theorem 5.3. *For* $0 < \delta \leqq 1$ *we have as* $s \to 0$

$$\int_0^\delta \psi(x) \Gamma_s(x, x) \chi(x) \, dx \sim \int_0^\delta \psi(x) \tilde{\Gamma}_s(x, x) \chi(x) \, dx + \sum_{j \geqq 1} \int_0^\delta \int_0^s \int_0^1 \psi(x) \tilde{\Gamma}_{s-t}(x, z) \chi(z) A_t^j(z, x) \, dz \, dt \, dx$$

where the remainder estimate is uniform in δ.

Using the terminology introduced in (4.5) we put $c_i := b\chi$, $i \geqq 1$, and find inductively for $0 < x, y, s \leqq 1$

$$A_s^j(x,y) = (-1)^j \frac{b(x)}{x} L_s^{j;c}(x,y)\chi(y), \quad j \geqq 1,$$

hence

$$\int_0^\delta \psi(x) \Gamma_s(x,x)\chi(x)\,dx \sim \sum_{j \geqq 1} (-1)^{j-1} \int_0^\delta \psi(x) L_s^{j;c}(x,x)\chi(x)\,dx. \tag{5.5}$$

6. Proof of the Main Theorem

Using the asymptotic representation of Theorem 5.3 we will now derive the asymptotic expansion of

$$\operatorname{tr} e^{-sT} = \int_0^1 \Gamma_s(x,x)\,dx$$

as $s \to 0$. To do so we choose $0 < \varepsilon \leqq 1/2$ and put $\sigma := s\varepsilon^{-2}$. Then we want to start with an asymptotic expansion of

$$I(\varepsilon, \sigma) := \varepsilon^2 \sigma \int_0^{1/2} \Gamma_{\varepsilon^2\sigma}(x,x)\,dx$$

as $\varepsilon^2 + \sigma^2 \to 0$. This will allow us the use Taylor expansions at $x = 0, 1$ and to separate the contributions from the singular points. We write

$$\begin{aligned} I(\varepsilon,\sigma) &= \left(\int_0^\varepsilon + \int_\varepsilon^{1/2} \right) \varepsilon^2 \sigma \Gamma_{\varepsilon^2\sigma}(x,x)\,dx \\ &=: I_1(\varepsilon,\sigma) + I_2(\varepsilon,\sigma). \end{aligned}$$

Choosing ψ and χ to be 1 on $[0, 1/2]$ we obtain from (5.5)

$$I_1(\varepsilon,\sigma) = \sum_{j=1}^N (-1)^{j-1} \int_0^\varepsilon \varepsilon^2 \sigma L_{\varepsilon^2\sigma}^{j;c}(x,x)\,dx + O_N((\varepsilon^2\sigma)^{\mu_N}) \tag{6.1}$$

where $\mu_N \to \infty$ as $N \to \infty$ i.e. the expansion (6.1) is ε, σ-asymptotic. To treat the individual terms in the sum we note that it follows from (4.6) that

$$L_{\varepsilon^2\sigma}^{j;c}(\varepsilon x, \varepsilon y) = \varepsilon^{j-2} L_\sigma^{j;c^\varepsilon}(x,y),$$

where $c_i^\varepsilon(x) := c_i(\varepsilon x)$. Thus we get

$$\begin{aligned} \varepsilon^2 \sigma \int_0^\varepsilon L_{\varepsilon^2\sigma}^{j;c}(x,x)\,dx &= \varepsilon^3 \sigma \int_0^1 L_{\varepsilon^2\sigma}^{j;c}(\varepsilon x, \varepsilon x)\,dx \\ &= \varepsilon^{j+1} \sigma \int_0^1 L^{j,c^\varepsilon}(x,x)\,dx. \end{aligned}$$

Now we plug in the Taylor expansion up to order N for each $c_i^\varepsilon=(\chi b)^\varepsilon$ around $x=0$. In view of the estimate (4.2) it follows that

$$\begin{aligned}&\varepsilon^2\sigma\int_0^\varepsilon L^{j,c}_{\varepsilon^2\sigma}(x,x)\,dx\\&=\sum_{\substack{0\leqq k\leqq N\\ \beta\in\mathbb{Z}_+^{j-1},|\beta|=k}}\varepsilon^{j+k+1}\sigma\frac{b^{(\beta_1)}(0)\dots b^{(\beta_{j-1})}(0)}{\beta_1!\dots\beta_{j-1}!}\int_0^1 L^{j,\beta-1}_\sigma(x,x)\,dx\\&+O_N(\varepsilon^{j+N+2})\end{aligned}\tag{6.2}$$

where we have written $\beta-1:=(\beta_1-1,\dots,\beta_{j-1}-1)$. Therefore, we only have to study the asymptotic expansion of the integrals on the right as $\sigma\to 0$. Using (4.3) with $\alpha=\frac{1}{2x}$ we obtain

$$\int_0^1 L^{j,\beta-1}_\sigma(x,x)\,dx=\int_0^1(2x)^{j+|\beta|-2}L^{j,\beta-1}_{\sigma/4x^2}(1/2,1/2)\,dx$$

and substituting $t:=\sigma/4x^2$

$$\int_0^1 L^{j,\beta-1}_\sigma(x,x)\,dx=\frac{1}{4}\sigma^{(j+|\beta|-1)/2}\int_{\sigma/4}^\infty t^{-(j+|\beta|+1)/2}L^{j,\beta-1}_t(1/2,1/2)\,dt\,.$$

Now we deduce from Theorem 4.2 with M sufficiently large

$$\begin{aligned}&\int_{\sigma/4}^{s_0} t^{-(j+|\beta|+1)/2}L^{j,\beta-1}_t(1/2,1/2)\,dt\\&=\left(\int_0^{s_0}-\int_0^{\sigma/4}\right)t^{-(j+|\beta|+1)/2}\left[L^{j,\beta-1}_t(1/2,1/2)-\sum_{l=0}^M t^{l+j-3/2}\frac{u_l^{j,\beta-1}(1/2,1/2)}{(4\pi)^{1/2}}\right]dt\\&+\int_{\sigma/4}^{s_0}\sum_{\substack{0\leqq l\leqq M\\ l\neq 1/2(|\beta|-j+2)}}t^{l-2+1/2(j-|\beta|)}u_l^{j,\beta-1}(1/2,1/2)(4\pi)^{-1/2}\,dt\\&+(4\pi)^{-1/2}u^{j,\beta-1}_{1/2(|\beta|-j+2)}(1/2,1/2)(\log 4s_0-\log\sigma)\,.\end{aligned}$$

The second integral has an asymptotic expansion in positive powers of $\sigma^{1/2}$ and the third has a (finite) asymptotic expansion involving only powers $\sigma^{m/2}$ with $m\geqq j-|\beta|-2$. Denoting the constant term in the expansion of

$$\int_{\sigma/4}^\infty t^{-(j+|\beta|+1)/2}L^{j,\beta-1}_t(1/2,1/2)\,dt$$

by

$$\text{p.f.}\int_0^\infty t^{-(j+|\beta|+1)/2}L^{j,\beta-1}_t(1/2,1/2)\,dt$$

(the "partie fini" of the integral) we arrive at the representation

$$\begin{aligned}&4\sigma\int_0^1 L^{j,\beta-1}_\sigma(x,x)\,dx\\&=:\sigma^{(j+|\beta|+1)/2}\left[\text{p.f.}\int_0^\infty t^{-(j+|\beta|+1)/2}L^{j,\beta-1}_t(1/2,1/2)\,dt\right.\\&\left.\quad-(4\pi)^{-1/2}u^{j,\beta-1}_{1/2(|\beta|-j+2)}(1/2,1/2)\log\sigma\right]+R_{j,\beta}(\sigma)\,,\end{aligned}\tag{6.3}$$

where we have the asymptotic expansion

$$R_{j,\beta}(\sigma) \sim \sum_{\substack{m \geq 1 \\ m \neq j+|\beta|+1}} R^m_{j,\beta} \sigma^{m/2} \quad \text{as} \quad \sigma \to 0 .$$

Lemma 6.1. *As $\varepsilon^2 + \sigma^2 \to 0$ we have the asymptotic expansion*

$$\begin{aligned} I_1(\varepsilon, \sigma) \sim & \sum_{\substack{j \geq 1, k \geq 0 \\ \beta \in \mathbb{Z}_+^{j-1}, |\beta| = k}} (-1)^{j-1} \varepsilon^{j+k+1} \frac{b^{(\beta_1)}(0) \dots b^{(\beta_{j-1})}(0)}{4\beta_1! \dots \beta_{j-1}!} \\ & \cdot \Bigg[\sigma^{(j+k+1)/2} \Bigg(\text{p.f.} \int_0^\infty t^{-(j+k+1)/2} L_t^{j,\beta-1}(1/2, 1/2)\, dt \\ & - (4\pi)^{-1/2} u^{j,\beta-1}_{1/2(k-j+2)}(1/2, 1/2) \log \sigma \Bigg) + \sum_{\substack{m \geq 1 \\ m \neq j+k+1}} R^m_{j,\beta} \sigma^{m/2} \Bigg] . \end{aligned}$$

The logarithmic terms occur only if $k - j$ is even and the term of lowest order is

$$\varepsilon^3 \sigma^{3/2} \log \sigma \left(\frac{b(0)}{8(\pi)^{1/2}} \right) .$$

Moreover, if b is odd at 0 i.e. $b^{(2m)} = 0$, $m \geq 0$, then all logarithmic terms do vanish.

Proof. The expansion follows from (6.1), (6.2), and (6.3). Certainly, logarithmic terms occur only if $k - j$ is even and the lowest order term occurs if $j = 2$, $k = 0$. It is easily calculated from (6.3) and (4.13). Now from $k = \beta_1 + \dots + \beta_{j-1}$ we see that at least one β_m must be even if $k - j$ is even hence no logarithmic terms occur if b is odd at 0. □

We turn to the expansion of I_2. Since $x \geq \varepsilon$ in the domain of integration we have $s/x^2 \leq \sigma$. Consequently, from the pointwise expansion of Γ_s given in Theorem 4.3 for $\varepsilon \leq x, y \leq 1/2$ we conclude

$$\begin{aligned} I_2(\varepsilon, \sigma) = & (4\pi)^{-1/2} \int_{j=0}^{N} (\varepsilon^2 \sigma)^{j+1/2} \int_\varepsilon^{1/2} \frac{w_j(x)}{x^{2j}(1-x)^{2j}} dx \\ & + O_N(\sigma^{N+1}) . \end{aligned}$$

Since $w_j \in C^\infty(\bar{I})$ it is easy to see that the integrals on the right have asymptotic expansions involving only the powers ε^m, $m \geq -2j+1$, and $\log \varepsilon$. We write

$$\begin{aligned} & \varepsilon^{2j+1} \int_\varepsilon^{1/2} \frac{w_j(x)}{x^{2j}(1-x)^{2j}} dx \\ & =: \varepsilon^{2j+1} \left(\text{p.f.} \int_0^{1/2} \frac{w_j(x)}{x^{2j}(1-x)^{2j}} dx + S_j \log \varepsilon \right) + T_j(\varepsilon) , \end{aligned}$$

where we have an expansion

$$T_j(\varepsilon) \sim \sum_{\substack{k \geq 1 \\ k \neq 2j+1}} T_{jk} \varepsilon^k .$$

Lemma 6.2. *As $\varepsilon^2+\sigma^2\to 0$ we have an asymptotic expansion*

$$I_2(\varepsilon,\sigma)$$
$$\sim \sum_{j\geq 0} \sigma^{j+1/2}(4\pi)^{-1/2}\left[\varepsilon^{2j+1}\left(\text{p.f.}\int_0^{1/2}\frac{w_j(x)}{x^{2j}(1-x)^{2j}}dx + S_j\log\varepsilon\right)+\sum_{\substack{k\geq 1\\ k\neq 2j+1}} T_{jk}\varepsilon^k\right].$$

Here w_j is a universal polynomial in the variables $a^{(l)}$, $l\geq 0$.

From these lemmas we deduce our principal result formulated in the introduction.

Proof of the Main Theorem. From the above lemmas we obtain an expansion of the following type:

$$\varepsilon^2\sigma\int_0^{1/2}\Gamma_{\varepsilon^2\sigma}(x,x)\,dx$$
$$\sim:\sum_{j\geq 1}(\varepsilon\sigma^{1/2})^j\left[A_j^1+A_j^2+B_j\log\varepsilon+C_j\log\sigma + \sum_{\substack{k\geq -j+1\\ k\neq 0}}(D_{jk}\varepsilon^k+E_{jk}\sigma^{k/2})\right]$$

where

$$A_j^1=\sum_{\substack{l\geq 1, m\geq 0, l+m+1=j\\ \beta\in\mathbb{Z}_+^{l-1}, |\beta|=m}}(-1)^{l-1}\cdot\frac{b^{(\beta_1)}(0)\dots b^{(\beta_{l-1})}(0)}{4\beta_1!\dots\beta_{l-1}!}\text{p.f.}\int_0^\infty t^{-j/2}L_t^{l,\beta-1}(1/2,1/2)\,dt,$$

$$A_{2j+1}^2=(4\pi)^{-1/2}\,\text{p.f.}\int_0^{1/2}\frac{w_j(x)}{x^{2j}(1-x)^{2j}}dx, \tag{6.4}$$

$$A_{2j}^2=0,$$

and

$$C_j=\sum_{\substack{l\geq 1, m\geq 0, l+m+1=j\\ \beta\in\mathbb{Z}_+^{l-1}, |\beta|=m}}(-1)^l\frac{b^{(\beta_1)}(0)\dots b^{(\beta_{l-1})}(0)}{4\beta_1!\dots\beta_{l-1}!}(4\pi)^{-1/2}u^{l,\beta-1}_{1/2(j+1-2l)}(1/2,1/2).$$

Substituting $\varepsilon^2:=s^{1/2}$, $\sigma:=s^{1/2}$, we get an expansion

$$s\int_0^{1/2}\Gamma_s(x,x)\,dx\sim:\sum_{j\geq 1}s^{j/4}(\tilde{\tilde{A}}_j+\tilde{\tilde{B}}_j\log s).$$

Resubstituting $s=\varepsilon^2\sigma$ and comparing the coefficients we see that

$$D_{jk}=E_{jk}=0,\qquad B_j=2C_j.$$

Hence the expansion takes the form

$$s\int_0^{1/2} \Gamma_s(x,x)\,dx \sim \sum_{j\geqq 1} s^{j/2}(\tilde{A}_j+\tilde{B}_j \log s),$$

where

$$\tilde{A}_j = A_j^1 + A_j^2, \quad \tilde{B}_j = C_j = 1/2 B_j.$$

Thus it follows for the logarithmic terms

$$\tilde{B}_{2j}=0, \quad \tilde{B}_{2j+1}=C_{2j+1}. \tag{6.5}$$

It remains to expand $\int_{1/2}^{1} \Gamma_s(x,x)\,dx$. The map $I \ni x \mapsto (1-x) \in I$ induces a unitary map U in $L^2(I)$. The operator

$$T' := U^* T U$$

is the Friedrichs extension of

$$\tau' := -d_x^2 + \frac{a'(x)}{x^2(1-x)^2}, \quad a'(x) := a(1-x).$$

Hence if Γ_s' denotes the kernel of $e^{-sT'}$ we have $\Gamma_s'(x,y) = \Gamma_s(1-x,1-y)$ and therefore

$$\int_0^{1/2} \Gamma_s'(x,x)\,dx = \int_{1/2}^{1} \Gamma_s(x,x)\,dx.$$

Noting the symmetry $w_j'(x) = w_j(1-x)$ the proof is completed. □

To conclude this section we want to point out a degenerate elliptic operator which is covered by our result. Consider the differential expression

$$\sigma := -d_x(x^a d_x), \quad 0 \leqq a < 2,$$

and denote by S the Friedrichs extension of $T_0(\sigma)$ in $L^2(I)$. Then we can apply the Main Theorem to compute $\operatorname{tr} e^{-sS}$ in view of the following fact [note that $A(a)$ below is $\geqq -1/4$ if $0 \leqq a < 2$].

Lemma 6.3. *Let* $A(a) := \dfrac{a(3a-4)}{4(2-a)^2}$. *Then S is unitarily equivalent to the Friedrichs extension of $T_0(\sigma')$ in $L^2([0,2/(2-a)])$ where*

$$\sigma' = -d_x^2 + \frac{A(a)}{x^2}.$$

Proof. Under the unitary transformation

$$\Phi_1 : L^2(I) \ni u \mapsto x^{a/4} u \in L^2(I, x^{-a/2}).$$

S is equivalent to the Friedrichs extension $\tilde{S}$ of

$$\tilde{\sigma} := -x^{a/2} d_x(x^{a/2} d_x) + \frac{a(3a-4)}{16} x^{a-2}$$

with domain $C_0^\infty(I)$ in $L^2(I, x^{-a/2})$. Since the principal part of $\tilde{\sigma}$ is the negative Laplacian of the metric $x^{-a}dx^2$ on $\mathbb{R}$ we have a unitary transformation

$$\Phi_2: L^2(I, x^{-a/2}) \ni u \to u \circ \psi^{-1} \in L^2([0, B(a)]),$$

where $\psi(x) := \int_0^x t^{-a/2}\,dt$ and $B(a) = \psi(1) = \frac{2}{2-a}$. An easy calculation completes the proof. □

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