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ON THE COMPACTNESS OF ISO-SPECTRAL POTENTIALS

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1. Introduction

Let M be a compact Riemannian manifold and Δ its Laplace-Beltrami operator. For a potential $q \in C^{\infty}(M)$ the Schrödinger operator $H_q := -\Delta + q$ is essentially self-adjoint in $L^2(M)$ and its spectrum consists of a sequence $\lambda_1 < \ldots < \lambda_j \to \infty$ of eigenvalues with finite multiplicity. Thus we may call $(\lambda_j)_{j \in \mathbb{N}}$ the spectrum of q. It is natural to ask to what extent the spectrum determines q. This question has attracted much interest in recent years but it does not seem to have a simple

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answer. First of all, if ψ is an isometry of M then q and $q\circ\psi$ are isospectral. If the set Is(q) of potentials isospectral with q contains only functions of this form q is called spectrally rigid. Then it is known that certain potentials on certain manifolds are spectrally rigid (cf. [2]) whereas the periodic solutions of the Korteweg - de Vries equation provide nontrivial isospectral deformations for potentials on S^1 (cf. [3]).

It is therefore interesting to investigate general structural properties of the set Is(q) for example compactness in various function spaces. If $M = S^1$ it is well known that Is(q) is compact in $C^\infty(S^1)$. In general dimensions, Gilkey has proved that compactness in $C^\infty(M)$ reduces to boundedness in certain Sobolev spaces ([1] Theorem 2.4). In this note we improve Gilkey's result which enables us to show that Is(q) is compact in $C^\infty(M)$ if dim $M \leq 3$. The proof is based on the asymptotic expansion of the trace of the heat kernel of H_q :

$$\mathrm{tr}\, e^{-sH_q} \underset{s\to 0}{\sim} (4\pi s)^{-n/2} \sum_{j\geq 0} s^j a_j(q) \ , \ n := \dim M \ .$$

With respect to the Riemannian measure on M the $a_j(q)$ are integrals of certain functions $u_j(q)$ over M which are universal polynomials in the covariant derivatives of q and of the curvature tensor of M. It can be shown ([1]) that for $j \geqq 2$ $a_j(q)$ equals the square of the Sobolev norm of q of order $j - 2$ plus integrals of products involving only lower order derivatives. This suggests the possibility of estimating the Sobolev norms of q in terms of the $a_j(q)$ hence in terms of the spectrum. In fact, using the Sobolev and Gagliardo-Nirenberg inequalities we show that Is(q) is bounded in every Sobolev space if it is bounded in the Sobolev space of order $3n_0 - 2$ where

$$n_0 := \inf \{m \in \mathbb{N} \mid m \geqq n/2\} \tag{1}$$

To verify the latter condition, however, and thus to prove compactness we have to introduce a restriction on the dimension.

I wish to thank Victor Guillemin and Marty Schwarz for several stimulating discussions on this subject.

2. Heat invariants and Sobolev norms

We will use Gilkey's result in the following form.

Theorem 1 a) *There are functions* $w_1, w_2 \in C^\infty(M)$ *depending only on the curvature of* M *such that*

$$a_0(q) = \text{vol } M, \quad a_1(q) = \int_M (q + w_1),$$

$$a_2(q) = \frac{1}{2} \int_M (q^2 + w_2) \ .$$

b) *For* $j \geq 3$ *we have*

$$a_j(q) = \frac{(-1)^j (j-1)!}{(2j-1)!} \int_M |D^{j-2} q|^2 + \sum_{k=1}^{j} \sum_{\substack{\alpha \in \mathbb{Z}_+^k \\ |\alpha| \leq \ell(k)}} \int_M P_{\alpha_1}^k (q) \ldots P_{\alpha_k}^k (q)$$

where D *denotes covariant derivative and* $(P_{\alpha_i}^k)_{1 \leq i \leq k}$ *is a family of differential operators with* C^∞ *coefficients depending only on the metric of* M*. Moreover, the orders satisfy the inequalities*

$$\text{ord } P_{\alpha_i}^k \leq j - 3, \quad \sum_{i=1}^{k} \text{ord } P_{\alpha_i}^k \leq 2(j-3) \ .$$

Proof a) This is proved in [1] Theorem 4.3.

b) This is contained in the proof of [1] Theorem 2.4. □

For $1 \leq p \leq \infty$ and $s \in \mathbb{R}$ we denote by $W_{s,p}(\mathbb{R}^n)$ the Sobolev space of distributions $u \in S'(\mathbb{R}^n)$ such that $\hat{u}$ is a function with $\widehat{k_s \hat{u}} \in L^p(\mathbb{R}^n)$ where $k_s(\xi) := (1+|\xi|^2)^{s/2}$; $W_{s,p}(\mathbb{R}^n)$ is a Banach space with the norm

$$\|u\|_{W_{s,p}(\mathbb{R}^n)} := \|\widehat{k_s \hat{u}}\|_{L^p(\mathbb{R}^n)} .$$

The space $W_{s,p}(M)$ consists of all distributions u on M such that $(fu) \circ \psi^{-1} \in W_{s,p}(\mathbb{R}^n)$ for every coordinate system (U,ψ) and every $f \in C_0^\infty(U)$. Choosing a finite atlas $(U_i,\psi_i)_{1\leq i\leq m}$ and a subordinate partition of unity $(f_i)_{1\leq i\leq m}$ we define a norm on $W_{s,p}(M)$ by

$$\|u\|_{W_{s,p}(M)} := \sum_{i=1}^{m} \|(f_i u) \circ \psi_i^{-1}\|_{W_{s,p}(\mathbb{R}^n)}. \tag{2}$$

This norm depends on the choices made but any two such norms are equivalent, and $W_{s,p}(M)$ is a Banach space with the norm (2). With n_0 defined in (1) we have the following consequences of the Gagliardo-Nirenberg inequalities (see [4] pp. 124).

Lemma 1 *For* $1 \leq p < \infty$ $W_{n_0,2}(M)$ *imbeds continuously into* $L^p(M)$ *i. e.*

$$\|u\|_{L^p(M)} \leq C \|u\|_{W_{n_0,2}(M)} , \quad u \in W_{n_0,2}(M), \tag{3}$$

where C *depends on* p *and* M. *If* $n_0 > n/2$ *then* (3) *also holds for* $p = \infty$.

Lemma 2 *Let* $n > 2$, $0 \leq \alpha \leq 1$, *and put*

$$\frac{1}{p} := \alpha \left(\frac{1}{2} - \frac{1}{n} \right) + \frac{1-\alpha}{2}.$$

Then $W_{1,2}(M) \subset L^p(M)$ *and*

$$\| u \|_{L^p(M)} \leqq C \; \| u \|^{\alpha}_{W_{1,2}(M)} \| u \|^{1-\alpha}_{L^2(M)}, \quad u \in W_{1,2}(M),$$

where C *depends on* M *and* α. *If* $n = 2$ *then the result is true for* $0 \leqq \alpha < 1$.

We are now able to reduce the compactness of Is(q) in $C^\infty(M)$ to boundedness in $W_{3n_o-2,2}(M)$. Gilkey ([1] Theorem 2.4) gives a reduction to boundedness in $W_{n',\infty}(M)$ where $n' := n$ if n is even, $n' := n-1$ if n is odd. To get this in terms of $W_{s,2}(M)$ norms one needs at least boundedness in $W_{3n_o,2}(M)$ which is a stronger condition.

Theorem 2 *If* Is(q) *is bounded in* $W_{3n_o-2,2}(M)$ *then it is compact in* $C^\infty(M)$.

Proof It is sufficient to show that Is(q) is bounded in $W_{j-2,2}(M)$ for every $j \geqq 3n_o$. This is true for $j = 3n_o$ by assumption so assume that Is(q) is bounded in $W_{j-3,2}(M)$ for some $j > 3n_o$. For $\tilde{q} \in$ Is(q) we obtain from Theorem 1 the estimate

$$\| \tilde{q} \|^2_{W_{j-2,2}(M)} \leqq C(1 + \sum_{\substack{1 \leqq k \leqq j \\ \alpha \in \mathbb{Z}^k_+}} \int_M | P^k_{\alpha_1}(\tilde{q}) \; \ldots \; P^k_{\alpha_k}(\tilde{q}) |$$

$$|\alpha|\leq \ell(k)$$

where C depends only on M and the spectrum of q and

$$\text{ord } P^k_{\alpha_i} \leq j-3 \ , \quad \sum_{i=1}^{k} \text{ord } P^k_{\alpha_i} \leq 2(j-3) \ . \tag{4}$$

We estimate the terms in the sum.

<u>1st case</u> We have ord $P^k_{\alpha_i} \leq j-3-n_0$ for all i. Then we deduce from Lemma 1 for $1 \leq p < \infty$, $1 \leq i \leq k$

$$\|P^k_{\alpha_i}(\tilde{q})\|_{L^p(M)} \leq C_p \|P^k_{\alpha_i}(\tilde{q})\|_{W_{n_0,2}(M)}$$

$$\leq C_p \|\tilde{q}\|_{W_{j-3,2}(M)} \ ,$$

hence the generalized Hölder inequality shows that the term considered is uniformly bounded on Is(q).

<u>2nd case</u> We have ord $P^k_{\alpha_i} \geq j-2-n_0$ for some i. But this can happen at most for two different values of i since otherwise by (4)

$$3(j-2-n_0) \leq \sum_{i=1}^{k} \text{ord } P^k_{\alpha_i} \leq 2(j-3)$$

and $j \leq 3n_0$, a contradiction. If there is only one such value, say i = 1, then we apply the generalized

Hölder inequality with $p_1 = 2$, and since ord $P^k_{\alpha_1} \leqq j-3$ the term is bounded as before.

Now assume ord $P^k_{\alpha_i} \geqq j-2-n_0$ for $i = 1,2$. We have to estimate

$$\int_M |P^k_{\alpha_1}(\tilde{q})\ P^k_{\alpha_2}(\tilde{q})\ F(\tilde{q})| \tag{5}$$

where by the argument of case 1 any L^p norm of $F(\tilde{q})$, $1 \leqq p < \infty$, is uniformly bounded on Is(q). Now choose $\varepsilon > 0$ such that

$$\frac{1}{2} - \frac{1}{n} < \frac{1}{2+\varepsilon} < \frac{1}{2}$$

and $0 < \alpha < 1$ such that

$$\frac{1}{2+\varepsilon} = \alpha \left(\frac{1}{2} - \frac{1}{n} \right) + \frac{1-\alpha}{2} .$$

We apply Lemma 2 and recall that ord $P^k_{\alpha_i} \leqq j-3$ to obtain the estimate

$$||P^k_{\alpha_1}(\tilde{q})||_{L^{2+\varepsilon}(M)} \leqq C\,||P^k_{\alpha_1}(\tilde{q})||^{\alpha}_{W_{1,2}(M)}\,||P^k_{\alpha_1}(\tilde{q})||^{1-\alpha}_{L^2(M)}$$

$$\leqq C\,||\tilde{q}||^{\alpha}_{W_{j-2,2}(M)}\ ||\tilde{q}||^{1-\alpha}_{W_{j-3,2}(M)}$$

hence by assumption

$$\| P^k_{\alpha_1}(\tilde{q}) \|_{L^{2+\varepsilon}(M)} \leq C \quad \|\tilde{q}\|^{\alpha}_{W_{j-2,2}(M)} \tag{6}$$

with C depending only on M and the spectrum of q. Applying the generalized Hölder inequality with $p_1 := 2+\varepsilon$, $p_2 := 2$, and $p_3 := \left(\frac{1}{2} - \frac{1}{2+\varepsilon} \right)^{-1}$ we see that the integral (5) can be estimated by the right hand side of (6), possibly with a different constant.

Summing up we arrive at the inequality

$$\|\tilde{q}\|^2_{W_{j-2,q}(M)} \leq C \ (1 + \ \|\tilde{q}\|^{\alpha}_{W_{j-2,2}(M)})$$

for $\tilde{q} \in Is(q)$ where $0 < \alpha < 1$ and C depends only on M and the spectrum of q. But this implies that Is(q) is bounded in $W_{j-2,2}(M)$. □

3. Compactness of Is(q)

To prove compactness of Is(q) in $C^{\infty}(M)$ we have to verify the condition of Theorem 2. The difficulty of doing this increases with the dimension of M. Our argument breaks down in dimensions greater 3 since then we do not control the L^p norms on Is(q) for $p > 4$.

Theorem 3 Is(q) *is compact in* $C^\infty(M)$ *if* dim $M \leq 3$.

Proof Suppose first that dim $M \leq 2$ such that $n_0 = 1$. By Theorem 2 we have to prove the boundedness of Is(q) in $W_{1,2}(M)$. It is clear from Theorem 1, a) that Is(q) is bounded in $L^2(M)$. From Lemma 2 we find $0 < \alpha \leq 1/3$ such that

$$\|\tilde{q}\|_{L^3(M)} \leq C \, \|\tilde{q}\|^{\alpha}_{W_{1,2}(M)} \, \|\tilde{q}\|^{1-\alpha}_{L^2(M)} \tag{7}$$

and from Theorem 1, b) with $j = 3$ we derive

$$\|\tilde{q}\|^2_{W_{1,2}(M)} \leq C \, (1 + \|\tilde{q}\|^3_{L^3(M)}) \tag{8}$$

where C depends only on M and the spectrum of q. Combining (7) and (8) we see that Is(q) is bounded in $W_{1,2}(M)$.

Now let dim $M = 3$ implying $n_0 = 2$. Thus we have to prove boundedness in $W_{4,2}(M)$. Arguing as before (with $\alpha = 1/2$ in (7)) we obtain the boundedness of Is(q) in $W_{1,2}(M)$. Using the full range of p in Lemma 2 with $n = 3$ we have for $2 \leq p \leq 6$ with $\alpha_p := 3/2 - 3/p$

$$\|\tilde{q}\|_{L^p(M)} \leq C_{M,p} \|\tilde{q}\|^{\alpha_p}_{W_{1,2}(M)} \|\tilde{q}\|^{1-\alpha_p}_{L^2(M)} , \quad \tilde{q} \in C^\infty(M) , \tag{9}$$

saying that Is(q) is bounded in $L^p(M)$ for $2 \leqq p \leqq 6$, too. Consider now a term in $a_4(\tilde{q})$ (Theorem 1,b)) which can be bounded by an integral

$$J := \int_M |P_1(\tilde{q})\ P_2(\tilde{q})\ \tilde{q}^k| \ , \quad \text{ord}\ P_i = 1,\ i = 1,2,$$

$k \leqq 2$.

Applying the generalized Hölder inequality with $p_1 = p_2 = p_3 = 3$ and using the inequalities (7) (with $\tilde{q}$ replaced by $P_i(\tilde{q})$ and $\alpha = 1/2$) and (9) it follows that J is bounded on Is(q). The remaining terms can be bounded by integrals

$$\int_M |P(\tilde{q})\ \tilde{q}^k| \ , \ \text{ord}\ P = 1,\ k \leqq 3\ ,$$

or

$$\int_M |\tilde{q}^k| \ , \ k \leqq 4\ ,$$

with obvious bounds in view of (9). Thus Is(q) is bounded in $W_{2,2}(M)$. Moreover, since $2 > 3/2$ $\|\tilde{q}\|_{L^\infty(M)}$ is bounded on Is(q), and by (9) the same is true for $\|P(\tilde{q})\|_{L^p(M)}$ if $2 \leqq p \leqq 6$ and P is any differential operator of order $\leqq 1$ with smooth coefficients. Examining the terms in $a_5(\tilde{q})$ it is then easy to show that Is(q) is bounded in $W_{3,2}(M)$. Observing that this implies the boundedness of $\| P(\tilde{q}) \|_{L^\infty(M)}$ and $\| P(\tilde{q}) \|_{L^p(M)}$, $2 \leq p \leq 6$, for differential operators of order 1 and 2, respectively, on Is(q), a similar study of $a_6(\tilde{q})$ leads to the conclusion that Is(q) is bounded in $W_{4,2}(M)$. The theorem is proved. □

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