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# THE ASYMPTOTIC EXPANSION OF MINAKSHISUNDARAM–PLEIJEL IN THE EQUIVARIANT CASE

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**1. Introduction.** In 1912 H. Weyl [24] determined the asymptotic behavior of the eigenvalues of the Laplacian for a compact domain in $\mathbb{R}^3$. Almost forty years later Minakshisundaram and Pleijel generalized this to a full asymptotic expansion for the corresponding Dirichlet series for a compact Riemannian manifold [20], giving rise to an extremely fruitful new development based on the identification of the coefficients in the expansion (see e.g. [16], [18], [4], [12], [2]). One should expect interesting information from an extension of these methods to singular spaces. In this paper we extend the Minakshisundaram–Pleijel expansion to the equivariant case. Even in the simplest nontrivial cases the structure of the coefficients becomes very complicated so we concentrate here on the existence of an expansion and the functions involved.

To describe the results let $M$ be a compact $n$-dimensional Riemannian manifold, $G$ a compact group of isometries, and $\rho$ a finite dimensional irreducible representation of $G$ in a complex vector space $V$. Denote by $E_\lambda$ the complexified eigenspace of the negative Laplacian $-\Delta$ with eigenvalue $\lambda$, $\lambda \geqslant 0$. Then $E_\lambda$ is $G$-invariant and we can consider the function

$$L_\rho(t) := \sum_{\lambda \geqslant 0} e^{-\lambda t} \dim \operatorname{Hom}_G(V, E_\lambda), \qquad t > 0.$$

Note that $\dim \operatorname{Hom}_G(V, E_\lambda)$ is the multiplicity of $\rho$ in $E_\lambda$ which is equal to $\dim E_\lambda^G$ in case $\rho$ is the trivial representation. Our main result (Theorem 4 below) states that $L_\rho$ has an asymptotic expansion as $t \to 0$ of the form

$$L_\rho(t) \sim (4\pi t)^{-m/2} \sum_{\substack{j \geqslant 0 \\ 0 \leqslant k \leqslant K_0 - 1}} a_{jk} t^{j/2} (\log t)^k$$

where $m := \dim {}^M/_G$ and $K_0$ is bounded by the number of different dimensions of $G$-orbits in M. Recall that the union of principal orbits, $M_0$, is open and dense in $M$ and that ${}^{M_0}/_G$ is a manifold whose dimension is $\dim {}^M/_G$ by definition. If $G$ is trivial Theorem 4 gives the classical result of Minakshisundaram and Pleijel mentioned above. If $G$ has no singular orbits (and hence $K_0 = 1$) it is contained

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in Donnelly [11]. The first order asymptotics have been obtained in [9] and [11] implying that $a_{00} = \operatorname{vol}^M / {}_G\dim V^H$, $H$ a principal isotropy group, and $a_{0k} = 0$, $k > 0$. Our result has been announced in [7]. It generalizes to transversally elliptic pseudodifferential operators acting on sections of bundles as will be shown in a subsequent paper.

We now sketch the main ideas leading to the proof of Theorem 4. For $t > 0$, the heat operator $e^{t\Delta}$ has a smooth kernel $\Gamma_t$ and it is well known that

$$\sum_{\lambda \geq 0} e^{-\lambda t} \dim E_\lambda = \operatorname{trace} e^{t\Delta} = \int_M \Gamma_t(p, p)\, dp. \tag{1}$$

The fundamental result of Minakshisundaram and Pleijel states that in a neighborhood of the diagonal $\Gamma_t$ has an asymptotic expansion

$$\Gamma_t(p,q) \sim (4\pi t)^{-n/2} e^{-d^2(p,q)/4t} \sum_{j=0}^{\infty} u_j(p,q) t^j, \qquad t \to 0. \tag{2}$$

This yields the asymptotic expansion of (1) immediately. In the equivariant case (1) generalizes to

$$L_\rho(t) = \frac{1}{\operatorname{vol} G} \int_{G \times M} \Gamma_t(p, gp) \bar{\chi}_\rho(g)\, dg\, dp$$

where $\chi_\rho$ denotes the character of $\rho$. Using the expansion (2) it is therefore enough to determine the asymptotic behavior of the integrals

$$\int_{G \times M} e^{-\varphi(g,p)/t} f(g, p)\, dg\, dp \tag{3}$$

with $\varphi(g, p) := d^2(p, gp)$ and $f(g, p) := u_j(p, gp)\bar{\chi}_\rho(g)$.

A very instructive special case arises if we consider an isometric action of the $k$-torus $T^k$ on Euclidean space $\mathsf{R}^{2n}$ (without trivial part). Splitting the representation of $T^k$ into irreducibles the integral (3) becomes

$$\int_{T^k} \int_{\mathsf{R}^{2n}} e^{-1/t \sum_{i=1}^{n} \sin^2 \omega_i(\theta) \|x_i\|^2} f(\theta, x)\, dx\, d\theta, \qquad f \in C_0^\infty(T^k \times \mathsf{R}^{2n}), \tag{4}$$

where each $\omega_i$ is a character of $T^k$. It is clear from the Weyl integration formula that the treatment of these integrals is decisive in understanding the general case. But it turns out that it is already a very difficult problem to determine the explicit asymptotic expansion of (4) in general. Leaving this open we attack the problem using the following more qualitative argument.

Obviously (3) depends only on the behavior of $\varphi$ and $f$ in an arbitrary neighborhood of the set

$$\mathscr{L} := \varphi^{-1}(0) = \{(g, p) \in G \times M \mid gp = p\}.$$

If $\mathscr{L}$ is a submanifold it is necessarily a nondegenerate critical submanifold of $\varphi$. Thus $\varphi$ can be replaced locally by $x_1^2$ in appropriate coordinates $(x_1, \ldots, x_n)$



and the asymptotic expansion follows readily. Unfortunately this happens only if the $G$ action has no singular orbits. More generally, if $\varphi$ is real analytic (locally in appropriate coordinates) we can apply Hironaka's theorem on the resolution of singularities (cf. Atiyah [1] and Bernshtein–Gelfand [5]) to replace $\varphi$ by a monomial. Thus we only have to deal with integrals of the type

$$\int_{\mathsf{R}_+^n} e^{-x_1^{\alpha_1} \cdots x_l^{\alpha_l}/t} h(x)\, dx, \qquad \alpha_i \in \mathsf{N}, \quad h \in C_0^\infty(\mathsf{R}_+^n), \quad \text{where}$$
$$\mathsf{R}_+^n := \{x \in \mathsf{R}^n \mid x_i \geq 0, 1 \leq i \leq n\}. \tag{5}$$

These have asymptotic expansions of the form

$$\sum_{\substack{j \geq 0 \\ 0 \leq k \leq l-1}} a_{jk}(f) t^{j/N} (\log t)^k$$

where $N := \text{l.c.m.}(\alpha_1, \ldots, \alpha_l)$ and the $a_{jk}$ are distributions with support in the set $\{x \in \mathsf{R}^n \mid x_1^{\alpha_1} \ldots x_n^{\alpha_n} = 0\}$. Thus we would get an asymptotic expansion of $L_\rho$ in case of an analytic action on an analytic manifold without information on $N$ and $l$, however. This has also been observed by Shafii-Dehabad [22]. To handle the general case we construct inductively and quite explicitly what we call a "weak resolution" of $\varphi(p, g) := d^2(p, g(p))$. A weak resolution is more general than a resolution but its definition (given in section 2) is somewhat technical. However, it is excellently suited for the present purpose and might be useful in other situations, too. Our explicit construction leads again to integrals of the type (5) without any analyticity assumptions. It also has the advantage of giving very precise information on $N$ and $l$ which can hardly be deduced from the general Hironaka theorem.

Thus, we have to expect logarithmic terms if $k_0 > 1$. Our first results in this direction were negative: there are no logarithmic terms for $S^1$ actions or more generally for actions of rank 1 groups (Theorem 7) and also not for *arbitrary* isometric group actions on the standard sphere (Theorem 5). This is surprising since for these actions the local structure of $\mathscr{L}$ can be as complicated as in the general case. However, since this paper has been written the first named author has obtained an example of a $T^2$-action on a (nonstandard) $S^6$ producing logarithmic terms. The example is based on the analysis of certain integrals of type (4), underlining their importance. Thus, our result is precise concerning the functions of $t$ involved in the expansion whereas the nature of the coefficients needs further clarification.

**2. Weak resolutions and asymptotic expansions.** We begin with the notion of resolution (cf. [1]). In the following a manifold is always assumed to be $C^\infty$ and paracompact.

*Definition* 1. Let $\hat{M}$ and $M$ be manifolds of dimension $n$ and $\phi : \hat{M} \to M$ a $C^\infty$ map.

(1) $\phi$ is called a *resolution* if $\phi$ is proper and there is a closed subset $X \subset M$ of measure 0 such that $\phi^{-1}(X)$ has measure 0 and $\phi$ restricts to a diffeomorphism $\hat{M}\backslash\phi^{-1}(X) \to M\backslash X$.

(2) $\phi$ is called a *resolution of* $\varphi \in C^\infty(M)$ (more precisely: of the set of zeros of $\varphi$) if

(a) $\phi$ is a resolution with $X := \varphi^{-1}(0)$,

(b) for each $p \in \hat{M}$ there is a coordinate system $(x_1, \ldots, x_n)$ centered at $p$ such that

$$\varphi \circ \phi(x) = k(x) \prod_{i=1}^{n} x_i^{\alpha_i} \qquad \text{near } p \text{ where } \alpha_i \in \mathbb{Z}_+$$

and $k$ is smooth and nowhere zero.

A typical example of a resolution is the following. Put

$$\hat{\mathsf{R}}^n := \{(\omega, x) \in \mathsf{R}P^{n-1} \times \mathsf{R}^n \mid x \in \omega\}$$

and define $\phi : \hat{\mathsf{R}}^n \to \mathsf{R}^n$ as restriction of the natural projection. $\hat{\mathsf{R}}^n$ results from "blowing up the origin" in $\mathsf{R}^n$ and coincides with the total space of the canonical line bundle on $\mathsf{R}P^{n-1}$.

As pointed out by Atiyah [1] the existence of resolutions has important applications in analysis. Our interest stems from the following essentially well known theorem on asymptotic expansions (cf. [6], [15]).

THEOREM 1. *Let $M$ be a manifold of dimension $n$ and $\varphi \in C^\infty(M)$, $\varphi \geqslant 0$. If there is a resolution $\phi : \hat{M} \to M$ of $\varphi$ then the integral*

$$I(f,t) := \int_M e^{-\varphi/t} f \tag{6}$$

*which is well defined for any smooth density $f$ with compact support has an asymptotic expansion as $t \to 0+$ of the form*

$$I(f,t) \sim \sum_{\substack{j \geqslant 0 \\ 0 \leqslant k \leqslant K}} a_{jk}(f) t^{j/J} \log^k t \tag{7}$$

*for some integers $J \geqslant 1$ and $K \geqslant 0$. The $a_{jk}$ are distributions on $M$ with support in $\varphi^{-1}(0)$.*

*Proof.* Denote by $(K_i)_{i \in I}$ and $(\hat{K}_i)_{i \in I}$ the components of $M\backslash\varphi^{-1}(0)$ and $\hat{M}\backslash\phi^{-1}(\varphi^{-1}(0))$ respectively. Then we have

$$\int_M e^{-\varphi/t} f = \sum_{i \in I} \int_{K_i} e^{-\varphi/t} f = \sum_{i \in I} \int_{\hat{K}_i} e^{-\varphi \circ \phi/t} \phi^* f.$$

By condition (2)(b) we see that each point in $\overline{\hat{K}_i}$ has a neighborhood $\hat{U}$ in $\hat{M}$ such that $\hat{U} \cap \overline{\hat{K}_i}$ is diffeomorphic to a (relatively) open subset of $\mathsf{R}^n_+ := \{x \in \mathsf{R}^n \mid x_j \geqslant 0, 1 \leqslant j \leqslant n\}$. Since $\phi$ is proper the same condition shows that $\operatorname{supp} \phi^* f \hat{K}_i \neq \emptyset$



only for finitely many $i$. Thus we see that $I(f,t)$ can be written as a finite sum of integrals of the form

$$\int_{\mathsf{R}^n_+} e^{-k(x) \prod_{j=1}^n x_j^{\alpha_j}} g(x)\, dx \tag{8}$$

where $k \in C^\infty(\mathsf{R}^n_+)$, $g \in C_0^\infty(\mathsf{R}^n_+)$, and $k(x) \neq 0$ for $x \in \operatorname{supp} g$. But the asymptotic expansion of the integrals (8) is known to exist and to be of the form (7) ([15] [8]). In particular, we can choose

$$J := \text{l.c.m.}\{\alpha_j \mid \alpha_j \neq 0\} \quad \text{and} \quad K := \#\{j \mid \alpha_j \neq 0\}. \quad \square$$

The existence of resolutions follows from the celebrated theorem of Hironaka in the analytic case. To get a result analogous to Theorem 1 in the $C^\infty$ case we will propose the notion of "weak resolution" which is still sufficient to yield the asymptotic expansion. In addition, it turns out to be explicitly constructable for the functions we are interested in. In particular, we will have good control on the integers $J$ and $K$ above.

We start with weakening Definition 1(1).

*Definition* 2. Let $M$ and $N$ be manifolds. A differentiable map $\phi : M \to N$ is called an *almost diffeomorphism* (more precisely: an almost diffeomorphism onto $\phi(M)$) if there is a closed subset $X \subset M$ of measure 0 such that $\phi$ restricts to a diffeomorphism $M\backslash X \to \phi(M\backslash X)$.

Certainly, every resolution is an almost diffeomorphism. Another typical example is

$$\phi : \mathsf{R} \times S^{n-1}_+ \ni (r, \omega) \mapsto r\omega \in \mathsf{R}^n$$

where $S^{n-1}_+ = \{(x_1, \ldots, x_n) \in S^{n-1} \mid x_n > 0\}$. Note that $\phi$ is not proper. We collect some obvious properties of almost diffeomorphisms.

LEMMA 1. *Let $\phi : M \to N$ be an almost diffeomorphism.*

(1) *If $K \subset N$ has measure 0 so has $\phi^{-1}(K)$.*

(2) *The restriction of $\phi$ to any open subset of $M$ is again an almost diffeomorphism.*

(3) *If $\phi' : N \to N'$ is an almost diffeomorphism then also $\phi' \circ \phi : M \to N'$.*

(4) *If $\phi : M \to N$ and $\tilde{\phi} : \tilde{M} \to \tilde{N}$ are almost diffeomorphism so is $\phi \times \tilde{\phi} : M \times \tilde{M} \to N \times \tilde{N}$.*

The weak analogue of Definition 1 now reads as follows.

*Definition* 3. Let $M$ be a manifold of dimension $n$.

(1) A family $(V_i, \phi_i)_{i \in I}$ of almost diffeomorphisms $\phi_i : V_i \to M$ is called a *weak resolution of* $M$ if for each $p \in M$ there are a finite subset $I' \subset I$ and functions $f_i \in C_0^\infty(V_i)$, $i \in I'$, such that

$$\sum_{i \in I'} f_i \circ \phi_i^{-1} = 1$$

a.e. in a neighborhood of $p$. Here the functions $f_i \circ \phi_i^{-1}$ (which are defined a.e. in $\phi_i(V_i)$) are extended by 0 to all of $M$.

(2) A family $(V_i, \phi_i)_{i \in I}$ of almost diffeomorphisms $\phi_i : V_i \to M$ is called a *weak resolution of* $\varphi \in C^\infty(M)$ (more precisely: of the set of zeros of $\varphi$) if

(a) $(V_i, \phi_i)_{i \in I}$ is a weak resolution with $V_i$ open in $\mathbb{R}^n$,

(b) $$\varphi \circ \phi_i(x) = k_i(x) \prod_{j=1}^{n} x_j^{\alpha_{ij}}$$

where $\alpha_{ij} \in \mathbf{Z}_+$ and $k$ is smooth and nowhere zero in $V_i$. We say that this weak resolution has *order at most* $L$ and *degrees contained in* $A \subset \mathbf{Z}_+$ if

$$\#\{j \mid \alpha_{ij} \neq 0\} \leqslant L \qquad \text{for all} \quad i \in I$$

and

$$\alpha_{ij} \in A \qquad \text{for all} \quad i \in I \quad \text{and} \quad 1 \leqslant j \leqslant n.$$

Before discussing this notion in more detail we want to show how a weak resolution of $\varphi$ leads to an asymptotic expansion of the integrals (6). All we need is the following simple fact the proof of which we omit.

LEMMA 2. *Let $(V_i, \phi_i)_{i \in I}$ be a weak resolution of $M$. Then for any compact subset $K$ of $M$ there are a finite subset $I' \subset I$ and functions $f_i \in C_0^\infty(V_i)$, $i \in I'$, such that*

$$\sum_{i \in I'} f_i \circ \phi_i^{-1} = 1$$

*a.e. in a neighborhood of $K$.*

Using property (1) of Definition 3 we can now repeat the proof of Theorem 1 to obtain the following result.

THEOREM 2. *Let $M$ be a manifold of dimension $n$ and $\varphi \in C^\infty(M)$, $\varphi \geqslant 0$. If there is a weak resolution $(V_i, \phi_i)_{i \in I}$ for $\varphi$ then we have for any smooth density with compact support the following asymptotic expansion as $t \to 0+$*

$$\int_M e^{-\varphi/t} f \sim \sum_{\substack{j \geqslant 0 \\ 0 \leqslant k \leqslant K}} a_{jk}(f) t^{j/J} \log^k t$$

*for some integers $J \geqslant 1$ and $K \geqslant 0$. The $a_{jk}$ are distributions on $M$ with support in $\varphi^{-1}(0)$.*

*If in addition the weak resolution has order at most $L$ and degrees contained in $A$, $A$ finite, then we can choose*

$$K = L - 1 \quad \text{and} \quad J = \text{l.c.m}\{a \mid a \in A \backslash \{0\}\}.$$



Thus the problem of expanding $I(f, t)$ has been reduced to the construction of a weak resolution of $\varphi$. We now collect some facts which are useful for this purpose and enable us to carry out the construction for $d^2(p, g(p))$ on $M \times G$. Note first of all that every family $(V_i, \phi_i)_{i \in I}$ of diffeomorphisms $\phi_i : V_i \to M$ satisfying $M = \bigcup_{i \in I} \phi_i(V_i)$ is a weak resolution of $M$. Secondly, every resolution $\phi : \hat{M} \to M$ is a weak resolution. More generally, we have the following fact.

LEMMA 3. *An almost diffeomorphism $\phi : M \to N$ is a weak resolution of $N$ if and only if $\phi$ is proper and surjective.*

*Proof.* 1. If $\phi$ is proper and surjective we choose for $p \in N$ a relatively compact neighborhood $U$ and $f \in C_0^\infty(M)$ such that $f \mid \phi^{-1}(\bar{U}) = 1$.

2. Let $\phi$ be a weak resolution of $N$ and $p \in N$. By the definition we can find functions $f_1, \ldots, f_k \in C_0^\infty(M)$ such that

$$\sum_i f_i \circ \phi^{-1} = 1$$

a.e. in a relatively compact neighborhood $U$ of $p$ i.e. we have for some measure zero subset $X$ of $N$

$$U \subset X \cup \bigcup_{i=1}^{k} \phi(\operatorname{supp} f_i).$$

Since $U$ is open we must have $U \cap X \subset \overline{U \backslash X}$ hence

$$U \subset \bigcup_{i=1}^{k} \phi(\operatorname{supp} f_i)$$

i.e. $\phi$ is surjective. We also have

$$\phi^{-1}(U) \subset X' \cup \bigcup_{i=1}^{k} \operatorname{supp} f_i$$

for some measure zero subset $X'$ of $M$. Reasoning as before we get

$$\phi^{-1}(U) \subset \bigcup_{i=1}^{k} \operatorname{supp} f_i .$$

Thus every point $p \in N$ has a compact neighborhood $V$ such that $\phi^{-1}(V)$ is compact i.e. $\phi$ is proper. □

The proof of the lemma shows that a weak resolution $(V_i, \phi_i)_{i \in I}$ of $M$ satisfies $\bigcup_{i \in I} \phi_i(V_i) = M$.

LEMMA 4. (1) *If $(V_i, \delta_i)_{i \in I}$ is a weak resolution of $M$ and $M' \subset M$ an open subset then $(V_i', \phi_i')_{i \in I}$ is a weak resolution of $M'$ where*

$$V_i' := V_i \cap \phi_i^{-1}(M'), \qquad \phi_i' := \varphi_i \mid V_i', \qquad i \in I.$$

(2) *If* $(V_i, \phi_i)_{i\in I}$ *is a weak resolution of* $M$ *and for each* $i$ $(V_{ij}, \phi_{ij})_{j\in I_i}$ *is a weak resolution of* $V_i$ *then* $(V_{ij}, \phi_i \circ \phi_{ij})_{i\in I, j\in I_i}$ *is a weak resolution of* $M$, *too.*

(3) *If* $(V_i, \phi_i)_{i\in I}$ *and* $(\tilde{V}_j, \tilde{\phi}_j)_{j\in J}$ *are weak resolutions of* $M$ *and* $\tilde{M}$ *respectively then* $(V_i \times \tilde{V}_j, \phi_i \times \tilde{\phi}_j)_{(i,j)\in I\times J}$ *is a weak resolution of* $M \times \tilde{M}$.

*Proof.* (1) and (3) are straightforward in view of Lemma 1. For the proof of (2) we note that $\phi_i \circ \phi_{ij} : V_{ij} \to M$ is an almost diffeomorphism by Lemma 1. Given $p \in M$ we find $f_i \in C_0^\infty(V_i)$, $i \in I'$, such that

$$\sum_{i\in I'} f_i \circ \phi_i^{-1} = 1 \qquad \text{a.e. in a neighborhood of } p.$$

According to Lemma 2 we can also find $f_{ij} \in C_0^\infty(V_{ij})$ for $j \in J_i'$, a finite subset of $J_i$, such that

$$\sum_{j\in J_i'} f_{ij} \circ \phi_{ij}^{-1} = 1 \text{ a.e. in a neighborhood of the compact set supp } f_i \subset V_i.$$

Put $\tilde{f}_{ij} := f_{ij} f_i \circ \phi_{ij} \in C_0^\infty(V_{ij})$ then

$$\sum_{\substack{i\in I'\\ j\in J_i'}} \tilde{f}_{ij} \circ (\phi_i \circ \phi_{ij})^{-1} = \sum_{i\in I'} (f_i \circ \phi_i^{-1}) \sum_{j\in J_i'} (f_{ij} \circ \phi_{ij}^{-1}) \circ \phi_i^{-1} = 1$$

a.e. on a neighborhood of $p$. □

We now turn to the weak resolution of functions. What we have done so far leads immediately to the following consequences which will be used below.

(1) A resolution of $\varphi$ gives rise to a weak resolution of $\varphi$: if $\phi : \hat{M} \to M$ is a resolution of $\varphi \in C^\infty(M)$ then every atlas $(V_i, \phi_i)_{i\in I}$ for $\hat{M}$ consisting of coordinate systems with property (2)(b) of Definition 1 defines a weak resolution of $\varphi$, namely $(V_i, \phi \circ \phi_i)_{i\in I}$.

(2) If $\phi : \hat{M} \to M$ is a resolution and $(V_i, \phi_i)_{i\in I}$ a weak resolution of $\varphi \circ \phi$ for some $\varphi \in C^\infty(M)$ then $(V_i, \phi \circ \phi_i)_{i\in I}$ is a weak resolution of $\varphi$. More generally, if $(V_i, \phi_i)_{i\in I}$ is a weak resolution of $M$ and $(V_{ij}, \phi_{ij})_{j\in J}$ is a weak resolution of $\varphi \circ \phi_i$ for some $\varphi \in C^\infty(M)$ and each $i$ then $(V_{ij}, \phi_i \circ \phi_{ij})_{i\in I, j\in J}$ is a resolution of $\varphi$.

(3) If $\pi : \tilde{M} \to M$ is a covering, $(V_i, \phi_i)_{i\in I}$ a weak resolution of $\varphi \circ \pi$ for some $\varphi \in C^\infty(M)$, and if $(U_j)_{j\in J}$ is an open cover of $\tilde{M}$ with $\pi | U_j$ injective for all $j$ then $(\phi_i^{-1}(U_j), \pi \circ (\phi_i | \phi_i^{-1}(U_j)))_{i\in I, j\in J}$ is a weak resolution of $\varphi$. In fact for $j$ fixed $(\phi_i^{-1}(U_j), \pi \circ (\phi_i | \phi_i^{-1}(U_j)))_{i\in I}$ is a weak resolution of $\varphi | \pi(U_j)$ by Lemma 4(1) and the previous remark.

(4) If $\varphi \in C^\infty(M)$ and $\varphi \in C^\infty(\tilde{M})$ have weak resolutions of order at most $L$ and $\tilde{L}$ and with degrees contained in $A$ and $\tilde{A}$ respectively then $\varphi \cdot \tilde{\varphi} : M \times \tilde{M} \to \mathbb{R}$ has a weak resolution of order at most $L + \tilde{L}$ and with degrees contained in $A \cup \tilde{A}$.

The following lemmas will be our main tools in constructing weak resolutions. The first allows to introduce polar coordinates.



LEMMA 5. *Let* $M$ *be a manifold and let* $\varphi \in C^\infty(\mathbb{R}^n \times M)$ *be homogeneous of degree* $k$ *in the space variable. If* $\tilde{\varphi} := \varphi | S^{n-1} \times M$ *has a weak resolution of order at most* $L$ *with degrees contained in* $A$ *then* $\varphi$ *has a weak resolution of order at most* $L + 1$ *and with degrees contained in* $A \cup \{k\}$.

*Proof.* Put $\phi : \mathbb{R} \times S^{n-1} \times M \ni (r, \omega, p) \to (r\omega, p) \in \mathbb{R}^n \times M$. Then $\varphi \circ \phi$ has a weak resolution of order at most $L + 1$ and with degrees contained in $A \cup \{k\}$ by remark 4 above. But $\phi$ is invariant under the natural $\mathbb{Z}_2$ action $(r, \omega, p) \to (-r, -\omega, p)$. This action is free and the quotient is diffeomorphic to $\hat{\mathbb{R}}^n \times M$ with $\hat{\mathbb{R}}^n$ described above. Hence we have a factorization $\phi = \hat{\phi} \circ \pi$ where $\hat{\phi}$ is a resolution and $\pi$ a covering. Thus the lemma follows from remarks (2) and (3). □

Though simple the next lemma is a powerful tool for our constructions. Note that—somewhat unexpectedly—we don't require $\varphi/\tilde{\varphi}$ below to be smooth.

LEMMA 6. *Let* $\varphi, \tilde{\varphi} \in C^\infty(M)$ *be nonnegative and satisfy the inequality*

$$\frac{1}{C}\tilde{\varphi} \leqslant \varphi \leqslant C\tilde{\varphi} \tag{9}$$

*for some positive constant* $C$. *If* $\tilde{\varphi}$ *has a weak resolution of order at most* $L$ *and with degrees contained in* $A$ *so has* $\varphi$.

*Proof.* Let $(V_i, \phi_i)_{i\in I}$ be a weak resolution for $\tilde{\varphi}$ and let

$$\tilde{\varphi} \circ \phi_i(x) = \tilde{k}_i(x) \prod_{j=1}^n x_j^{\alpha_{ij}}, \qquad x \in V_i,$$

with $\tilde{k}_i$ smooth and $\tilde{k}_i(x) \neq 0$. The right hand inequality in (9) implies that $\varphi \circ \phi_i$ is $C^\infty$-divisible by $\prod_{j=1}^n x_j^{\alpha_{ij}}$. In fact this is obvious if $\beta_i := \sum_{j=1}^n \alpha_{ij}$ is equal to zero or one and follows in general by induction on $\beta_i$. Hence

$$\varphi \circ \phi_i(x) = k_i(x) \prod_{j=1}^n x_j^{\alpha_{ij}}, \qquad x \in V_i,$$

for some $k_i \in C^\infty(V_i)$. But the left hand inequality in (9) implies that $k_i$ is nonzero. □

Finally we can "sum" weak resolutions under special circumstances.

LEMMA 7. *Suppose that the nonnegative functions* $\varphi \in C^\infty(M)$ *and* $\tilde{\varphi} \in C^\infty(\tilde{M})$ *have weak resolutions of order at most* $L$ *and* $\tilde{L}$ *respectively but both with degrees contained in* $\{0, d\}$ *for some* $d \in \mathbb{N}$. *Then* $\varphi + \tilde{\varphi} \in C^\infty(M \times \tilde{M})$ *has a weak resolution of order at most* $L + \tilde{L} - 1$ *and with degrees contained in* $\{0, d\}$.

*Proof.* By the functional properties of weak resolutions described above it is sufficient to construct a weak resolution of the functions

$$\varphi(x, y) := \prod_{i=1}^n x_i^{\alpha_i} + \prod_{j=1}^m y_j^{\beta_j}, \qquad (x, y) \in \mathbb{R}^{n+m},$$

where $\alpha_i, \beta_j \in \{0, d\}$ hence $\sum_i \alpha_i = kd$, $\sum_j \beta_j = ld$ for some $k, l \in \mathbb{Z}_+$. To do so we use induction on $k + l$. If $k + l \leqslant 1$ (or $k = 0$ or $l = 0$) there is nothing to prove. Now suppose $\alpha_{i_0} = \beta_{j_0} = d$ and consider the resolution $\phi : \hat{\mathbb{R}}^2 \times \mathbb{R}^{n+m-2} \to \mathbb{R}^{n+m}$ obtained by blowing up the origin in the $(x_{i_0}, y_{j_0})$ plane while leaving all other coordinates unchanged. On $\hat{\mathbb{R}}^2$ we can locally introduce coordinates $x_{i_0}, \bar{x}_{i_0}$ with $y_{j_0} = \bar{x}_{i_0} x_{i_0}$ or $y_{j_0}, \bar{y}_{j_0}$ with $x_{j_0} = \bar{y}_{j_0} y_{j_0}$. Thus we can use the induction hypothesis to get a weak resolution of $\varphi \circ \phi$ which gives a weak resolution of $\varphi$, too. Obviously this resolution has order at most $L + \tilde{L} - 1$ and degrees contained in $\{0, d\}$. □

**3. Weak resolution of the square of the Riemannian distance.** We will now apply the techniques of the previous section to construct a weak resolution of the function $\varphi_0 \in C(M \times G)$ given by $\varphi_0(p, g) := d^2(p, g(p))$ where $d$ denotes Riemannian distance. More precisely we are going to show that every point $(p, g_0) \in \mathscr{L} = \varphi_0^{-1}(0)$ has a neighborhood $U$ in $M \times G$ such that $\varphi_0 | U$ is $C^\infty$ and has a weak resolution thus defining a weak resolution of $\varphi_0$ in some neighborhood of $\mathscr{L}$.

To do so we equip $G$ with a biinvariant metric by choosing an $\mathrm{Ad}\, G$-invariant scalar product on the Lie algebra $\mathfrak{g}$ of $G$. For $p \in M$ we denote by $G_p$ the isotropy group and by $\mathfrak{m}_p$ an orthogonal complement to its Lie algebra in $\mathfrak{g}$. $Gp$ denotes the $G$-orbit of $p$ and $N_p$ the orthogonal complement of its tangent space in $T_pM$. Finally, let exp and $\exp_p$ be the exponential maps on $\mathfrak{g}$ and $T_pM$ respectively. With these notations we define a map $\phi : N_p \times \mathfrak{m}_p \times \mathfrak{m}_p \times G_p \to M \times G$ by

$$\phi(x, y, z, g) := \left(\exp z \exp_p x, \exp z \exp y g (\exp z)^{-1}\right)$$

and a function $f : N_p \times \mathfrak{m}_p \times \mathfrak{m}_p \times G_p \to \mathbb{R}_+$ by

$$f(x, y, z, g) := |x - dg_p(x)|^2 + |y|^2.$$

Lemma 8. *For every $g_0 \in G_p$ there is a neighborhood $W$ of $(0, 0, 0, g_0)$ in $N_p \times \mathfrak{m}_p \times \mathfrak{m}_p \times G_p$ such that*

$$\phi | W \quad \text{is a diffeomorphism}$$

*and*

$$\frac{1}{C} f \leqslant \varphi_0 \circ \phi \leqslant Cf \tag{10}$$

*on $W$ for some $C > 0$.*

*Proof.* The image of $d\phi(0, 0, 0, g_0)$ clearly contains $N_p, dR_{g_0}(\mathfrak{g})$ where $R_{g_0}$ denotes right translation by $g_0$, and the tangent space of $G_p$ in $M$. Thus $d\phi$ is surjective hence also injective since the dimensions are equal. Hence $\phi$ is a local diffeomorphism.



Since $G$ acts by isometries we have

$$\varphi_0 \circ \phi(x, y, z, g) = d^2(\exp_p x, \exp y \exp_p dg_p(x))$$
$$=: d^2(\phi_1(x, 0), \phi_1(dg_p(x), y))$$

where $\phi_1 : N_p \times \mathfrak{m}_p \to M$ is given by $\phi_1(x, y) = \exp y \exp_p x$. Now $\phi_1$ is a diffeomorphism in a neighborhood of $(0, 0)$ hence the pull back of $d^2$ under $\phi$ is equivalent to the Euclidean metric on $N_p \times \mathfrak{m}_p$ near $(0, 0)$ which is the inequality (10). □

It is now easy to see that $\mathscr{L}$ is a submanifold of $M \times G$ if and only if the $G$ action has no singular orbits. In fact, in the local coordinates above we have $\phi^{-1}(\mathscr{L}) = \{(x, y, z, g) \in W \mid dg_p x = x\}$ i.e., $\mathscr{L}$ is a submanifold of $M \times G$ if and only if $\overline{\mathscr{L}} := \{(x, g) \in N_p \times G_p \mid dg_p(x) = x\}$ is a submanifold of $N_p \times G_p$. But then the components of $\overline{\mathscr{L}}$ and $N_p \times G_p$ containing $(0, e)$ must be equal meaning that the identity component of $G_p$ acts trivially on $N_p$ which in turn is equivalent to the nonexistence of singular orbits in view of the slice theorem.

We need one more definition. For a $G$-invariant open set $U \subset M$ we put

$$K(U, G) := \#\{\dim Gq \mid q \in U\}$$

and for $p \in M$

$$K(p, G) := \inf\{K(U, G) \mid U \ G\text{-invariant}, p \in U\}.$$

Finally, let

$$K(M, G) := \sup_{p \in M} K(p, G). \tag{11}$$

We can now construct the desired weak resolution of $\varphi_0$.

Theorem 3. *There is a neighborhood $U$ of $\mathscr{L}$ such that $\varphi_0 | U$ is $C^\infty$ and has a weak resolution of order at most $K(M, G)$ and with degrees contained in $\{0, 2\}$.*

*Proof.* It is sufficient to prove existence of such a weak resolution in a neighborhood $V$ of any given point $(p, g_0) \in \mathscr{L}$. Choosing $W$ as in Lemma 8 and using Lemma 6 we need only construct a weak resolution for $|x - dg_p(x)|^2 + |y|^2$ in $W$. Picking a product neighborhood $V \subset W$ and applying Lemma 7 repeatedly we can further reduce to $|x - dg_p(x)|^2$ in a neighborhood of $(0, g_0)$ in $N_p \times G_p$. Now denote by $G^0$ the identity component of $G_p$ and put

$$\tilde{N} := \{x \in N_p \mid dg_p(x) = x \text{ for } g \in G^0\},$$
$$N_1 := \{x \in \tilde{N} \mid dg_{0,p}(x) = x\}.$$

We decompose orthogonally

$$\tilde{N} := N_1 \oplus N_2, \qquad N := N_1 \oplus N_2 \oplus N_3$$

and note that $\tilde{N}$ is $G_p$-invariant. Writing $x = x_1 + x_2 + x_3$ accordingly we find in a connected neighborhood of $(0, g_0)$

$$|x - dg_p(x)|^2 = |(x_1 + x_2) - dg_{0,p}(x_1 + x_2)|^2 + |x_3 - dg_p(x_3)|^2$$
$$= |(id - dg_{0,p})(x_2)|^2 + |x_3 - dg_p(x_3)|^2.$$

Since $id - dg_{0,p}$ is nonsingular on $N_2$ we can use Lemma 7 again and are left with $f(x_3, g) := |x_3 - dg_p(x_3)|^2$ in a neighborhood of $(0, g_0)$ in $N_3 \times G_p$.

We now use induction on $K(M, G)$. If $K(M, G) = 1$ we have no singular orbits hence $N_3 = 0$ and the proof is complete in this case. Assume then that the theorem is true for all compact manifolds $M'$ with isometric $G'$ actions such that $K(M', G') < K(M, G)$. Denoting by $S$ the unit sphere in $N_3$ we have a $G_p$ action on $S$ with $K(S, G_p) < K(M, G)$ since $G_p$ has no orbits of dimension 0 on $S$. The induction hypothesis gives a weak resolution of $\tilde{f} := f \mid S \times G_p$ of order at most $K(M, G) - 1$ and with degrees contained in $\{0, 2\}$: namely, in a neighborhood of $\tilde{f}^{-1}(0) = \{(\omega, g) \in S \times G_p \mid dg_p(\omega) = \omega\}$ $\tilde{f}$ is equivalent to $d_s^2(\omega, dg_p(\omega))$ with $d_s$ the standard distance on the sphere, hence the assertion follows from Lemma 6. Now introducing polar coordinates by

$$\phi : \mathrm{R} \times S \times G_p \ni (r, \omega, g) \mapsto (r\omega, g) \in N_3 \times G_p$$

we have $f \circ \phi(r, \omega, g) = r^2 \tilde{f}(\omega, g)$. Applying Lemma 5 the proof is completed. □

**4. The asymptotic expansion.** We now derive the asymptotic expansion of

$$L_\rho(t) = \sum_\lambda e^{-\lambda t} \dim \operatorname{Hom}_G(V, E_\lambda).$$

With an orthonormal basis $(\varphi_i^\lambda)$ for $E_\lambda$ we have for the character $\chi_\lambda$ of the representation of $G$ in $E_\lambda$

$$\chi_\lambda(g) = \sum_i \int_M \varphi_i^\lambda(p) \bar{\varphi}_i^\lambda(g(p))\, dp.$$

Denoting by $\chi_\rho$ the character of $\rho$ we obtain from the orthogonality relations ([25], p. 189).

$$\dim \operatorname{Hom}_G(V, E_\lambda) = \frac{1}{\operatorname{vol} G} \sum_i \int_{M \times G} \bar{\chi}_\rho(g) \varphi_i^\lambda(p) \bar{\varphi}_i^\lambda(g(p))\, dp\, dg.$$

On the other hand, the kernel $\Gamma_t$ of $e^{t\Delta}$ is given by the convergent series ([4], p. 205)

$$\Gamma_t(p, q) = \sum_\lambda e^{-\lambda t} \sum_i \varphi_i^\lambda(p) \bar{\varphi}_i^\lambda(q), \qquad p, q \in M, \quad t > 0.$$

The Cauchy–Schwarz inequality gives for every $N \in \mathrm{N}$

$$\left| \sum_{\lambda \leqslant N} e^{-\lambda t} \sum_i \varphi_i^\lambda(p) \bar{\varphi}_i^\lambda(q) \right|^2 \leqslant \Gamma_t(p, p) \Gamma_t(q, q)$$



hence the Lebesgue–Fatou Lemma implies the identity

$$L_\rho(t) = \frac{1}{\operatorname{vol} G} \int_{M \times G} \bar{\chi}_\rho(g) \Gamma_t(p, g(p))\, dp\, dg.$$

Now we have the estimate ([18], p. 50)

$$|\Gamma_t(p, q)| \leqslant C_1 t^{-n/2} e^{-C_2(d^2(p,q)/t)}, \qquad p, q \in M, \quad t > 0,$$

with some positive constants $C_1, C_2$. Thus for any $f \in C^\infty(M \times G)$ with $f \equiv 1$ in a neighborhood of $\mathscr{L}$ we obtain as $t \to 0$

$$L_\rho(t) \sim \frac{1}{\operatorname{vol} G} \int_{M \times G} \bar{\chi}_\rho(g) \Gamma_t(p, g(p)) f(p)\, dp\, dg.$$

Now we choose a neighborhood $U$ of $\mathscr{L}$ in $M \times G$ such that Theorem 3 holds in $U$ and the map $(p, g) \to (p, g(p))$ maps $U$ into a neighborhood of the diagonal in $M \times M$ where the Minakshisundaram–Pleijel expansion (2) is valid. Combining these facts with Theorem 2 and Theorem 3.1 in [9] we have proved the main result of this paper.

THEOREM 4. *We have the following asymptotic expansion as* $t \to 0$

$$L_\rho(t) \sim (4\pi t)^{-m/2} \sum_{\substack{j \geqslant 0 \\ 0 \leqslant k \leqslant K(M,G) - 1}} a_{jk} t^{j/2} \log^k t$$

*where* $m := \dim M/G$ *and* $K(M, G)$ *was defined in* (11). *Moreover*

$$a_{00} = \operatorname{vol} M/G \dim V^H,$$

*H a principal isotropy group, and*

$$a_{0k} = 0, \qquad k > 0.$$

*Remark.* If $G$ is connected we may reduce the integral over $G$ in the formula for $L_\rho$ to an integral over $T$, $T$ a maximal torus, by Weyl's integration formula. In particular, we may replace $K(M, G)$ by $K(M, T)$ in Theorem 4 showing that the exponent of the logarithmic factor is at most $\dim T = \operatorname{rank} G$. But $K(M, T) \geqslant K(M, G)$ is possible, e.g., if $G = \mathrm{SO}(3)$ acts on $S^2$ in the standard way.

**5. The case of the sphere.** As mentioned in the introduction, the logarithmic terms do occur, but so far their dependence on the geometry is not clear. In the remainder of this paper we present some nonexistence results.

It is clear from the proof of Theorem 4 that the log terms are somehow related to the singularities of $\mathscr{L}$. By Lemma 8 $\mathscr{L}$ is locally (up to Euclidean factors) diffeomorphic to

$$\{(x, g) \in \mathrm{R}^n \times \bar{G} \mid g(x) = x\}$$

where $\overline{G} \subset O(n)$ is an isotropy group of $G$. Thus the local singularities of $\mathscr{L}$ can be realized already by isometric actions on the standard sphere $S^n$ (by imbedding $\overline{G}$ in $O(n+1)$ in the usual way). Somewhat surprisingly we have the following result which seems to indicate a strong dependence of the log terms on the Riemannian metric.

THEOREM 5. *For $M = S^n$ the standard sphere, $G$ a closed subgroup of $O(n+1)$, and any finite dimensional representation $\rho$ of $G$ on a complex vector space $V$ we have an asymptotic expansion of the form*

$$L_\rho(t) \sim t^{-n/2} \sum_{j=0}^{\infty} a_j t^{j/2}, \qquad t \to 0.$$

The proof is independent of Theorem 4 and will follow from the next two propositions recalling that $\lambda_k(S^n) = k(k+n-1)$.

PROPOSITION 1. *There exist an integer $m \geq 1$ and polynomials $P_0, \ldots, P_{m-1} \in \mathbb{Q}[x]$ of degree at most $n-1$ such that*

$$\dim \operatorname{Hom}_G(V, E_{\lambda_k}(S^n)) = P_r(k)$$

*if $k \equiv r \bmod m$ and $k$ is sufficiently large.*

PROPOSITION 2. *For $m, r, s \in \mathbb{R}$ and $v \in \mathbb{Z}$ with $m > 0$ and $v, r, s \geq 0$ we have with certain $a_k \in \mathbb{R}$*

$$\sum_{k=0}^{\infty} e^{-(km+r)(kn+s)} k^v \sim t^{-(v+1)/2} \sum_{k=0}^{\infty} a_k t^{k/2}$$

*as $t \to 0$.*

We begin with the proof of Proposition 1. Denote by $\mathscr{P}$ the polynomial ring $\mathbb{C}[x_1, \ldots, x_{n+1}]$ and by $\mathscr{P}_k$ the homogeneous polynomials of degree $k \geq 0$ in $\mathscr{P}$. Put $P_{-1} := \mathscr{P}_{-2} := 0$. The Laplace operator $\Delta$ in $\mathbb{R}^{n+1}$ maps $\mathscr{P}$ to itself. We denote by $\Delta_k : \mathscr{P}_k \to \mathscr{P}_{k-2}$ the restriction to $\mathscr{P}_k$. It is well known that $E_{\lambda_k}(S^n) \simeq \mathscr{H}_k := \ker \Delta_k$ the isomorphism being given by restriction to the sphere. A direct computation shows the following.

LEMMA 9. *$\Delta_k : \mathscr{P}_k \to \mathscr{P}_{k-2}$ is surjective, $k \geq 0$.*

Now let $G \subset O(n+1)$ be a closed subgroup and $\rho: G \to GL(V)$ a finite dimensional representation of $G$. Then $G$ acts naturally on $\mathscr{P}$ leaving $\mathscr{P}_k$ and $\mathscr{H}_k$ invariant and the isomorphism $\mathscr{H}_k \simeq E_{\lambda_k}(S^n)$ is $G$-equivariant. Since $G$ is compact the exact sequence

$$0 \to \operatorname{Hom}(V, \mathscr{H}_k) \to \operatorname{Hom}(V, \mathscr{P}_k) \to \operatorname{Hom}(V, \mathscr{P}_{k-2}) \to 0$$

implies the exact sequences

$$0 \to \operatorname{Hom}_G(V, \mathscr{H}_k) \to \operatorname{Hom}_G(V, \mathscr{P}_k) \to \operatorname{Hom}_G(V, \mathscr{P}_{k-2}) \to 0.$$



Thus we have

LEMMA 10.

$$\dim \operatorname{Hom}_g(V, E_{\lambda_k}(S^n)) = \dim \operatorname{Hom}_G(V, \mathscr{P}_k) - \dim \operatorname{Hom}_G(V, \mathscr{P}_{k-2}), \qquad k \geq 0.$$

Now we come to the crucial point.

LEMMA 11. *The Poincaré series of $\bigoplus_{k=0}^{\infty} \operatorname{Hom}_G(V, E_{\lambda_k}(S^n))$ has the form*

$$\sum_{k=0}^{\infty} \dim \operatorname{Hom}_G(V, E_{\lambda_k}(S^n)) z^k = \frac{f(z)}{(1-z^{d_1}) \ldots (1-z^{d_s})}$$

*where $f \in \mathbb{Z}[z]$ and $d_1, \ldots, d_s$ are certain positive integers (namely the degrees of a set of generators for $\bigoplus_{k=0}^{\infty} \operatorname{Hom}_G(V, \mathscr{P}_k)$).*

*Proof.* Since by Lemma 5

$$\sum_{k=0}^{\infty} \dim \operatorname{Hom}_G(V, E_{\lambda_k}(s^n)) z^k = (1-z^2) \sum_{k=0}^{\infty} \dim \operatorname{Hom}_G(V, \mathscr{P}_k) z^k$$

it is enough to consider the Poincaré series of $\bigoplus_{k=0}^{\infty} \operatorname{Hom}_G(V, \mathscr{P}_k)$ for which the corresponding statement is essentially known. In fact by a classical result of Weyl [25] (cf. also [14], Ch. 10, Theorem 5) $\mathscr{P}^G$, the subspace of $G$-invariant polynomials, is a finitely generated algebra over $\mathbb{C} = \mathscr{P}_0^G$. Since $A := \bigoplus_{k=0}^{\infty} \operatorname{Hom}(V, \mathscr{P}_k)$ is a finitely generated graded $\mathscr{P} = \bigoplus_{k=0}^{\infty} \mathscr{P}_k$-module it follows from [23], 2.4.14 that $A^G = \bigoplus_{k=0}^{\infty} \operatorname{Hom}_G(V, \mathscr{P}_k)$ is a finitely generated graded $\mathscr{P}^G$-module.

But then by a result of Hilbert–Serre ([23], Proposition 2.5.4 or [3], Theorem 11.1) the Poincaré series of $\operatorname{Hom}_G(V, \mathscr{P}_k)$ is a rational function of the desired form. □

LEMMA 12. *Let $(a_k)_{k \geq 0}$ be a sequence of integers and assume*

$$\sum_{k=0}^{\infty} a_k z^k = \frac{f(z)}{(1-z^{d_1}) \ldots (1-z^{d_s})}$$

*for some $f \in \mathbb{Z}[z]$ and some integers $d_1, \ldots, d_s > 0$. Put $m := \text{l.c.m.}(d_1, \ldots, d_s)$. Then there exist $P_0, \ldots, P_{m-1} \in \mathbb{Q}[z]$ such that $a_k = P_r(k)$ if $k \equiv r \bmod m$ and $k$ is sufficiently large.*

*Proof.* Since $(1-z^{d_i})^{-1} = g_i(z)(1-z^m)^{-1}$ for some $g_i \in \mathbb{Z}[z]$ we may assume that $d_1 = \cdots = d_s = m$. If

$$f(z) =: \sum_{\mu=0}^{N} b_\mu z^\mu$$

it follows from

$$\frac{(s-1)!}{(1-z)^s} = \left(\frac{1}{1-z}\right)^{(s-1)} = \sum_{v=0}^{\infty} \binom{v+s-1}{s-1} z^v$$

that

$$\sum_{k=0}^{\infty} a_k z^k = \sum_{\mu=0}^{N} \sum_{v=0}^{\infty} b_\mu \binom{v+s-1}{s-1} z^{\mu+mv}$$

$$= \sum_{k=0}^{\infty} z^k \sum_{\substack{0\leqslant\mu\leqslant\min\{k,N\}\\ \mu\equiv k \bmod m}} b_\mu \binom{(k-\mu)/m+s-1}{s-1}.$$

Hence if $k \geqslant N$ and $k \equiv r \bmod m$

$$a_k = \sum_{\substack{0\leqslant\mu\leqslant N\\ \mu\equiv r \bmod m}} b_\mu \binom{(k-\mu)/m+s-1}{s-1} =: P_r(k)$$

and the lemma follows. □

The proof of Proposition 1 now follows from Lemma 11 and Lemma 12 noting that $\dim \operatorname{Hom}_G(V, E_{\lambda_k}(S^n)) \leqslant \dim V \dim E_{\lambda_k}(S^n) \leqslant Ck^{n-1}$ hence $\deg P_r \leqslant n-1$, $0 \leqslant r \leqslant m-1$.

For the proof of Proposition 2 we need the following result which follows easily from [10], Lemmas 8.1 and 8.5.

Lemma 13. *We have the following asymptotic expansions as* $t \to 0$.

(1) $$\sum_{k=0}^{\infty} e^{-k^2t} \sim \left(\frac{\pi}{4t}\right)^{1/2} + \frac{1}{2}.$$

(2) $$\sum_{k=0}^{\infty} e^{-k^2t}k^{2v} \sim (-1)^v \frac{(2v)!}{v!\,2^{2v+1}} \pi^{1/2} t^{-(2v+1)/2}, \qquad v \in \mathbb{N}.$$

(3) $$\sum_{k=0}^{\infty} e^{-k^2t}k^{2v+1} \sim \frac{v!}{2t^{v+1}} + \sum_{k=0}^{\infty} \frac{(-1)^k B_{2(k+v+1)}}{(2k)!\,(k+v+1)} t^k,$$

$v \in \mathbb{Z}_+$, *where* $B_n$ *denotes the* $n$th *Bernoulli number.*

*Proof of Proposition* 2. We have

$$\sum_{k=0}^{\infty} e^{-(km+r)(km+s)t} k^v = e^{-rst} \sum_{k=0}^{\infty} e^{-k(k+\alpha)m^2t} k^v$$

where $\alpha = (r+s)/m$. Thus it suffices to prove such an expansion for



$F_v(\alpha, t) =: \sum_{k=0}^{\infty} e^{-k(k+\alpha)t} k^v$, $\alpha \geqslant 0$. Since

$$\frac{\partial^j F_v}{\partial \alpha^j}(\alpha, t) = (-t)^j F_{v+j}(\alpha, t)$$

and hence for $N \in \mathbb{N}$

$$F_v(\alpha, t) = \sum_{j=0}^{N} \frac{\alpha^j}{j!}(-t)^j F_{v+j}(0, t) + \frac{\alpha^{N+1}}{N!}(-t)^{N+1} \int_0^1 F_{v+N+1}(s\alpha, t)(1-s)^N\, ds,$$

the result follows from $F_{v+N+1}(s\alpha, t) \leqslant F_{v+N+1}(0, t)$ and Lemma 8. □

Theorem 5 generalizes easily to spaces which are the image of a sphere under a Riemannian submersion with minimal fibres and with decktransformations acting transitively on each fibre. Thus we get the following generalization.

Theorem 6. *Let* $M$ *be a spherical space form,* $\mathbb{C}P^n$, *or* $\mathbb{H}P^n$. *Then for any closed subgroup* $G$ *of* $I(M)$ *and any finite dimensional complex representation* $\rho$ *of* $G$ *we have an asymptotic expansion of the form*

$$L_\rho(t) \sim t^{-n/2} \sum_{j=0}^{\infty} a_j t^{j/2}, \qquad t \to 0.$$

We conjecture that Theorem 5 generalizes to all compact symmetric spaces.

## 6. $S^1$ actions.

We are going to show that the asymptotic expansion of $L_\rho$ does not contain logarithmic terms when $G = S^1$. The basic observation leading to the proof of this fact is the following.

Lemma 14. *Let* $f \in C_0^\infty(\mathbb{R}_+^2)$ *satisfy*

$$\frac{\partial^{2j} f}{\partial x^j \partial y^j}(0, 0) = 0, \qquad j \geqslant 0. \tag{12}$$

*Then the asymptotic expansion of*

$$I(t) := \int_{\mathbb{R}_+^2} e^{-x^2y^2/t} f(x, y)\, dx\, dy$$

*for* $t \to 0$ *does not contain logarithmic terms.*

*Proof.* It is known (cf. e.g., [8], Theorem 1) that there is an asymptotic expansion

$$I(t) \sim \sum_{\substack{j\geqslant 0\\ 0\leqslant k\leqslant 1}} a_{jk} t^{j/2} \log^k t.$$

We put $\sigma := t/\epsilon^2$; then upon substituting this in the above expansion we get an asymptotic expansion with respect to the functions $(\epsilon^j\sigma^{j/2}\log^k\sigma)_{j\geq 1, k=0,1}$ and $(\epsilon^j\sigma^{j/2}\log^k\epsilon^2)_{j\geq 1, k=0,1}$ as $\epsilon^2 + \sigma^2 \to 0$, and in this expansion the coefficients of $\epsilon^j\sigma^{j/2}\log\sigma$ and $\epsilon^j\sigma^{j/2}\log\epsilon^2$ must be equal. Now write

$$I(t) = \int_0^\infty dx \left(\int_0^\epsilon + \int_\epsilon^\infty\right) dy\, e^{-x^2y^2/t} f(x, y) =: I_1(\sigma, \epsilon) + I_2(\sigma, \epsilon).$$

An easy calculation gives as $\epsilon^2 + \sigma^2 \to 0$

$$I_1(\sigma, \epsilon) \sim \sum_{j \geq 0} \frac{\epsilon^{j+1}}{j!} \int_0^1 \int_0^\infty e^{-x^2y^2/\sigma} y^j \frac{\partial^j f}{\partial y^j}(x, 0)\, dx\, dy$$

which has no terms of the form $\epsilon^j\sigma^{k/2}\log\epsilon^2$. On the other hand, since $t/y^2 \leq \sigma$ we get from Taylor's formula

$$I_2(\sigma, \epsilon) \sim \sum_{j \geq 0} t^{(j+1)/2} \int_0^\infty e^{-x^2} x^j / j!\, dx \int_\epsilon^\infty \frac{\partial^j f}{\partial x^j}(0, y)/y^{j+1}\, dy.$$

By Taylor's formula again we find

$$\int_\epsilon^\infty \frac{\partial^j f}{\partial x^j}(0, y)/y^{j+1}\, dy = C_j + \sum_{k=0}^{j} \frac{\partial^{j+k} f}{\partial x^j \partial y^k}(0, 0)/k!$$

$$\cdot \int_\epsilon^1 y^{k-j-1}\, dy + R_j(\epsilon)$$

where

$$C_j := \int_1^\infty \frac{\partial^j f}{\partial x^j}(0, y)/y^{j+1}\, dy$$

and $R_j(\epsilon)$ has an asymptotic expansion in nonnegative powers of $\epsilon$. The assumption (12) shows that $I_2(\sigma, \epsilon)$ contains no term $\epsilon^j\sigma^{k/2}\log\epsilon^2$, too, and the Lemma is proved. □

THEOREM 7. *Let $M$ be a compact Riemannian manifold of dimension $n$ with an effective isometric $G$-action, where $G$ is connected of rank one. Then we have the asymptotic expansion*

$$L_\rho(t) \sim (4\pi t)^{(-n+1)/2} \sum_{j \geq 0} a_j t^{j/2}$$

*as $t \to 0$, where $a_0 = \mathrm{vol}^M/_{S^1}$.*

*Proof.* We only need to show that there are no logarithmic terms. The remark after Theorem 4 shows that we may assume $G = S^1$. Fix $(p, g_0) \in \mathscr{L}$ and



assume that we can find a diffeomorphism $\phi: W \to U$ where $U$ is a neighborhood of $(p, g_0)$ in $M \times S^1$ and $W$ an open subset of $\mathsf{R}^{m_1} \times \mathsf{R}^{m_2} \times \mathsf{R}^{2k} \times \mathsf{R}$ such that

$$\varphi_0 \circ \phi(y_1, y_2, z, \theta) = |y_2|^2 + |z|^2\theta^2. \tag{13}$$

Here $\varphi_0(p, t) = d^2(p, g(p))$ as before. As in the proof of Theorem 4 we have to determine the asymptotic expansion of the integrals

$$I(f, t) := \int_{\mathsf{R}^{m_2} \times \mathsf{R}^{2k} \times \mathsf{R}} e^{-(|y_2|^2 + |z|^2\theta^2)/4t} f(y_2, z, \theta)\, d\theta\, dz\, dy_2$$

as $t \to 0$ where $f \in C_0^\infty(\mathsf{R}^{m_2} \times \mathsf{R}^{2k} \times \mathsf{R})$. Introducing polar coordinates $(r, \omega)$ in $\mathsf{R}^{m_2}$ and expanding the $r$-integral we do not encounter logarithmic terms. Hence we are left with the integrals

$$I_1(f, t) := \int_{\mathsf{R}^{2k} \times \mathsf{R}} e^{-|z|^2\theta^2/4t} f(z, \theta)\, d\theta\, dz.$$

Here $f \in C_0^\infty(\mathsf{R}^{2k} \times \mathsf{R})$ and we can assume that $f$ is invariant under the transformations $z \mapsto -z$ and $\theta \mapsto -\theta$. Introducing polar coordinates $(\rho, \sigma)$ in $\mathsf{R}^{2k}$ we find

$$I_1(f, t) = \int_{\mathsf{R}_+^2} e^{-\rho^2\theta^2/4t} \rho^{2k-1} \tilde{f}(\rho, \theta)\, d\theta\, d\rho$$

where $\tilde{f} \in C_0^\infty(\mathsf{R}_+^2)$ and its Taylor series around 0 contains only even powers of $\rho$ and $\theta$ by the above invariance property. But then

$$\frac{\partial^{2j}}{\partial\rho^j \partial\theta^j}(\rho^{2k-1}\tilde{f})(0, 0) = 0, \qquad j \geq 0,$$

and the proof is complete in view of Lemma 14.

For the proof of (13) we consider first the case that $S_p^1$ is finite. Using Lemma 8 we get a coordinate system $\phi: W \to U$, where $U$ is a neighborhood of $(p, g_0)$ in $M \times S^1$ and $W$ open in $\mathsf{R}^{m_1} \times \mathsf{R}^{m_2}$, such that

$$\frac{1}{C}|y_2|^2 \leq \varphi_0 \circ \phi(y_1, y_2) \leq C|y_2|^2$$

which corresponds to the case $k = 0$ in (13). But then the critical set of $\varphi_0 \circ \phi$ is given by $y_2 = 0$ and $\varphi_0 \circ \phi$ is nongenerate with Hess $\varphi_0 \circ \phi(y_1, 0) \mid \{0\} \times \mathsf{R}^{m_2}$ positive definite. The generalized Morse Lemma (cf. [19]) proves (13) in this case. Now assume $S_p^1 = S^1$. We decompose $N_p = T_pM$ orthogonally as $T_pM = V_1 \oplus \tilde{V}$ where $\tilde{V} := \{x \in T_pM \mid dg_{0,p}(x) = x\}$ and $\tilde{V}$ orthogonally as $\tilde{V} := V_2 \oplus V_3$ where $V_2$ is the subspace that carries the trivial part of the orthogonal representation of $S^1$ on $\tilde{V}$. In particular, $\dim V_3 = 2k$ for some $k \in \mathsf{Z}_+$. Using Lemma 8 again and the explicit form of orthogonal $S^1$ actions we can find a coordinate system $\phi_1 : W \to U$, $W$ open in $\mathbb{R}^{m_1} \times \mathbb{R}^{m_2} \times \mathbb{R}^{2k} \times \mathbb{R}$, such that $\varphi_0 \circ \phi_1$ satisfies

$$\frac{1}{C}(|y_2|^2 + |z|^2\theta^2) \leqslant \varphi_0 \circ \phi_1(y_1, y_2, z, \theta) \leqslant C(|y_2|^2 + |z|^2\theta^2). \tag{14}$$

Since $S^1$ can be covered by two coordinate systems centered at 1 and $-1$ we may assume $g_0 = \pm 1$ and consequently that $\varphi_0 \circ \phi_1$ is even in $\theta$. Also, by our previous considerations it is clear that we have to prove (13) for $\varphi_0 \circ \phi_1$ in a neighborhood of the points $(y_1, 0, 0, 0)$ only. We fix a point $(y_1^0, 0, 0, 0)$ and write

$$\varphi_0 \circ \phi_1(y_1, y_2, z, \theta) =: \varphi_1(y_1, y_2, z) + \theta^2 \varphi_2(y_1, y_2, z, \theta)$$

since $\varphi_0 \circ \phi_1$ is even in $\theta$. From (14) we get

$$\frac{1}{C}|y_2|^2 \leqslant |\varphi_1(y_1, y_2, z)| \leqslant C|y_2|^2. \tag{15}$$

Using Taylor's formula we obtain

$$\varphi_2(y_1, y_2, z, \theta) =: \varphi_2(y_1, 0, z, \theta) + 2\langle \Phi_1(y_1, \theta)(y_2) \,|\, z \rangle$$
$$+ \langle \Phi_2(y_1, y_2, z, \theta)(z) \,|\, z \rangle + \varphi_3(y_1, y_2, z, \theta)$$

since $\operatorname{grad}_{y_2} \varphi_2(y_1, 0, 0, \theta) = 0$ by (14). Here $\Phi_1, \Phi_2$ are smooth matrix functions, $\|\Phi_2(y_1, y_2, z, \theta)\| = O(|y_2|)$, and $\varphi_3 = O(|y_2|^2)$. From (14) again we have

$$\frac{1}{C}|z|^2 \leqslant |\varphi_2(y_1, 0, z, \theta)| \leqslant C|z|^2 \tag{16}$$

hence

$$\varphi_2(y_1, 0, z, \theta) = \langle \Phi_3(y_1, z, \theta)(z) \,|\, z \rangle$$

with $\Phi_3$ smooth and positive definite. Thus we find

$$\varphi_2(y_1, y_2, z, \theta) =: \langle \Phi(y_1, y_2, z, \theta)(z + \varphi_4(y_1, y_2, z, \theta)) \,|\, z + \varphi_4(y_1, y_2, z, \theta) \rangle$$
$$+ \varphi_5(y_1, y_2, z, \theta)$$

with $\Phi$ again smooth and positive definite, $\varphi_4$ linear in $y_2$, and $\varphi_5 = O(|y_2|^2)$. This gives

$$\varphi_0 \circ \phi_1(y_1, y_2, z, \theta) := \varphi_6(y_1, y_2, z, \theta) + \theta^2 \langle \Phi(z + \varphi_4) \,|\, z + \varphi_4 \rangle (y_1, y_2, z, \theta)$$

where $\varphi_6$ also satisfies (15). Now changing coordinates $y_1' = y_1$, $y_2' = y_2$, $z' = z + \varphi_4$, $\theta' = \theta$ we find a local diffeomorphism $\phi_2$ such that

$$\varphi_0 \circ \phi_2(y_1, y_2, z, \theta) =: \varphi_7(y_1, y_2, z, \theta) + \theta^2 \varphi_8(y_1, y_2, z, \theta)$$



where $\varphi_7$ and $\varphi_8$ satisfy (15) and (16), respectively. Applying the generalized Morse Lemma we may assume $\varphi_7(y_1, y_2, z, \theta) = |y_2|^2$ while $\varphi_8$ still satisfies (16). Applying this argument to $\varphi_8$ we obtain (13) since the required change of coordinates leaves $y_2$ and $\theta$ unchanged.

**Concluding remarks.** It is clear from the proof of Theorem 7 that the asymptotic expansion of the integral

$$\int_{M \times G} \Gamma_t(p, g(p)) f(p)\, dp\, dg$$

does not contain logarithmic terms if $G_p \subset S^1$ for $p \in \operatorname{supp} f$. We can use the arguments also to prove nonexistence in several other special situations, for example if $\dim {}^{M}/_{G} \leqslant 1$.

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