SPECTRAL ANALYSIS OF SINGULAR STURM-LIOUVILLE PROBLEMS WITH OPERATOR COEFFICIENTS

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We give a report on work in progress which has been done largely in collaboration with Bob Seeley.

1. Motivating examples

a) Consider a surface of revolution M in $\mathbb{R}^3$ homeomorphic to S^2, generated by a smooth curve

$$c(t) = (c_1(t), 0, c_3(t)), \ t \in [0,L],$$

parametrized by arc length. In the natural coordinate system

$$(0,L) \times (0,2\pi) \ni (t,\varphi) \mapsto (c_1(t)\cos\varphi, c_1(t)\sin\varphi, c_3(t)) \in M$$

the metric of M takes the form

$$g(t,\varphi) = \begin{pmatrix} 1 & 0 \\ 0 & c_1(t)^2 \end{pmatrix}$$

and the Laplacian Δ (which we define to be positive) becomes

$$= -\frac{1}{c_1(t)} \frac{\partial}{\partial t} (c_1(t) \frac{\partial}{\partial t}) - \frac{1}{c_1(t)^2} \frac{2}{\partial\varphi^2} .$$

The isometric S^1 action on M induces a unitary S^1 action on all eigenspaces of Δ and hence subdivides the spectrum according to the irreducible unitary representations of S^1. Let us denote the spectrum by

σ and by σ_κ the subset belonging to the representation κ. It turns out that the invariant spectrum σ_1 is determined by σ and contains already interesting geometric information on M (see [4]). Moreover, the invariant spectrum is precisely the spectrum of the Friedrichs extension T of the operator

$$-d_t^2 + \frac{2c_1(t)c_1''(t) - c_1'(t)^2}{4c_1(t)^2} =: -d_t^2 + q(t)$$

in $L^2([0,L])$ with domain $C_o^\infty((0,L))$ (see [4]). By construction we must have

$$c_1(0) = c_1(L) = 0, \; c_1'(0) = -c_1'(L) = 1.$$

Thus the potential q can be written

$$q(t) =: \frac{a(t)}{t^2(L-t)^2}$$

where $a \in C^\infty([0,L])$ and $a(0), a(L) = -1/4$ i.e. the analysis of the invariant spectrum reduces to the spectral analysis of a singular Sturm-Liouville problem. One may ask whether the method of heat invariants also generalizes to the spectra σ_κ, i.e. whether there is an asymptotic expansion of

$$\sum_{\lambda\in\sigma_\kappa} e^{-\lambda s} \text{ as } s \to 0 \quad .$$

This is in fact so (see [3]) and we have an asymptotic expansion of the form

$$\sum_{\lambda\in\sigma_\kappa} e^{-\lambda s} \sim (4\pi s)^{-1/2} \sum_{j\geq o} a_j^\kappa s^{j/2} \quad ,$$

where $a_o^\kappa = L$. For $\kappa = 1$ this implies the existence of an asymptotic expansion for tr e^{-sT} as $s \to 0$ and we are lead to ask whether this remains true for more general singular Sturm-Liouville operators, which cannot be handled by the group action approach.

b) It seems desirable to extend the spectral theory of the Laplacian on compact smooth manifolds to more general spaces allowing singularities. The most simple type of singularity is a cone over a smooth compact manifold N: we put

$$C_r(N) := (0,r) \times N$$

equipped with the metric

$$dx \otimes dx + x^2 g, \text{ g a smooth Riemannian metric on N,}$$

and call it a metric cone over N. As a special case we obtain $C_r(S)$, the n+1-ball of radius r. Cheeger ([7]) has developed a precise functional calculus for the Laplacian on $C_r(N)$ in terms of functions of the Laplacian on N, based on separation of variables and the Hankel transform. It is, however, possible to attack this problem without using separation of variables. Let us denote by $A^p(M)$ the smooth p-forms on a differentiable manifold M and by $L^2_p(M)$ the square integrable p-forms. Then we have a bijective linear map

$$\psi_p : C^\infty((0,r), A^p(N)) \times C^\infty((0,r), A^{p-1}(N)) \to C^\infty(C_r(N))$$

given by

$$\psi_p(\phi_1,\phi_2) = \pi^*\phi_1 + \pi^*\phi_2 \wedge dx$$

where $\pi : C_r(N) \to N$ is the natural projection. By a cumbersome but straightforward computation one finds that

$$\begin{aligned} &\psi_p^{-1} \circ \Delta \circ \psi_p \; (\phi_1,\phi_2) \qquad (1) \\ &= (-x^{2p-n} \frac{\partial}{\partial x} (x^{n-2p} \frac{\partial \phi_1}{\partial x}) + x^{-2}\Delta_N \phi_1 + (-1)^p \frac{2}{x} d_N \phi_2 \; , \\ &-\frac{\partial}{\partial x} (x^{2p-n-2} \frac{\partial}{\partial x} (x^{n+2-2p} \phi_2)) + x^{-2}\Delta_N \phi_2 + (-1)^p \frac{2}{x^3} \partial_N \phi_1) \; , \end{aligned}$$

where $d_N, \partial_N, \Delta_N$ denote the intrinsic operations on N. Regarding Δ as a symmetric operator in $L^2_p(C_r(N))$ with domain the smooth functions with compact support we see that Δ is unitarily equivalent to the operator

(1) with domain $C_0^\infty((0,r),A^p(N)) \times C_0^\infty((0,r),A^{p-1}(N)$ in the Hilbert space $L^2((0,r),L_p^2(N),x^{n-2p}dx) \oplus L^2((0,r),L_{p-1}^2(N),x^{n-2p+2}dx)$. The obvious transformation brings us to the operator

$$\Big(-\frac{\partial^2}{\partial x^2}\phi_1 + (\tfrac{n}{2}-p)(\tfrac{n}{2}-p-1)\ x^{-2}\phi_1 + x^{-2}\Delta_N\phi_1 + (-1)^p\,\frac{2}{x^2}\,d_N\phi_2\ ,$$

$$-\frac{\partial^2}{\partial x^2}\phi_2 + (\tfrac{n}{2}+2-p)(\tfrac{n}{2}+1-p)\ x^{-2}\phi_2 + x^{-2}\phi_2 + x^{-2}\Delta_N\phi_2 + (-1)^p\,\frac{2}{x^2}\,\partial_N\phi_1\Big)$$

in the Hilbert space $L^2((0,r),L_p^2(N),dx) \oplus L^2((0,r),L_{p-1}^2(N),dx)$ $\simeq L^2((0,r),L_p^2(N) \oplus L_{p-1}^2(N),dx)$ with domain $C_0^\infty((0,r),A^p(N) \times A^{p-1}(N))$. Setting

$$A := \begin{pmatrix} \Delta_N + (\frac{n}{2}-p)(\frac{n}{2}-p-1) & (-1)^p\ 2d_N \\ (-1)^p\ 2\partial_N & \Delta_N + (\frac{n}{2}+2-p)(\frac{n}{2}+1-p) \end{pmatrix}$$

we see that A is an elliptic formally selfadjoint operator on $A^p(N) \times A^{p-1}(N)$ hence is essentially selfadjoint in $L_p^2(N) \oplus L_{p-1}^2(N)$ and the unique selfadjoint extension has a pure point spectrum. It is also not hard to see that $A \geq -1/4$, and that $-1/4$ is in fact an eigenvalue if there are harmonic $(n+1)/2$ forms on N. Therefore, the selfadjoint extension of Δ on forms with compact support on $C_r(N)$ are unitarily equivalent to selfadjoint extensions of

$$-d_x^2 + \frac{A}{x^2} \tag{2}$$

in $L^2((0,r),L_p^2(N) \oplus L_{p-1}^2(N),dx)$ with domain $C_0^\infty((0,r),A^p(N) \times A^{p-1}(N))$. We will see that the operator (2) is bounded below so its Friedrichs extension T exists. As pointed out by Cheeger there is a choice of boundary conditions if the middle cohomology $H^{(n+1)/2}(N)$ does not vanish but we will restrict attention to T in what follows. Our problem is now to express spectral invariants of T by spectral invariants of A. Away from the singular point the analysis of Δ is classical so this will amount to expressing spectral invariants of Δ by spectral invariants of A. In particular, we are interested in the asymptotic expansion of tr e^{-sT} as $s \to 0$. A general calculus would also allow us to replace A by a more general operator, assuming for example that N already has cone-like singularities i.e. we could attack the analysis of iterated cone singularities as done in [8].

c) The calculus developed by Cheeger is special in the sense that it does not allow pertubations of the metric. A very natural extension of the metric cone would be a metric on $(0,r) \times N$ of the form

$$dx \otimes dx + x^2 g_x \tag{3}$$

where g_x is a smooth family of metrics on $[0,r]$ (this includes for example all normal geodesic balls); a metric of this type is called asymptotically cone-like. In the approach described above this leads to a singular operator Sturm-Liouville problem of the form

$$-d_x^2 + \frac{A(x)}{x^2} \tag{4}$$

where $A(x)$ is a smooth family of selfadjoint elliptic operators with $A(0) \geq -1/4$. If $A(x)$ is scalar this is essentially the problem discussed in a).

2. The heat equation: scalar case

We start with the problem discussed in 1a): consider an intervall $I = (0,L)$, $a \in C^\infty(\bar{I})$ with $a(0), a(L) \geq -1/4$, and the singular Sturm-Liouville operator

$$-d_x^2 + \frac{a(x)}{x^2(L-x)^2}\ , \ x \in I. \tag{5}$$

This operator is symmetric in $L^2(I)$ with domain $C_0^\infty(I)$, and by Hardy's inequality it is bounded below (here we need the $-1/4$ condition). Hence the Friedrichs extension T does exist and we want to determine the asymptotic expansion of tr e^{-sT} as $s \to 0$ (if it exists). This has been done in [6], [5], and [1] with three different methods. We give a short description of the approach in [1]. If $a(0), a(L) > -1/4$ then the domain $\mathcal{D}(T)$ of T is contained in $H^1(I)$ so the Sobolev inequality implies the existence of the heat kernel,

$$e^{-sT} u(x) = \int_I \Gamma_s(x,y)\, u(y)\, dy, \quad u \in L^2(I) .$$

Using the fact that (5) is a differential equation with regular singularities this follows in general and we also obtain good estimates of $\Gamma_s(x,y)$ near the singular points. In particular, e^{-sT} is trace class and

$$\operatorname{tr} e^{-sT} = \int_I \Gamma_s(x,x)\, dx .$$

By a simple reflection argument one may restrict attention to the left endpoint i.e. it suffices to expand

$$I(s) := \int_0^{L/2} \Gamma_s(x,x)\, dx .$$

Now choose $0 < \varepsilon < L/2$ and split the integral at $x = \varepsilon$,

$$I(s) = \int_0^{\varepsilon} + \int_{\varepsilon}^{L/2} \Gamma_s(x,x)\, dx =: I_<(s,\varepsilon) + I_>(s,\varepsilon) .$$

Away from 0 we obtain an expansion of $\Gamma_s(x,x)$ by the classical method of Minakshisundaram-Pleijel,

$$\Gamma_s(x,x) \sim (4\pi s)^{-1/2} \sum_{j\geq 0} s^j \frac{w_j(x)}{x^{2j}(L-x)^{2j}} ,$$

where w_j is a universal polynomial in the variables $a^{(k)}$, $0 \leq k \leq 2j$, and with coefficients in $C^\infty(\bar{I})$. This gives an expansion for $I_>(s,\varepsilon)$ as $s \to 0$ but we are not allowed to let $\varepsilon \to 0$ because the resulting integrals are divergent beyond the first one. The arbitrariness of ε suggests that we treat ε as an additional variable and the homogeneity of the equation forces us to introduce $\sigma := s/\varepsilon^2$ as second variable. Then it is easy to see that $I_>(\varepsilon^2\sigma, \varepsilon)$ has an asymptotic expansion with respect to the system of functions $(\sigma^{i/2}\varepsilon^{j}\log^{k}\varepsilon)_{\substack{i\geq -1\\ j\geq 0\\ 0\leq k\leq 1}}$ as $\varepsilon^2 + \sigma^2 \to 0$.

To achieve a similar expansion for $I_<$ near 0 we compare Γ_s with the heat kernel $\bar{\Gamma}_s$ of the Friedrichs extension of

$$-d_x^2 + \frac{a(0)}{x^2} \tag{6}$$

in $L^2(\mathbb{R}_+)$, using Duhamel's principle. Thus we obtain an asymptotic representation of $\Gamma_s(x,x)$ near 0 by a Neumann series built from $\bar{\Gamma}_s$. Since ε is now a variable we can use Taylor expansion on the coefficients rendering the terms in the series universal expressions in the variables $a^{(k)}(0)$, $k \geq 1$. It remains to expand certain convolution integrals in $\bar{\Gamma}_s$ the simplest one being

$$\int_0^{\varepsilon} \bar{\Gamma}_s(x,x)\, dx \,.$$

Now the homogeneity of (6) causes the following homogeneity of $\bar{\Gamma}_s$:

$$\bar{\Gamma}_s(x,y) = \alpha\, \bar{\Gamma}_{\alpha^2 s}(\alpha x,\alpha y)\,,\quad \alpha,s,x,y > 0\,. \tag{7}$$

Substituting $x = \varepsilon u$ and using (7) with $\alpha = \varepsilon$ we find

$$\int_0^{\varepsilon} \bar{\Gamma}_s(x,x)\, dx = \int_0^1 \varepsilon\, \bar{\Gamma}_{\varepsilon^2\sigma}(\varepsilon u,\varepsilon u)\, du = \int_0^1 \bar{\Gamma}_\sigma(u,u)\, du\,.$$

Again with (7) and $\alpha = 1/u$ and with the substitution $x = \sigma/u$ we obtain

$$\int_0^1 \bar{\Gamma}_\sigma(u,u)\, du = \int_0^1 \frac{1}{u}\, \bar{\Gamma}_{\sigma/u^2}(1,1)\, du = 2\int_\sigma^\infty \frac{1}{x}\, \bar{\Gamma}_x(1,1)\, dx\,. \tag{8}$$

So all we need is the asymptotic expansion of $\bar{\Gamma}_s$ at $(1,1)$ as $s \to 0$ which is classical. The other terms in the Neumann series can be treated in a similar way leading to an expansion of $I_<$ in terms of the functions $(\sigma^{i/2}\varepsilon^{j}\log^{k}\sigma)_{\substack{i\geq -1\\ j\geq 0\\ 0\leq k\leq 1}}$ (the appearance of logarithmic terms is no surprise in view of the $\frac{1}{x}$ factor in (8)).

Summing the two expansions leads to an expansion of I(s) of the following form:

$$I(s) \sim (4\pi s)^{-1/2} \sum_{j\geq o} s^{j/2}(A_j + B_j + C_j \log s) \ . \tag{9}$$

Here A_j is the "interior term",

$$A_{2j} = \text{constant term in the expansion of} \tag{10}$$

$$\int_{\varepsilon}^{L-\varepsilon} \frac{w_j(x)}{x^{2j}(L-x)^{2j}}\, dx \quad \text{as } \varepsilon \to 0 \ ,$$

$$A_{2j+1} = 0,\ j \geq 0 \ .$$

B_j and C_j are "boundary terms" i.e. universal expressions in $a^{(k)}(0)$, $k \geq 0$. In particular,

$$C_o = C_1 = 0,\ C_2 = -\frac{b_0(0) + b_L(0)}{8\sqrt{\pi}} \ ,$$

where we have written $\dfrac{a(x)}{x^2(L-x)^2} = \dfrac{a_i}{(i-x)^2} + \dfrac{b_i(i-x)}{(i-x)}$ as $x \to i$, $i = 0,L$.

It is interesting to note that all C_j vanish if b_0 and b_L are odd at 0 which is the case in example 1a) above.

3. The heat equation: operator coefficients

We now consider the following situation. Let H be a Hilbertspace, H_1 a dense subspace, and A(x) a family of semibounded selfadjoint operators with common domain H_1, $x \in \bar{I}$. We assume that each A(x) has pure point spectrum and that $A(0) \geq -1/4$. Moreover, we require that the map $\bar{I} \ni x \mapsto A(x) \in L(H_1,H)$, the space of bounded linear operators from H_1 with the graph norm of A(0) to H, is smooth. Then

$$-d_x^2 + \frac{A(x)}{x^2} \tag{11}$$

is a symmetric operator in $L^2(I,H)$ with (dense) domain $C_o^\infty(I,H_1)$, and

Hardy's inequality shows again that this operator is bounded from below. So the Friedrichs extension T exists and we ask for an asymptotic expansion of tr e^{-sT} as $s \to 0$ if it exists. A natural condition for this to hold is that $e^{-sA(x)}$ is trace class for each $x \in \bar{I}$ and that expansions

$$\operatorname{tr} e^{-sA(x)} \sim s^{-\alpha} \sum_{\substack{j \geq 0 \\ 0 \leq k \leq k_0}} s^{\mu_j} \log^k s \; a_{jk}(x)$$

do exist for some $\alpha > 0$ and with $\mu_j \to \infty$. In fact, we pose the stronger condition that expansions of the form

$$\operatorname{tr} p(A, A', \dots, A^{(k)})(x) \; e^{-sA(x)} \sim s^{-\alpha} \sum_{\substack{j \geq 0 \\ 0 \leq k \leq k_0}} s^{\mu_j} \log^k s \; a_{jk}(p,x) \tag{12}$$

do exist for $x \in \bar{I}$ and any (noncommutative) polynomial p in A and its derivatives. In this general setting one can prove the existence of an expansion

$$\operatorname{tr} e^{-sT} \sim s^{-\alpha-1/2} \sum_{\substack{j \geq 0 \\ 0 \leq k \leq k_0+1}} s^{\gamma_j}(A_j + B_j + C_{jk} \log^k s) \; . \tag{13}$$

Here A_j is again an "interior contribution" built similar to (10) from the functions $a_{jk}(p,x)$ in (12) for suitable p. B_j and C_{jk} are "boundary terms" depending only on the derivatives of A at 0 and L.

The proof of (13) can be done following essentially the pattern of the scalar case. The technical difficulties are, however, considerable. The first step is to prove the existence of an operator valued heat kernel for T i.e.

$$e^{-sT} u(x) = \int_I \Gamma_s(x,y)(u(y)) \; dy$$

for $x,y \in I$, $s > 0$, and $u \in L^2(I,H)$. Thus $\Gamma_s(x,y)$ is a bounded linear operator in H and the arguments for the scalar case can be generalized to yield good estimates for the operator norm. If Γ_s were trace class in H we would get

$$\operatorname{tr} e^{-sT} = \int_I \operatorname{tr}_H \Gamma_s(x,x)\, dx$$

as expected. The crucial step to prove this is a modification of the Minakshisundaram-Pleijel expansion:

$$\Gamma_s(x,y) \sim (4\pi s)^{-1/2} e^{-\frac{(x-y)^2}{4s}} \sum_{j\geq o} s^j U_j(x,y)\, e^{-s\frac{A(y)}{y^2}} . \tag{14}$$

Here the U_j are polynomials in $A(x)$, $A(y)$, and their derivatives, and the expansion is uniform with respect to the trace norm in compact subsets of $I \times I$. In a neighborhood of 0 we compare with the heat kernel $\bar{\Gamma}_s$ of the Friedrichs extension of

$$-d_x^2 + \frac{A(0)}{x^2}$$

which again enjoys the scaling property (7). The treatment of the Neumann series is more complicated due to the unboundedness of the operators $A(x)$; here we need regularity theorems for weak solutions of (11).

The situation is much simpler if A is a constant function as in example 1b). Then it is enough to derive the expansion for

$$\int_o^{L/2} \operatorname{tr} \Gamma_s(x,x)\, dx .$$

Using (7) and the analogous properties of $\bar{\Gamma}_s$ we find as in the scalar case

$$\int_o^{L/2} \operatorname{tr} \Gamma_s(x,x)\, dx \sim \int_o^{L/2} \operatorname{tr} \bar{\Gamma}_s(x,x)\, dx = \int_o^{L/2} \frac{1}{x} \operatorname{tr} \bar{\Gamma}_{s/x^2}(1,1)\, dx$$

$$= 2 \int_{4s/L^2}^{\infty} \frac{1}{u} \operatorname{tr} \bar{\Gamma}_u(1,1)\, du .$$

Thus the expansion follows from (14) and the expansions for

$$\operatorname{tr} A^k e^{-sA},\ k \geqq 0 ,$$

whose coefficients are linear combinations of the coefficients in the expansion with $k = 0$. The contributions to the constant term in the expansion are (nonlocal) spectral invariants of A leading to Cheeger's formulas in the case of cone-like singularities.

4. The resolvent

For a semibounded selfadjoint operator T we can also study the resolvent $(T + z)^{-1}$, $\operatorname{Im} z \neq 0$. If γ is a suitably chosen path around the spectrum of T and if the resolvent has modest growth at infinity we have

$$e^{-sT} = \frac{1}{2\pi i} \int_{\gamma} e^{2z} (T + z)^{-1}\, dz .$$

Thus an asymptotic expansion of $\operatorname{tr}(T + z)^{-1}$ for large z implies an asymptotic expansion of $\operatorname{tr} e^{-sT}$ for small s. For nonsingular elliptic operators this expansion is well known ([9]). In the simplest singular case, the Friedrichs extension $\bar{T}$ in $L^2(\mathbb{R}_+)$ of the operator

$$-d_x^2 + \frac{a}{x^2} ,\ a \geqq -1/4 ,$$

the kernel of the resolvent $(\bar{T} + z^2)^{-1}$ can be determined explicitly,

$$(\bar{T} + z^2)^{-1}(x,y) = (xy)^{1/2}\, I_\nu(xz)\, K_\nu(yz) ,$$

where $0 < x \leqq y$, $\nu := (a + 1/4)^{1/2}$, $\operatorname{Im} z \neq 0$, and I_ν, K_ν are modified Bessel functions (cf. [6]). This operator is not trace class but we can define a "distributional trace" by

$$\int_0^\infty \varphi(x)\, x\, I_\nu(xz)\, K_\nu(xz)\, dx,\ \varphi \in C_0^\infty(\mathbb{R}) .$$

The well known asymptotic expansion of Bessel functions for large arguments suggests the following generalization. Consider a "symbol"

$\sigma(x,\zeta)$ where $x \in \mathbb{R}$ and $\zeta \in C := \{z \in \mathbb{C} \mid |\arg z| < \pi-\varepsilon\}$ for some $\varepsilon > 0$. We require σ to be smooth in x and rapidly decreasing and to have an asymptotic expansion as $\zeta \to \infty$ in C,

$$\sigma(x,\zeta) \sim \sum_{\alpha,m} \sigma_{\alpha m}(x)\ \zeta^{\alpha} \log^{m} \zeta \quad ,$$

where α runs through a discrete set of complex numbers, $\operatorname{Re}\sigma \to -\infty$, each m is a nonnegative integer, and there are only finitely many $\sigma_{\alpha m} \not\equiv 0$ for fixed α. Then we may ask whether there is an asymptotic expansion of

$$\int_0^\infty \sigma(x,xz)\ dx \tag{15}$$

as $z \to \infty$ in C. This expansion has been determined in [5] and has the following form:

$$\int_0^\infty \sigma(x,xz)\ dx \sim \sum_{k\geq 0} z^{-k-1} \int_0^\infty \frac{\zeta^k}{k!}\ \sigma^{(k)}(0,\zeta)\ d\zeta \tag{16}$$

$$+ \sum_{\alpha,m} \int_0^\infty \sigma_{\alpha m}(x)\ (xz)^{\alpha} \log^{m}(xz)\ dx$$

$$+ \sum_{\substack{\alpha,m \\ -\alpha\in\mathbb{N}}} z^{\alpha} \log^{m+1} z\ \sigma_{\alpha m}^{(-\alpha-1)}(0) \Big/ (m+1)(-\alpha-1)!\ .$$

The first and third sum contains "boundary" terms and the middle sum "interior" terms all of which can be viewed as "moments" of σ. The integrals, however, may be divergent and have to be defined suitably. For example, the integral

$$\int_0^\infty \varphi(x)\ x^{\alpha} \log^{m} x\ dx =: I(\alpha)$$

with $\varphi \in C_0^\infty(\mathbb{R})$ is analytic in α if $\operatorname{Re}\alpha > -1$. Integrating by parts we see that $I(\alpha)$ extends meromorphically to the whole complex plane. At a pole, we substract the singular part and call the resulting value the "regular analytic extension". In this way all the integrals in (16) are defined.

The more general case of the operator

$$-d_x^2 + \frac{a(x)}{x^2}$$

can also be handled by this approach. To do so we use the resolvent of $\bar{T}$ as parametrix and obtain a convergent Neumann series. Each term in the series turns out ot be of the form (15) and the expansion follows from (16).

The method can also be extended to handle (constant) operator coefficients. We choose $m \in \mathbb{N}$ such that for a given selfadjoint operator $A \geqq -1/4$ with discrete spectrum

$$\sum_{\nu^2-1/4 \in \text{spec } A} \nu^{-2m} < \infty .$$

Denoting by $k_\nu^m(x,y,z)$ the kernel of

$$(-d_x^2 + \frac{\lambda}{x^2} + z^2)^{-m}, \quad \nu = \sqrt{\lambda+1/4} \, ,$$

we find that the distributional trace of

$(-d_x^2 + \frac{A}{x^2} + z^2)^{-m}$ is given by

$$\int_0^\infty \varphi(x) \; x^{2m-1} \sum_\nu k_\nu^m(1,1,xz) \; dx \, .$$

To apply (16) we have to show that

$$\sigma(x,\zeta) := x^{2m-1} \sum_\nu k_\nu^m(1,1,\zeta)$$

has an asymptotic expansion as $\zeta \to \infty$. This is done by means of an expansion for $(-d_x^2 + \frac{A}{x^2} + z^2)^{-m}$ in terms of $(A + \mu)^{-\ell}$. This argument parallels the expansion (14) and uses the calculus of pseudodifferential operators with operator coefficients.

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