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Regular Singular Asymptotics

JOCHEN BRÜNING

University of Ausburg, D-8900 Augsburg, West Germany

AND

ROBERT SEELEY

University of Massachusetts, Boston, Massachusetts 02125

INTRODUCTION

Various asymptotic expansions reduce to an integral of the form

$$\int_0^\infty \sigma(x, xz)\, dx, \qquad z \to \infty \tag{0.1}$$

where $\sigma(x, \zeta)$ has an expansion as $\zeta \to \infty$ in terms of the functions $\zeta^\alpha (\ln \zeta)^m$, α complex, m integer $\geqslant 0$. For instance, the singular differential operator which is the Friedrichs extension in $L^2(0, \infty)$ of

$$A = -\partial_x^2 + x^{-2} a, \tag{0.2}$$

where a is constant $\geqslant -\frac{1}{4}$, has resolvent with a symmetric kernel involving Bessel functions,

$$(xy)^{1/2} I_\nu(xz)\, K_\nu(yz), \qquad x \leqslant y, \quad \nu = (a + \tfrac{1}{4})^{1/2}.$$

Callias [3] treats this for $a \geqslant \frac{3}{4}$, in which case A is essentially self-adjoint. The resolvent does not have finite trace, but it does have a "distributional trace" given by the integral

$$\int_0^\infty \phi(x)\, x I_\nu(xz)\, K_\nu(xz)\, dx$$

for ϕ in the Schwartz space $\mathscr{S}(R^1)$. This is an example of (0.1), and the function $\sigma(x, \zeta) = x\phi(x)\, I_\nu(\zeta)\, K_\nu(\zeta)$ has an expansion in terms of $\zeta^{-1}, \zeta^{-2},\ldots$ as $\zeta \to \infty$ in any open sector $|\arg \zeta| < \pi/2 - \varepsilon$. This example is developed in Section 3 below, and Section 4 generalizes it to the case where the term a in (0.2) is a C^∞ function with $a(0) \geqslant -\frac{1}{4}$ and $|a(x)| \leqslant C(1 + x)$. This is an

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alternate approach to recent results of Brüning [1], who constructs the heat kernel for an operator like (0.2), but on $L^2(0, 1)$ rather than $L^2(0, \infty)$. Section 5 outlines another application of (0.1), rederiving Uhlmann's expansion for conical refraction [8].

A significant but difficult generalization would be to the case where $a(x)$ is an appropriate unbounded operator on a Hilbert space. If this operator is independent of x, the generalization does not seem too hard, and it allows a derivation of the expansions obtained by Cheeger at conical singularities of "product" type.

A very similar general expansion has been given by Callias, Markenscoff, and Uhlmann in [3–5], and of course the basic idea occurs in [8, 3]; similar phenomena are found in the work of Brüning and Heintze [2]. These papers consider

$$\int_0^\infty f(s/x, x)\, x^{-w} dx, \qquad \mathrm{Re}(w)<0, \quad s \to 0+$$

which corresponds essentially to (0.1) with

$$\sigma(x, \zeta) = f(1/\zeta, x)\, \zeta^{-w}, \qquad s = 1/z.$$

The proof in [5] uses the Mellin transform, which is similar to the analytic continuation method given here.

The expansions in [1, 3] are for the heat kernel; allowing z to be complex in our expansion of the resolvent permits a passage to the heat kernel by contour integration.

Earlier treatments of (0.1) in the case where $\sigma(x, \zeta)$ has the form $f(x)\, h(x\zeta)$, and with more general conditions on $h(x)$ at $x=0$, are found in [10, 11].

The operator (0.2) has a "regular singular" point at $x=0$. It seems natural to expect this kind of singularity to lead to a resolvent with an expansion in terms of xz; hence the name "regular singular asymptotics."

Thanks are due to Callias, Uhlmann, and Richard Melrose for useful discussions related to this topic.

1. The Expansion

Suppose that $\sigma(x, \zeta)$ is defined on $R^1 \times C$, where C is an open wedge

$$C = \{|\arg \zeta| < \pi - \varepsilon\}$$

in the complex ζ plane, $\varepsilon > 0$. Suppose that σ is C^∞ in x, with derivatives analytic in ζ for ζ in C, and that σ has an expansion

$$\sigma(x, \zeta) \sim \sum \sigma_{\alpha m}(x)\, \zeta^\alpha (\ln \zeta)^m, \qquad \zeta \to \infty \tag{1.1}$$

where each $\sigma_{\alpha m}$ is in $\mathscr{S}(R^1)$, α runs through a discrete set of complex numbers, $\mathrm{Re}(\alpha) \to -\infty$, each m is an integer $\geqslant 0$, and there are at most finitely many m for each α. The relation (1.1) is taken to mean that for all $x \geqslant 0$, ζ in C, and non-negative integers J, K, M,

$$\left| x^J \partial_x^K \left[\sigma(x, \zeta) - \sum_{\mathrm{Re}(\alpha) \geqslant -M} \sigma_{\alpha m}(x)\, \zeta^\alpha \ln^m \zeta \right] \right| \leqslant C_{JMK} |\zeta|^{-M}, \qquad |\zeta| \geqslant 1, \quad 0 \leqslant x \leqslant |\zeta|/c_0 \tag{1.2a}$$

for some constant c_0. (We are really interested in $\sigma(x, xz)$, which may be defined only for $|z| \geqslant c_0$, or $|\zeta|/x \geqslant c_0$.) For small ζ we assume that

$$\int_0^1 \int_0^1 s^K |\sigma^{(K)}(\theta s t, s\xi)|\, ds\, dt \leqslant C_K \tag{1.2b}$$

where $\sigma^{(K)}(x, \xi) = \partial_x^K \sigma(x, \xi)$, and C_K is independent of θ for $0 \leqslant \theta \leqslant 1$, and of $\xi \in C$ for $|\xi| = c_0$, a constant. (By rescaling z, we can assume that $c_0 = 1$.) Condition (1.2b) would follow if for $x \leqslant s$, $|\sigma^{(K)}(x, s\xi)| \leqslant g_K(s)\, s^{-K}$ with $\int_0^1 g_K(s)\, ds < \infty$, a condition given in [5].

Theorem. *Assuming (1.2a)–(1.2b), then the expansion of*

$$\int_0^\infty \sigma(x, xz)\, dx, \qquad z \to \infty \text{ in } C$$

has three kinds of terms. There are "boundary terms"

$$\sum_{k \geqslant 0} z^{-k-1} \int_0^\infty \frac{\zeta^k}{k!} \sigma^{(k)}(0, \zeta)\, d\zeta \tag{Ia}$$

where $\sigma^{(k)}(x, \zeta) = \partial_x^k \sigma(x, \zeta)$. Further, each term $\sigma_{\alpha m}(x)\, \zeta^\alpha \ln^m \zeta$ in the given expansion (1.1) contributes an "interior term"

$$\int_0^\infty \sigma_{\alpha m}(x) (xz)^\alpha \ln^m (xz)\, dx. \tag{Ib}$$

Finally, if α is a negative integer, there is a further contribution

$$z^\alpha (\ln z)^{m+1} (m+1)^{-1} \sigma_{\alpha m}^{(-\alpha-1)}(0)/(-\alpha-1)!. \tag{Ic}$$

The integrals in (Ia) and (Ib), involving certain "moments" of $\sigma(x, \zeta)$ and of the terms in its expansion (1.1), may be divergent, and must be appropriately defined. The intent in (Ib) is to expand the integral as a combination of terms

$$\int \sigma_{\alpha m}(x)\, x^\alpha \ln^k x\, dx, \qquad 0 \leqslant k \leqslant m.$$

Since $\sigma_{\alpha m}$ is in $\mathcal{S}$, these latter integrals can be (and are) defined by the standard identification of the functions

$$x_+^\alpha \ln^m x = x^\alpha (\ln x)^m, \qquad x>0$$
$$= 0, \qquad x<0$$

as distributions on $\mathcal{S}(R^1)$, by analytic continuation, as follows. For ϕ in $\mathcal{S}(R^1)$, the integral

$$\int_0^\infty \phi(x)\, x^\alpha \ln^m x\, dx \tag{1.3}$$

is analytic for $\mathrm{Re}(\alpha) > -1$. Extend this meromorphically to all α; for instance, integrate by parts, using the primitives of $x^\alpha \ln^m x$ which vanish at $x=0$. Define the integral (1.3) for $\mathrm{Re}(\alpha) \leqslant -1$ by means of this meromorphic extension; if α is a pole, subtract the singular part and evaluate just the regular part. We call this the "regular analytic extension." For example,

$$\int_0^\infty \phi(x)\, x^\alpha dx = -\int_0^\infty \phi'(x) \frac{x^{\alpha+1}}{\alpha+1} dx$$
$$= -\int_0^\infty \phi'(x) \frac{x^{\alpha+1}-1}{\alpha+1} dx + \frac{\phi'(0)}{\alpha+1}.$$

The first equation defines the integral for $\mathrm{Re}(\alpha) > -2$, $\alpha \neq -1$. The second equation defines it for $\alpha = -1$: drop the singular part $\phi'(0)/(\alpha+1)$ and take the limit as $\alpha \to -1$ in the other term, finding

$$\int_0^\infty \phi(x)\, x^{-1} dx = -\int_0^\infty \phi'(x) \ln x\, dx.$$

This agrees with Hadamard's "partie fini" interpretation. Incidentally, the terms (Ic) above can be seen as the result of this "regularizing" process at the poles—see the discussion after (1.7).

The integral in (Ia) can likewise be defined by regular analytic continuation. The integral for $0 \leqslant \zeta \leqslant 1$ converges by (1.2b) with $\theta = 0$; the integral for $1 \leqslant \zeta < \infty$ can be written as a finite sum of terms of the form

$$c \int_1^\infty \zeta^k \zeta^\alpha \ln^m \zeta\, d\zeta \tag{1.4}$$

plus a convergent remainder, in view of (1.2a). As for (1.4), we have for $\mathrm{Re}(\beta) < -1$

$$\int_\varepsilon^\infty \zeta^\beta \ln^m \zeta\, d\zeta = \partial_\beta^m(-\varepsilon^{\beta+1}/(\beta+1)) \tag{1.5a}$$

and this continues analytically to all $\beta \neq -1$. For $\beta = -1$, the Taylor expansion of $\varepsilon^{\beta+1}$ in powers of $\beta+1$ gives the regular analytic continuation

$$\int_\varepsilon^\infty \zeta^{-1} \ln^m \zeta\, d\zeta = -(\ln \varepsilon)^{m+1}/(m+1). \tag{1.5b}$$

Similarly, the integral $\int_0^\varepsilon \zeta^\beta \ln^m \zeta\, d\zeta$ converges for $\mathrm{Re}(\beta) > -1$, and regular analytic continuation gives

$$\int_0^\varepsilon \zeta^\beta \ln^m \zeta\, d\zeta = \partial_\beta^m [\varepsilon^{\beta+1}/(\beta+1)], \qquad \beta \neq -1 \tag{1.6a}$$
$$= (\ln^{m+1} \varepsilon)/(m+1), \qquad \beta = -1. \tag{1.6b}$$

In particular, for all β and m

$$\int_0^\infty \zeta^\beta \ln^m \zeta\, d\zeta = 0. \tag{1.7}$$

For future reference we need a formula for change of scale. By the same understanding as for (Ib),

$$\int_\varepsilon^\infty (x/\varepsilon)^\beta \ln^m (x/\varepsilon)\, dx = \varepsilon^{-\beta} \sum C_j^m \left(\ln \frac{1}{\varepsilon}\right)^j \int_\varepsilon^\infty x^\beta \ln^{m-j} x\, dx$$
$$= \varepsilon^{-\beta} \sum C_j^m \left(\ln \frac{1}{\varepsilon}\right)^j \int_\varepsilon^\infty x^\alpha \ln^{m-j} x\, dx \Big|_{\alpha=\beta} \tag{1.8}$$

where the evaluation symbol indicates regular analytic extension from $\mathrm{Re}(\alpha) < -1$ to $\alpha = \beta$. For $\mathrm{Re}(\alpha) < -1$, setting $x = \varepsilon\zeta$ in (1.8) gives

$$\varepsilon^{\alpha-\beta+1} \int_1^\infty \zeta^\alpha \sum C_j^m \left(\ln \frac{1}{\varepsilon}\right)^j \ln^{m-j}(\varepsilon\zeta)\, d\zeta = \varepsilon^{\alpha-\beta+1} \int_1^\infty \zeta^\alpha \ln^m \zeta\, d\zeta. \tag{1.9}$$

If $\beta \neq -1$, the analytic continuation to $\alpha = \beta$ gives the expected:

$$\int_\varepsilon^\infty (x/\varepsilon)^\beta \ln^m (x/\varepsilon)\, dx = \varepsilon \int_1^\infty \zeta^\alpha \ln^m \zeta\, d\zeta \Big|_{\alpha=\beta}. \tag{1.10a}$$

But if $\beta = -1$ the regular analytic continuation of (1.9) is, by (1.5a),

$$\varepsilon[\varepsilon^{\alpha+1}(-1)^{m+1} m!\, (\alpha+1)^{-m-1}]_{\alpha=-1} = \varepsilon(-\ln \varepsilon)^{m+1}/(m+1).$$

Since $\int_1^\infty \zeta^{-1} \ln^m \zeta\, d\zeta = 0$, we get in this case

$$\int_\varepsilon^\infty (x/\varepsilon)^{-1} \ln^m (x/\varepsilon)\, dx = \varepsilon \left[\int_1^\infty \zeta^{-1} \ln^m \zeta\, d\zeta + (-\ln \varepsilon)^{m+1}/(m+1)\right]. \tag{1.10b}$$

The last term in (1.10b) is the source of the terms in (Ic).

Remark. If (1.2) holds only for $\zeta \geqslant 0$, then (Ia)–(Ic) is valid for $z>0$, $z \to +\infty$. In this case "analyticity in ζ" can be replaced by measurability.

2. The Proof

The proof will be given for the case $z>0$; the general case follows by letting $z \to \infty$ along a ray, as indicated below.

Suppose at first that each α in (1.1) has $\mathrm{Re}(\alpha) < -J-1$. Then we obtain (I) up to terms in z^{-J} from the Taylor expansion

$$\sigma(x,\zeta) = \sum_0^{J-2} \sigma^{(k)}(0,\zeta)\, x^k/k! + O\left(x^{J-1}\int_0^1 |\sigma^{(J-1)}(tx,\zeta)|\, dt\right).$$

This gives

$$\int_0^\infty \sigma(x,xz)\,dx = \int_0^\infty \sum_0^{J-2} \sigma^{(k)}(0,xz)\,x^k dx/k! + O\left(\left(\int_0^{1/z} + \int_{1/z}^\infty\right) x^{J-1}\int_0^1 |\sigma^{(J-1)}(tx,xz)|\,dt\,dx\right)$$
$$= \sum_0^{J-2} (z^{-k-1}/k!)\int_0^\infty \sigma^{(k)}(0,xz)(xz)^k d(xz) + O(z^{-J})$$

by (1.2b) and (1.2a). This gives the terms in (Ia) up to $O(z^{-J})$. The corresponding terms in (Ib) and (Ic) are zero since, in the present case, $\sigma_{\alpha m}=0$ for $\mathrm{Re}(\alpha) \geqslant -J-1$.

In the general case, write $\sigma(x,\zeta)$ as a finite sum of terms having the form

$$\psi(\zeta)\,\sigma_{\alpha m}(x)\,\zeta^\alpha(\ln\zeta)^m, \qquad \mathrm{Re}(\alpha) \geqslant -J-1$$

plus a remainder, where ψ is C^∞ with $\psi(r)=1$ for $r \geqslant 1$, $\psi(r)=0$ for $r \leqslant 1/2$. We have just seen that the remainder has the desired expansion (I), up to $O(z^{-J})$; so we need only check the terms in the finite sum, up to this order. Thus, consider a function σ such that

$$|x^J\sigma^{(K)}(x,\zeta)| \leqslant C_{JK} \qquad \text{for } \zeta \leqslant 1$$

and

$$\sigma(x,\zeta) = \phi(x)\,\zeta^\alpha(\ln\zeta)^m, \qquad \zeta \geqslant 1, \quad \phi \text{ in } \mathscr{S}(R^1). \tag{2.1}$$

Split our integral at $x=\varepsilon<1$:

$$\int_0^\infty \sigma(x,xz)\,dx = I_{>\varepsilon} + I_{<\varepsilon}$$

and assume $\varepsilon z > 1$. We find

$$I_{>\varepsilon} = \int_\varepsilon^\infty \sigma(x,xz)\,dx = \int_\varepsilon^\infty \phi(x)(xz)^\alpha(\ln xz)^m dx.$$

This will give the (Ib) term. Add and subtract the divergent integral $\int_0^\varepsilon \phi(x)(xz)^\alpha(\ln xz)^m dx$ and expand the integral in the subtracted term to get

$$\int_\varepsilon^\infty \sigma(x,xz)\,dx = \int_0^\infty \phi(x)(xz)^\alpha(\ln xz)^m dx - \int_0^\varepsilon \sum_0^J \frac{1}{j!}\phi^{(j)}(0)\,x^j(xz)^\alpha(\ln xz)^m dx + O(\varepsilon^{J+2}|\varepsilon z|^{|\mathrm{Re}\,\alpha|+1}). \tag{2.2}$$

The divergent integrals are interpreted as combinations of $\int \phi^{(j)}(0)\,x^\alpha \ln^k x\,dx$, and we take $J > -\mathrm{Re}(\alpha)-2$. Next, by (2.1)

$$|\sigma^{(j)}(x,\zeta)| = O((1+\zeta)^{\mathrm{Re}(\alpha)}(1+\ln^m\zeta))$$

so

$$I_{<\varepsilon} = \int_0^\varepsilon \sigma(x,xz)\,dx = \int_0^\varepsilon \sum_0^J \sigma^{(j)}(0,xz)\frac{x^j}{j!}dx + O(\varepsilon^{J+2}(\varepsilon z)^{|\mathrm{Re}(\alpha)|+1}).$$

Add and subtract the appropriate integral from ε to ∞, and use (2.1) in the subtracted term, where $xz \geqslant \varepsilon z > 1$:

$$I_{<\varepsilon} = \int_0^\infty \sum_0^J \sigma^{(j)}(0,xz)\frac{x^j}{j!}dx - \int_\varepsilon^\infty \sum_0^J \phi^{(j)}(0)\frac{x^j}{j!}(xz)^\alpha \ln^m xz\,dx + O(\varepsilon^{J+2}|\varepsilon z|^{|\mathrm{Re}\,\alpha|+1}). \tag{2.3}$$

Since by (1.7) $\int_0^\infty \zeta^\beta \ln^k\zeta\,d\zeta = 0$, (2.2) and (2.3) combine to

$$I_{>\varepsilon} + I_{<\varepsilon} = \int_0^\infty \phi(x)(xz)^\alpha \ln^m xz\,dx + \sum_0^J \int_0^\infty \sigma^{(j)}(0,xz)\frac{x^j}{j!}dx + O(\varepsilon^{J+2}|\varepsilon z|^{|\mathrm{Re}\,\alpha|+1}). \tag{2.4}$$

Now choose $\varepsilon = z^{-\delta}$, $0<\delta<1$, δ close to 1. Then $\varepsilon z = z^{1-\delta} > 1$ and

$$\varepsilon^{J+2}|\varepsilon z|^{|\mathrm{Re}(\alpha)|+1} = z^{-\delta(J+2)+(1-\delta)(|\mathrm{Re}\,\alpha|+1)}.$$

When δ is close enough to 1, this is $O(z^{-J-1})$. Hence the terms in (2.4) are valid up to that order. The first integral in (2.4) is precisely (Ib), since here $\sigma_{\alpha m}=\phi$. And each term in the sum $\sum_0^J$ in (2.4) is converted by a change of variable into the corresponding term of (Ia), plus (Ic) if $\alpha+j=-1$. In fact, if $\alpha+j\neq -1$ we have, using (1.10a),

$$\int_0^\infty \sigma^{(j)}(0, xz)\frac{x^j}{j!}\,dx=\int_0^{1/z}\sigma^{(j)}(0, xz)\frac{x^j}{j!}\,dx$$

$$+z^{-j}\int_{1/z}^\infty \phi^{(j)}(0)(xz)^\alpha \ln^m(xz)\frac{(xz)^j}{j!}\,dx$$

$$=z^{-j-1}\int_0^1 \sigma^{(j)}(0,\zeta)\frac{\zeta^j}{j!}\,d\zeta+z^{-j-1}\int_1^\infty \phi^{(j)}(0)\,\zeta^{\alpha+j}\ln^m\zeta\,d\zeta$$

$$=z^{-j-1}\int_0^\infty \sigma^{(j)}(0,\zeta)\frac{\zeta^j}{j!}\,d\zeta.$$

If $\alpha+j=-1$ we use (1.10b) instead, which gives

$$\int_0^\infty \sigma^{(j)}(0, xz)\frac{x^j}{j!}\,dx=z^{-j-1}\left[\int_0^\infty \sigma^{(j)}(0,\zeta)\frac{\zeta^j}{j!}\,d\zeta+\frac{\phi^{(j)}(0)\ln^{m+1}z}{(m+1)\,j!}\right].$$

The additional term here is precisely (Ic), since $\phi=\sigma_{\alpha m}$ and $j=-\alpha-1$.

This completes the proof for $z>0$. The general case follows by considering the limit along a ray $z=tz_0$, $|z_0|=1$. Then

$$\int \sigma(x, xz)\,dx=\int \sigma_{z_0}(x, xt)\,dx$$

where $\sigma_{z_0}(x,\zeta)=\sigma(x, z_0\zeta)$. This has an expansion as $t\to+\infty$, and the terms (Ib) and (Ic) for σ_{z_0} as $t\to\infty$ reduce directly to those for σ as $z\to\infty$. The terms (Ia) are

$$t^{-k-1}\int_0^\infty (\tau^k/k!)\,\sigma_{z_0}^{(k)}(0,\tau)\,d\tau$$

$$=z^{-k-1}\left[\int_0^{z_0}(\zeta^k/k!)\,\sigma^{(k)}(0,\zeta)\,d\zeta\right.$$

$$\left.+\int_{z_0}^{\infty z_0}(\zeta^w/k!)\,\sigma^{(k)}(0,\zeta)\,d\zeta\big|_{w=k}\right].$$

The integral with ζ^w converges for $\mathrm{Re}(w)\ll 0$, and the contour can be deformed into a contour Γ, from z_0 to 1 and thence to $+\infty$. The continuation to $w=k$ gives $\int_\Gamma(\zeta^k/k!)\,\sigma^{(k)}(0,\zeta)\,d\zeta$, and this plus the integral from 0 to z_0 gives (Ia), by a final contour deformation.

3. The Motivating Example

We return to the example in the Introduction, the Friedrichs extension of the singular operator

$$A=-\partial_x^2+x^{-2}a \qquad \text{on} \quad L^2(0,\infty).$$

Here a is a constant $\geqslant -\frac{1}{4}$. (In the next section, a is a function with $a(0)\geqslant -\frac{1}{4}$.) Callias [3] treats $a\geqslant\frac{3}{4}$, which makes A essentially self-adjoint; however, that case is very similar to the Friedrichs extension when $a\geqslant -\frac{1}{4}$, and the case $a=-\frac{1}{4}$ arises naturally in treating the Laplace operator on $L^2(R^2)$ in polar coordinates.

The resolvent $(A+z^2)^{-1}$ has kernel [3]

$$(xy)^{1/2}I_\nu(xz)\,K_\nu(yz), \qquad x\leqslant y, \quad \nu=(a+\tfrac{1}{4})^{1/2}.$$

The distributional tace is $\int_0^\infty \sigma(x, xz)\,dx$ where

$$\sigma(x,\zeta)=\phi(x)\,xI_\nu(\zeta)\,K_\nu(\zeta), \qquad \phi \text{ in } \mathscr{S}(R^1).$$

The asymptotic expansion as $\zeta\to\infty$ is [9, pp. 202–203]

$$\sigma(x,\zeta)\sim x\phi(x)\left[\frac{1}{2}\zeta^{-1}-\frac{4\nu^2-1}{16}\zeta^{-3}+\cdots\right), \qquad |\arg\zeta|<\pi/2 \tag{3.1}$$

and this gives the "interior terms" in (Ib):

$$z^{-1}\left(\frac{1}{2}\right)\int_0^\infty \phi(x)\,dx+z^{-3}(-a/4)\int_0^\infty x^{-2}\phi(x)\,dx+\cdots \tag{3.2}$$

where $\int_0^\infty x^{-2}\phi(x)\,dx$ means $-\int\phi''(x)\ln x\,dx$. The terms (Ic) give

$$\ln z\left(z^{-3}\phi'(0)(-\tfrac{a}{4})+ \text{ higher odd powers of } z^{-1}\right). \tag{3.3}$$

For the "boundary terms" in (Ia) we need the analytic continuation of the integral

$$I(w)=\int_0^\infty \zeta^w I_\nu(\zeta)\,K_\nu(\zeta)\,d\zeta, \qquad -1<\mathrm{Re}(w)<0.$$

From [6, pp. 96, 91, 6]

$$I(w)=\frac{1}{4\sqrt{\pi}}\,\frac{\Gamma\left(\nu+\frac{w+1}{2}\right)\Gamma\left(-\frac{w}{2}\right)\Gamma\left(\frac{w+1}{2}\right)}{\Gamma\left(\nu-\frac{w-1}{2}\right)}. \tag{3.4}$$

Hence the terms in (Ia) are

$$z^{-2}\phi(0)\left(-\frac{\nu}{2}\right)+\sum_{k=2}^{\infty} z^{-k-1}I(k)\,\phi^{(k-1)}(0)/(k-1)! \tag{3.5}$$

where $I(k)$ denotes the regular analytic continuation of (3.4), if k is an even integer.

The trace of the resolvent gives the trace of the heat operator

$$e^{-At}=\frac{-1}{2\pi i}\int_\Gamma e^{\lambda t}(A+\lambda)^{-1}d\lambda$$

where Γ is the arc $\{|\lambda|=1,\ |\arg\lambda|\leqslant\pi/2+\varepsilon\}$ together with the rays $\{|\lambda|\geqslant 1,\ \arg\lambda=\pm(\pi/2+\varepsilon)\}$ run "upwards." The distributional trace of this operator on a test function ϕ in $\mathscr{S}(R^1)$ is

$$T_\phi(t)=\frac{-1}{2\pi i}\int_\Gamma e^{\lambda t}\int\phi(x)\,k(x,x,\sqrt{\lambda})\,dx\,d\lambda.$$

Integrating the above expansions (3.2), (3.3), (3.5) with $\sqrt{\lambda}=z$ gives an expansion in terms of $t^{j/2}$ and $t^{j/2}\ln t$, $j\geqslant -1$. This was obtained in a different way by Callais [3].

4. Variable Coefficients

Consider now the operator

$$A=-\partial_x^2+x^{-2}a(x)\qquad\text{on}\quad L^2(0,\infty) \tag{4.1}$$

where a is a real-valued C^∞ function with $a(0)\geqslant-\frac{1}{4}$. We derive the expansion for the trace of the resolvent of the Friedrichs realization of (4.1). Let $\nu=(a(0)+\frac{1}{4})^{1/2}$ and denote by k_0 the kernel of $(-\partial_x^2+x^{-2}a(0)+z^2)^{-1}$:

$$k_0(x,y,z)=(xy)^{1/2}I_\nu(xz)\,K_\nu(yz),\qquad x\leqslant y. \tag{4.2}$$

For $x>y$ let $k_0(x,y,z)=\bar{k}_0(y,x,\bar{z})$.

Lemma 1. *Acting on $L^p(R^1)$, $1\leqslant p\leqslant\infty$, the operator with kernel $(xy)^{-1/2}k_0(x,y,z)$ has norm $O(|z|^{-1})$. In any sector $|\arg z|\leqslant\pi/2-\varepsilon$, the operators with kernels $x^{-1/2}k_0$ and $k_0y^{-1/2}$ have norm $O(|z|^{-3/2})$. For $p=2$ and ϕ in $\mathscr{S}(R^1)$, the Hilbert–Schmidt norms of $x^{-1/2}k_0\phi(y)$ and $\phi(x)\,k_0\,y^{-1/2}$ are $O(|z|^{-1}\ln|z|)$.*

Proof. The norm of an integral operator on L^p with kernel $k(x,y)$ is bounded by the larger of

$$\sup_y\int|k(x,y)|\,dx\qquad\text{and}\qquad\sup_x\int|k(x,y)|\,dy.$$

Consider the kernel, with $|\arg z|\leqslant\pi/2-\varepsilon$:

$$\begin{aligned}k(x,y)=y^{-1/2}k_0(x,y,z)&=x^{1/2}I_\nu(xz)\,K_\nu(yz),&&x\leqslant y\\&=x^{1/2}K_\nu(xz)\,I_\nu(yz),&&y\leqslant x.\end{aligned}$$

The standard estimates for $K_\nu(\zeta)$ and $I_\nu(\zeta)$ as $\zeta\to 0$ and $\zeta\to\infty$ give estimates for $k(x,y)$

$$\begin{aligned}y^{-1/2}|k_0(x,y,z)|&\leqslant C|z|^{-1/2}\frac{1+|\ln|yz||}{(1+y|z|)^{1/2}}e^{-(y-x)c|z|},&&x\leqslant y\\&\leqslant C|z|^{-1/2}\frac{1+|\ln|xz||}{(1+y|z|)^{1/2}}e^{-(x-y)c|z|},&&y\leqslant x.\end{aligned}$$

Integrating separately for $0\leqslant x\leqslant y$ and $y\leqslant x$;

$$\begin{aligned}\int_0^\infty k(x,y)|\,dx\leqslant{}&C|z|^{-1/2}\frac{1+|\ln|yz||}{(1+y|z|)^{1/2}}\frac{1-e^{-yc|z|}}{|z|}\\&+C|z|^{-3/2}\frac{e^{cy|z|}}{(1+y|z|)^{1/2}}\int_{y|z|}^\infty(1+|\ln\zeta|)\,e^{-c\zeta}d\zeta.\end{aligned}$$

Setting $y|z|=\eta$, the last line is

$$C|z|^{-3/2}\int_\eta^\infty e^{\eta-\zeta}\frac{1+|\ln\zeta|}{(1+\eta)^{1/2}}d\zeta\leqslant C'|z|^{-3/2}$$

since $1+|\ln\zeta|\leqslant C(1+\eta)^{1/2}(1+|\ln(\zeta-\eta)|)$ for $\zeta\geqslant\eta$.

Similar estimates establish the rest of the lemma.

Now return to the operator A in (4.1), and let K_0 be the integral operator with kernel k_0 in (4.2). Since k_0 is the kernel of $(-\partial_x^2+a(0)\,x^{-2}+z^2)^{-1}$, we have as an identity on $C_c^2(0,\infty)$

$$K_0(A+z^2)=I+K_0x^{-1}b(x)$$

where

$$b(x)=x^{-1}[a(x)-a(0)].$$

Assume that b is bounded. Then by Lemma 1, for $n \geqslant 1$

$$\|(K_0 x^{-1} b)^n K_0\| = \|K_0 x^{-1/2}(b x^{-1/2} K_0 x^{-1/2})^{n-1} b x^{-1/2} K_0\| \leqslant C|z|^{-n-2}.$$

Hence for z sufficiently large, the Neumann series

$$K = \sum_0^\infty (-K_0 x^{-1} b)^n K_0$$

converges, and

$$K(A + z^2) = I \quad \text{on} \quad C_c^2(0, \infty). \tag{4.3}$$

For z real, K is self-adjoint; from (4.3) it has dense range, so it is 1–1. Hence the self-adjoint operator $K^{-1} - z^2$ can be defined (by the spectral theorem) and on C_c^2

$$(K^{-1} - z^2)\,\phi = (K^{-1} - z^2)\,K(A + z^2)\,\phi = A\phi$$

by (4.3); so we have a self-adjoint extension of A, with K as its resolvent. (A further argument shows that this is in fact the Friedrichs extension.) Finally, since K is analytic in z, it agrees with the resolvent of the Friedrichs extension for $\mathrm{Re}(z) > 0$. Lemma 1 shows further that the trace norm of

$$\phi(x)\,K_0(x^{-1} b K_0)^n \psi(y), \qquad \phi, \psi \in \mathscr{S}(R^1)$$

is $O((|z|^{-1} \ln |z|)^2 |z|^{1-n})$. Hence the Neumann series gives an expansion of the trace of $\phi(A + z^2)^{-1}\psi$. If $\psi \equiv 1$ on the support of ϕ, this is the "distribution trace" of $(A + z^2)^{-1}$ on the test function ϕ. Letting $k(x, y, z)$ be the kernel of the resolvent, this distributional trace is

$$\int_0^\infty \phi(x)\,k(x, x, z)\,dx = \int_0^\infty \phi(x)\,k_0(x, x, z)\,dx - \int_0^\infty \int_0^\infty \phi(x)\,k_0(x, y, z)\,y^{-1} b(y)\,k_0(y, x, z)\,dy\,dx + \cdots. \tag{4.4}$$

The asymptotics of the first term, as $z \to \infty$, are discussed in Section 3 above. The second term can be written in polar coordinates as $\int_0^\infty \sigma_2(r, rz)\,dr$, where

$$\sigma_2(r, \zeta) = -r^2 \int_0^{\pi/4} \cos\theta\, I_\nu^2(\zeta \sin\theta)\, K_\nu^2(\zeta\cos\theta)[\phi(r\cos\theta)\, b(r\sin\theta) + \phi(r\sin\theta)\, b(r\cos\theta)]\, d\theta. \tag{4.5}$$

For simplicity, suppose that $\phi \equiv 1$ in a neighborhood of $x = 0$. [Typically, ϕ will be part of a partition of unity isolating the trace of $(A + z^2)^{-1}$ near $x = 0$.] With such a ϕ, we find the following derivatives at $x = 0$:

$$\sigma_2(0, \zeta) = \sigma_2^{(1)}(0, \zeta) = 0$$

$$\sigma_2^{(2)}(0, \zeta) = -4b(0) \int_0^{\pi/4} \cos\theta\, I_\nu^2(\zeta\sin\theta)\, K_\nu^2(\zeta\cos\theta)\, d\theta$$

$$\sigma_2^{(3)}(0, \zeta) = -6b'(0) \int_0^{\pi/4} I_\nu^2(\zeta\sin\theta)\, K^2(\zeta\cos\theta)\, P(\sin\theta, \cos\theta)\, d\theta$$

for some polynomial P. This generates a term of type (Ia)

$$-2z^{-3} b(0) \int_0^\infty \int_0^{\pi/4} \zeta^2 \cos\theta\, I_\nu^2(\zeta\sin\theta)\, K_\nu^2(\zeta\cos\theta)\, d\theta\, d\zeta$$

plus others in $z^{-4}, \ldots$ which combine with terms from later integrals in the Neumann series (4.4).

The asymptotic expansion of σ_2 as $\zeta \to \infty$ can be derived from the expansions of I_ν and K_ν. The part of the integral in (4.5) for $0 \leqslant \theta \leqslant \pi/8$ is $O(e^{-c|\zeta|})$, since for $|\arg \zeta| < \pi/2 - \varepsilon$

$$I_\nu^2(\zeta\sin\theta)\, K_\nu^2(\zeta\cos\theta) = O(e^{-c|\zeta|(\cos\theta - \sin\theta)}).$$

So as $\zeta \to \infty$

$$\sigma_2(r, \zeta) \sim -r^2 \int_{\pi/8}^{\pi/4} \cos\theta\, e^{-2\zeta(\cos\theta - \sin\theta)} \left[\frac{1}{4\zeta^2 \cos\theta\sin\theta} + \cdots\right] \times [\phi(r\cos\theta)\, b(r\sin\theta) + \phi(r\sin\theta)\, b(r\cos\theta)]\, d\theta.$$

Integrate by parts, integrating $e^{-2\zeta(\cos\theta - \sin\theta)}(\cos\theta + \sin\theta)$:

$$\sigma_2(r, \zeta) \sim -r^2 \left[\frac{1}{4\zeta^3}\phi\,(r/\sqrt{2})\, b(r/\sqrt{2})\right] + O(r^2/\zeta^4). \tag{4.6}$$

This generates a term of type (Ib)

$$-z^{-3} \int_0^\infty \frac{r^2}{4} \phi(r/\sqrt{2})\, b(r/\sqrt{2})\, r^{-3} dr = z^{-3} \int_0^\infty \frac{1}{4} (\phi b)'(x) \ln x\, dx \qquad [x = r/\sqrt{2}]$$

and another of type (Ic)

$$z^{-3} \ln z\,(-\tfrac{1}{4} b(0)). \tag{4.7}$$

The other terms in (4.4) can be similarly analyzed, thus showing that the trace of $\phi(A+z^2)^{-1}$ has the form

$$\int_0^\infty \sigma(x, xz)\,dx$$

for a symbol σ as in Section 1 above. The terms analyzed above in Sections 3 and 4 give the expansion

$$\begin{aligned}
\operatorname{tr}(\phi(A+z^2)^{-1}) \sim{}& z^{-1}\int_0^\infty \frac{1}{2}\phi(x)\,dx + z^{-2}\phi(0)\left(-\frac{\nu}{2}\right) + z^{-3}\phi'(0)\,c_\nu + \cdots\\
&-2z^{-3}b(0)\int_0^\infty\int_0^{\pi/4} \zeta^2\cos\theta\, I_\nu^2(\zeta\sin\theta)\,K_\nu^2(\zeta\cos\theta)\,d\theta\,d\zeta\\
&+z^{-3}\int_0^\infty\left[-\frac{a(0)}{4}x\phi(x)-\frac{x^2}{4}b(x)\,\phi(x)\right]x^{-3}dx\\
&+z^{-3}\ln z\left(-\frac{1}{4}b(0)\right)+O(z^{-4}\ln z)\\
={}& z^{-1}\int_0^\infty\frac{1}{2}\phi(x)\,dx + z^{-3}\int_0^\infty\left(-\frac{1}{4}\right)\phi(x)\,a(x)\,x^{-2}dx\\
&+z^{-2}(-\nu/2)+z^{-3}(-2a'(0))\int_0^\infty\int_0^x xI_\nu^2(y)\,K_\nu^2(x)\,dy\,dx\\
&+(z^{-3}\ln z)(-a'(0)/4)+O(z^{-4}\ln z) \qquad (4.8)
\end{aligned}$$

assuming that $\phi(x)\equiv 1$ for x near 0. In particular, there is a $z^{-3}\ln z$ term if $a'(0)\neq 0$, as noted by Brüning [1].

Remark. The expansion (4.4) gives the kernel $k(x, y)$ of $(A+z^2)^{-1}$ as $z\to\infty$. For $x=y\neq 0$, this kernel can be expanded by the standard (and simpler) method in [7]. Hence the terms of types (Ib) and (Ic) might be determined more easily using the simpler expansion. However, those in (Ia) cannot be obtained this way—the successive terms in the classical expansion can all be written in the form $\sigma(x, xz)$, but each such $\sigma(x,\zeta)$ has a simple zero at $x=0$, so each contributes to the z^{-2} term in (Ia).

5. Uhlmann's Expansion

Uhlmann [8] gives an asymptotic expansion for an integral

$$I(s)=\int_0^\infty\int_{|\theta|=1}\int_{R^n} e^{iq(y)/sR}a(y)\,dy g(R\theta)\,d\theta\,dR \qquad (5.1)$$

where $a\in C_c^\infty(R^n)$, $g\in C_c^\infty(R^k)$, and

$$q(y)=\langle Q^{-1}y, y\rangle$$

is a non-degenerate quadratic form. In this case

$$\phi(R)=\int g(R\theta)\,d\theta\in C_c^\infty(R^1)$$

is an even function, and we can write (5.1) as

$$I(s)=\int_0^\infty \phi(R)\,\rho(\sqrt{sR})\,dR=\int\phi(x^2)\,\rho(x\sqrt{s})\,2x\,dx \qquad (5.2)$$

where

$$\rho(\xi)=\int e^{iq(y)/\xi^2}a(y)\,dy. \qquad (5.3)$$

The exponential series give an expansion as $\xi\to\infty$, and ρ is bounded. So ρ is a symbol, and from (5.3) and (Ia)–(Ic), I(s) has an expansion

$$\sum_{k\geq 0} s^{-(k+1)/2}\gamma_k\int_0^\infty\frac{\xi^k}{k!}\rho(\xi)\,d\xi+\sum_{j\geq 0}s^{-j/2}\int_0^\infty 2x^{1-j}\phi(x^2)\rho_{-j}dx$$

$$-\sum_{k\geq 0}s^{-(k+1)/2}\ln\sqrt{s}\,\gamma_k\rho_{-k-1}/k!$$

where $\rho(\xi)\sim\sum_0^\infty\rho_{-j}\xi^{-j}$ and

$$\gamma_k=(\partial_x)^k(2x\phi(x^2))|_{x=0}.$$

Since ϕ is even, it follows that $\gamma_k=0$ unless $k\equiv 1 \bmod 4$. And from (5.3), $\rho_{-j}=0$ unless j is even. So the expansion involves only integer powers s^m and $s^m\ln s$.

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Characteristic-Free Representation Theory of the General Linear Group

KAAN AKIN*

Purdue University, West Lafayette, Indiana 47907, and Massachusetts Institute of Technology, Cambridge, Massachusetts 02139

AND

DAVID A. BUCHSBAUM†

Brandeis University, Waltham, Massachusetts 02254

1. INTRODUCTION

In attempting to apply the techniques of Schur functors and complexes as developed in [3] to extend the results on resolutions of determinantal ideals described in [2], we became increasingly aware of the need to study systematically $\mathbb{Z}$-forms of rational representations of the general linear group. These $\mathbb{Z}$-forms arise in a number of different contexts and, to make clear the kinds of things we are talking about, we shall first illustrate with some simple examples.

If F is a free module over a commutative ring R, and p is a positive integer, we have the $GL(F)$-module $\Lambda^p F \otimes F$. Now for any integer k, consider the map of $\Lambda^{p+1}F$ into $\Lambda^p F \otimes F \oplus \Lambda^{p+1}F$ which sends $\Lambda^{p+1}F$ into $\Lambda^p F \otimes F$ by diagonalization and $\Lambda^{p+1}F$ into $\Lambda^{p+1}F$ by multiplication by k. We will denote the cokernel of this map by $H_k(p, 1)$. Clearly, when $R = \mathbb{Q}$, $H_k(p, 1)$ is isomorphic to $\Lambda^p F \otimes F$ as a $GL(F)$-module, but this is not true in general when $R = \mathbb{Z}$. In fact, if we consider the exact sequence

$$0 \to \Lambda^{p+1}F \to \Lambda^p F \otimes F \to L_{(p,1)}F \to 0, \tag{1}$$

then the map $\Lambda^{p+1}F \to^k \Lambda^{p+1}F$ induces the exact sequence

$$0 \to \Lambda^{p+1}F \to H_k(p, 1) \to L_{(p,1)}F \to 0 \tag{2}$$

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