

INDEX THEORY FOR REGULAR SINGULAR OPERATORS AND APPLICATIONS

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1. This is a report on joint work with R. Seeley. In dealing with singular elliptic problems which admit separation of variables one frequently encounters regular singularities in the classical sense i.e. one has to solve ordinary differential equations of the type

$$(\partial_x + x^{-1}a(x))u(x) = f(x)$$

or

$$(-\partial_x^2 + x^{-2}a(x))u(x) = f(x), x > 0,$$

where a is smooth in $x \geq 0$. Cheeger [Ch] used this approach systematically to study the geometric operators on manifolds with cone-like singularities. In a series of papers [B+S1,2,3] we have developed the notion of first and second order regular singular operators abstractly, derived the asymptotic expansion of the trace of the resolvent in the second order case, and applied this to prove an index theorem for first order regular singular operators. In the following we will describe how these techniques can be used to calculate the L^2 index of the geometric operators on complete manifolds with finitely many ends all of which are warped products; the full details will appear elsewhere. The resulting index theorem will then be applied to the Gauß-Bonnet operator.

Let us recall first the notion of a regular singular first order differential operator on a Riemannian manifold M (cf. [B+S,3] §1) which we present here in a slightly more general form. So let $D : C^\infty(E) \rightarrow C^\infty(F)$ be a first order elliptic differential operator between the smooth sections of two hermitian vector bundles E and F over M . We think of M as a singular Riemannian manifold with singularities in an open subset U such that $M \setminus U$ is a smooth compact manifold with boundary. The nature of the singularities of course influences the structure of the geometric operators on U . From this fact we abstract certain axioms concerning D ; it will be called a regular singular differential operator if the following is true.

(RS 1) There is a compact Riemannian manifold N , with $\dim N + 1 = \dim M$, and a hermitian vector bundle G over N such that there are bijective linear maps

$$\begin{aligned}\Phi_E &: C_0^\infty(E | U) \rightarrow C_0^\infty(I, C^\infty(G)), \\ \Phi_F &: C_0^\infty(F | U) \rightarrow C_0^\infty(I, C^\infty(G)),\end{aligned}$$

where $I := (0, \varepsilon)$ for some ε , $0 < \varepsilon \leq 1$.

(RS 2) Φ_E and Φ_F extend to unitary maps $L^2(E | U) \rightarrow L^2(I; L^2(G))$ and $L^2(F | U) \rightarrow L^2(I, L^2(G))$, respectively.

(RS 3) For $\varphi \in C^\infty(I)$ with φ constant near 0 and ε let M_φ be the multiplication operator on $L^2(I, L^2(G))$. Then $\Phi_E^* M_\varphi \Phi_E = \Phi_F^* M_\varphi \Phi_F = M_{\bar{\varphi}}$ for some $\bar{\varphi} \in C^\infty(M)$, and $\bar{\varphi} \in C_0^\infty(M)$ if φ vanishes in a neighborhood of 0.

(RS 4) On $C_0^\infty(E | U)$ we have

$$\Phi_F D \Phi_E^* = \partial_x + x^{-1}(S_0 + S_1(x))$$

where

- a) S_0 is a self-adjoint first order elliptic differential operator on $C^\infty(G)$, and $\text{spec } S_0 \cap \{-1/2, 1/2\} = \emptyset$;
- b) $S_1(x)$ is a first order differential operator depending smoothly on $x \in (0, \varepsilon)$;
- c) $\|S_1(x)(|S_0| + 1)^{-1}\| + \|(|S_0| + 1)^{-1}S_1(x)\| = o(1)$ as $x \rightarrow 0$.

The main example for this situation is a manifold with asymptotically cone-like singularities. In this case we assume that U above is isometric to $(0, \varepsilon) \times N$ with metric $dx^2 + x^2 ds_N(x)^2$ where $\varepsilon > 0$, x is the standard coordinate on $(0, \varepsilon)$, N is a compact (not necessarily connected) Riemannian manifold, and $ds_N(x)^2$ is a family of metrics on N varying smoothly in $[0, \varepsilon]$. It is then readily verified that the geometric operators on M are regular singular in the above sense.

2. Now let M be a complete Riemannian manifold with finitely many ends. We assume that there is an open $U \subset M$ such that $M \setminus U$ is a smooth compact manifold with boundary and $U = \bigcup_{i=1}^k U_i$ where each U_i is isometric to a warped product $(y_{0i}, \infty) \times_{f_i} N_i$, $1 \leq i \leq k$. To simplify the exposition we will assume that $k = 1$ so U is $(y_0, \infty) \times N$, for some $y_0 > 0$ and some compact Riemannian manifold N , equipped with the metric $dy^2 + f(y)^2 ds_N^2$, where ds_N^2 is the metric on N and f is some positive function in $C^\infty[y_0, \infty)$. A lengthy but straightforward calculation shows that the geometric operators on U are unitarily equivalent to

$$(1) \quad \partial_{\mathbf{y}} + \frac{1}{f(\mathbf{y})} S_0 + \frac{f'(\mathbf{y})}{f(\mathbf{y})} S_1$$

in the sense of RS4) where S_0 is a suitable self-adjoint first order differential operator on $C^\infty(G)$, G a bundle over N , and S_1 is a zero order differential operator on $C^\infty(G)$ (cf. Section 5 below for the example of the Gauß-Bonnet operator).

We therefore consider a first order elliptic differential operator $D : C^\infty(E) \rightarrow C^\infty(F)$ between the smooth sections of two hermitian bundles E, F over M which are unitarily equivalent to an operator of the form (1) over U in the above sense. It is natural to investigate the L^2 -index of D i.e. the quantity

$$(2) \quad L^2\text{-ind } D := \dim \ker D \cap L^2(E) - \dim \ker D' \cap L^2(F)$$

where $D' : C^\infty(F) \rightarrow C^\infty(E)$ is the formal adjoint of D , defined by $(Du, v) = (u, D'v)$ for all $u \in C_0^\infty(E)$, $v \in C_0^\infty(F)$. Note that D' has automatically similar properties as D , in particular

$$(3) \quad D' \simeq -\partial_{\mathbf{y}} + \frac{1}{f(\mathbf{y})} S_0 + \frac{f'(\mathbf{y})}{f(\mathbf{y})} S'_1.$$

There are various L^2 -index theorems applying to this situation, dealing e.g. with cylinders [A+P+S], asymptotically Euclidean spaces ([S] Theorem 1), or Riemannian manifolds with cusps ([S] Theorem 2). We will present an L^2 -index theorem unifying and extending these results; the main point is to link the L^2 -index with the index of a regular singular operator in the sense of RS1) - RS4). To do so we need of course a condition on f since in general the L^2 -index will not be finite; a counterexample is provided by the Gauß-Bonnet operator on $\mathbb{R}^n = [0, \infty) \times S^{n-1}$ with a rotationally invariant metric $d\mathbf{y}^2 + f(\mathbf{y})^2 ds_{S^{n-1}}^2$ such that $\int_1^\infty \frac{d\mathbf{y}}{f(\mathbf{y})} < \infty$ (cf. [D]). The condition we impose is

$$(4) \quad \lim_{\mathbf{y} \rightarrow \infty} f'(\mathbf{y}) = 0.$$

implying

$$(5) \quad f(y) = o(y), \quad y \rightarrow \infty.$$

It is well known that all warped products are conformally equivalent to Riemannian products i.e. cylinders; elaborating on this idea we show that a weighted version of D , i.e. gDg for a suitable positive function $g \in C^\infty(M)$, is regular singular if (4) holds. To do so, define

$$(6) \quad F(y) := \int_{y_0}^y \frac{du}{f(u)}$$

such that $F \in C^\infty(y_0, \infty)$; in view of (5) we have the estimate

$$(7) \quad F(y) \geq \log y^N - c_N$$

for all $N > 0$. Next pick a positive function $g \in C^\infty(M)$ such that $g|_U$ depends on y only and

$$(8) \quad g^2(y) = f(y)e^{F(y)} \quad \text{for } y \text{ sufficiently large.}$$

Then the function

$$(9) \quad s(y) := \int_{y_0}^y \frac{du}{g^2(u)}$$

equals $e^{-F(y)}$ for large y and defines a diffeomorphism from (y_0, ∞) to $(0, x_1)$ for some $x_1 > 0$. Thus we obtain a linear transformation $\Phi : C_0^\infty((0, x_1), C^\infty(G)) \rightarrow C_0^\infty((y_0, \infty), C^\infty(G))$ given by

$$(10) \quad \Phi u(y) := \frac{1}{g(y)} u(s(y)).$$

Clearly, Φ extends to a unitary map $L^2((0, x_1), L^2(G)) \rightarrow L^2((y_0, \infty), L^2(G))$, and it is easily calculated that $D_g := gDg$ transforms as

$$(11) \quad \Phi^* D_g \Phi = -\partial_x + \frac{g^2}{f} \circ s^{-1}(x)(S_0 + f' \circ s^{-1}(x)S_1).$$

The definition of g and s and (4) then imply

LEMMA 1 D_g is a regular singular differential operator.

The discussion of the closed extensions of D_g and their Fredholm properties can now be carried out essentially along the lines of [B+S,3] §§2 and 3. The only difference lies in the fact that we have relaxed condition RS4,c) above where in [B+S,3] we required instead

$$(12) \quad \|S_1(x)(|S_0| + 1)^{-1}\| + \|(|S_0| + 1)^{-1}S_1(x)\| = O(x^\alpha)$$

as $x \rightarrow 0$ for some $\alpha > 1/2$,

whereas the elimination of the $\pm 1/2$ eigenvalues in RS4,a) was not necessary. In the case at hand we may assume that the restriction on $\text{spec } S_0$ is satisfied; otherwise we replace S_0 by μS_0 and f by μf for a suitable $\mu > 0$ which will not affect condition (4). Then we obtain the following result.

THEOREM 1 *The closed extensions of D_g in $L^2(E)$ are classified by the subspaces of the finite dimensional space $W_0 := \mathcal{D}(D_{g,\max}) / \mathcal{D}(D_{g,\min})$. All closed extensions are Fredholm operators, and if $D_{g,W}$ denotes the closed extension corresponding to $W \subset W_0$ we have*

$$\text{ind } D_{g,W} = \text{ind } D_{g,\min} + \dim W.$$

3. The next task is to compare $\text{ind } D_{g,W}$ with L^2 - $\text{ind } D$ for a suitably chosen W . If $u \in \ker D \cap L^2(E)$ then clearly $0 = Du = \frac{1}{g} D_g \frac{1}{g} u$. It is easy to see from (8) and (4) that $\frac{1}{g} \in L^\infty(M)$ so we obtain an injection

$$(13) \quad \ker D \cap L^2(E) \ni u \mapsto \frac{1}{g} u \in \ker D_{g,\max}.$$

This map is bijective onto $\ker D_{g,\max} \cap \frac{1}{g} L^2(E)$ so we would like to define

$$\begin{aligned}\mathcal{D}(D_{g,W}) &:= \mathcal{D}(D_{g,\max}) \cap \frac{1}{g}L^2(E), \\ D_{g,W} &:= D_{g,\max} \upharpoonright \mathcal{D}(D_{g,W}).\end{aligned}$$

With the modifications of [B+S,3] §2 mentioned above and the crucial condition (4) it then follows that

$$(14) \quad \mathcal{D}(D_{g,\min}) \subset \mathcal{D}(D_{g,\max}) \cap \frac{1}{g}L^2(E).$$

This implies that $D_{g,W}$ is a closed extension of D_g hence a Fredholm operator in view of Theorem 1. It is also not difficult to see that under the map analogous to (13) we obtain an injection

$$(15) \quad \ker D' \cap L^2(F) \hookrightarrow \ker D_{g,W}^*.$$

We define

$$(16) \quad h_0 := \dim W, \quad h_1 := \dim \ker D_{g,W}^* - \dim \ker D' \cap L^2(F).$$

Using Theorem 1 we arrive at the following L^2 -index theorem.

THEOREM 2

$$(17) \quad \begin{aligned}L^2\text{-ind } D &= \text{ind } D_{g,W} + h_1 \\ &= \text{ind } D_{g,\min} + h_0 + h_1.\end{aligned}$$

It is now necessary to describe the terms on the right hand side of (17) more explicitly. The calculation of $\text{ind } D_{g,\min}$ is largely parallel to the index calculation in [B+S,3] and will be carried out in the next section. To clarify the role of h_0 and h_1 we need an additional assumption which is also satisfied by the geometric operators (cf. Section 5), namely: if Q denotes the orthogonal projection in $H := L^2(G)$ onto $\ker S_0$ we have

$$(18) \quad S_1 \text{ is symmetric in } H \text{ and } (I - Q)S_1Q = 0.$$

If $u \in C^1((y_0, \infty), H)$ solves $Du = 0$ we obtain from (1) and (18)

$$(19) \quad (Qu)'(y) + \frac{f'(y)}{f(y)} QS_1 Qu(y) = 0, \quad y \in (y_0, \infty).$$

We now write the spectral decomposition of QS_1Q in the form

$$(20) \quad QS_1Q = \bigoplus_{t \in \mathbb{R}} t Q_t$$

where of course only finitely many Q_t are nonzero. Then the general solution of (19) is

$$(21) \quad Qu(y) = \sum_{t \in \mathbb{R}} \left(\frac{f(y)}{f(y_0)} \right)^{-t} Q_t u(y_0),$$

and since we are only interested in L^2 -solutions of D and D' it is natural to decompose further

$$Q = Q_0 \oplus Q'_0 \oplus Q_1$$

where

$$(22) \quad \begin{aligned} Q_0 &:= \bigoplus_{f^{-t} \in L^2} Q_t, \\ Q'_0 &:= \bigoplus_{f^t \in L^2} Q_t, \\ Q_1 &:= \bigoplus_{f^{-t}, f^t \notin L^2} Q_t. \end{aligned}$$

The analysis of h_0 requires a good description of $\mathcal{D}(D_{g,\min})$ which is provided by a result analogous to [B+S,3] Lemma 3.2 namely

$$\mathcal{D}(D_{g,\min}) = \{u \in \mathcal{D}(D_{g,\max}) \mid \|\Phi^* u(x)\| = O(x^{1/2}) \text{ as } x \rightarrow 0\}.$$

Analyzing the solutions of the transformed equation along the lines of [B+S,3] Lemma 3.2 then proves

LEMMA 2

$$h_0 = \dim Q_0 .$$

In dealing with h_1 it seems advantageous to study the original equation directly. In fact, under the isomorphism $v \mapsto \tilde{v} := gv$ we have

$$\begin{aligned} \ker D_{g,W}^* &= \{ \tilde{v} \in C^\infty(F) \mid D'\tilde{v} = 0, \frac{1}{g}\tilde{v} \in L^2(F), \\ &\quad (D\tilde{u}, \tilde{v}) = 0 \text{ for all } \tilde{u} \in L^2(E) \text{ with } gD\tilde{u} \in L^2(F) \} \\ &=: \mathcal{K}_W^* . \end{aligned}$$

The homogeneous equation $D'\tilde{v}(y) = 0$ is conveniently transformed by the change of variables

$$y(z) := F^{-1}(z), \quad \tilde{w}(z) := \tilde{v}(y(z)), \quad z \in (0, \infty),$$

leading to

$$(23) \quad [\partial_z - S_0 - f'(F^{-1}(z))S_1]\tilde{w}(z) = 0 .$$

The L^2 -solutions of this equation can be studied by standard methods. Then it follows that

$$L^2\text{-ker } D' = \{ \tilde{v} \in \mathcal{K}_W^* \mid Q_1 \tilde{v}(y) = 0, y \geq y_0 \} .$$

Introducing the map

$$\tau_y : \mathcal{K}_W^* \ni \tilde{v} \mapsto Q_1 \tilde{v}(y) \in Q_1 H ,$$

defined for $y \geq y_0$, we therefore find

LEMMA 3 For all $y \geq y_0$

$$\begin{aligned} h_1 &= \dim \operatorname{im} \tau_y = \dim \{ Q_1 \tilde{v}(y) \mid \tilde{v} \in \mathcal{K}_W^* \} \\ &\leq \dim Q_1 . \end{aligned}$$

In particular, $h_1 = 0$ if $Q_1 = 0$ which is the case e.g. if $f(y) = e^{-y}$, that is if M is a manifold with a cusp. It seems, however, very difficult to compute h_1 in general. We will give an example below with $h_1 > 0$, cf. Theorem 5.

4. It remains to compute $\text{ind } D_{g,\min}$. This is parallel to the work in [B+S,3] §4 though now the manifold may have infinite volume. The above discussion shows that $\text{ind } D_{g,\min}$ is the same for all g satisfying (8) for x sufficiently large. Thus it is natural that we define g to be constant on the part of M where $y \leq R$ for some large R in order to obtain the regularized interior contribution to the index independent of g . Taking the limit $R \rightarrow \infty$ in this approach is, however, technically somewhat delicate, and we are lead to impose a further condition on the growth of f , namely

$$(24) \quad \begin{aligned} & \text{if } Q_\alpha f := f^{\alpha_0} (f')^{\alpha_1} \dots (f^{(k)})^{\alpha_k}, \alpha_i \geq 0, \text{ is any} \\ & \text{monomial such that } \alpha_0 \leq \sum_{j \geq 2} (j-1)\alpha_j \text{ then} \\ & \lim_{y \rightarrow \infty} Q_\alpha f(y) = 0. \end{aligned}$$

Note that this condition contains (4) and that it is satisfied if $f(y) = e^{-y}$ or $f(y) = y^\beta$, $\beta < 1$, for large y . Also, (24) can be viewed as the analogue of condition (4.31) in [B+S,3] for the case under consideration. Then g_R will be a positive function in $C^\infty(M)$ satisfying

$$(25) \quad \begin{aligned} g_R^2(y) &= f(R) \quad \text{if } y \leq R + \frac{1}{2}f(R), \\ g_R^2(y) &= f(y)e^{F(y)} \quad \text{if } y \text{ is sufficiently large,} \\ \int_R^\infty \frac{du}{g_R(u)^2} &= 1, \end{aligned}$$

and we define

$$s_R(y) := \int_y^\infty \frac{du}{g_R(u)^2}.$$

Then an isometry Φ_R is defined as in (11) which transforms $D_{g_R} | C_0^\infty((R, \infty), H)$ to

$$-\partial_x + a_R(x)S_0 + b_R(x)S_1$$

on $C_0^\infty((0, 1), H)$ where $a_R(x) = 1/x$ near $x = 0$ and $\lim_{x \rightarrow 0} b_R(x) = 0$. The condition (24) ensures that uniformly on $[1/2, 1]$

$$(26) \quad \begin{aligned} \lim_{R \rightarrow \infty} a_R(x) &= 1, \\ \lim_{R \rightarrow \infty} a_R^{(j+1)}(x) &= \lim_{R \rightarrow \infty} b_R^{(j)}(x) = 0, \quad j \geq 0. \end{aligned}$$

We can then modify $D_{g_R, \min}$ to an operator $D_R : \mathcal{D}(D_{g_R, \min}) \rightarrow L^2(F)$ in such a way that $b_R(x) = 0$ if $x \in [0, 1/2]$ and $a_R(x) = 1$ near $x = 1/2$, and $\text{ind } D_R = \text{ind } D_{g_R, \min}$. Using suitable cut-off functions and computing separately the contributions to the constant term in the asymptotic expansion of

$$\text{tr}(e^{-tD_R^* D_R} - e^{-tD_R D_R^*})$$

coming from $y \leq R$, $R \leq y \leq R + \frac{1}{2}f(R)$, and $y \geq R + \frac{1}{2}f(R)$, we obtain three terms. Since the sum gives the index of $D_{g_R, \min}$ and hence does not depend on R we can take the limit $R \rightarrow \infty$. The first contribution involves only the “index form” ω_D of D and equals

$$(27) \quad \lim_{R \rightarrow \infty} \int_{y \leq R} \omega_D$$

proving in particular the existence of the limit. The index form is obtained as follows: the operators $e^{-tD^* D}$ and $e^{-tD D^*}$ have kernels with respect to the given Riemannian measure which when restricted to the diagonal in $M \times M$ yield smooth sections of the bundles $\text{Hom}(E, E)$ and $\text{Hom}(F, F)$, respectively. These kernels have pointwise asymptotic expansions as $t \rightarrow 0$ and ω_D denotes the difference of the fiber traces of the constant terms in this expansion. The second contribution turns out to be $o(1)$ as $R \rightarrow \infty$ in view of (26) since $(-\partial_x + S_0)(\partial_x + S_0) = (\partial_x + S_0)(-\partial_x + S_0)$. The third contribution is computed in [B+S,4]; it is independent of R and equals

$$(28) \quad \frac{1}{2}(\eta(S_0) - \dim \ker S_0)$$

where

$$\eta_{S_0}(z) := \sum_{s \in \text{spec } S_0 \setminus \{0\}} \text{sgn } s |s|^{-z}$$

is the η -function introduced in [A+P+S]. For general elliptic operators η_{S_0} is known to be meromorphic in \mathbb{C} with only simple poles, and 0 is a regular value. Then

$$\eta(S_0) := \eta_{S_0}(0).$$

Combining Theorem 2 with (27) and (28) we obtain

THEOREM 3 (L^2 -index theorem) *Let M be a complete Riemannian manifold as in Section 2, such that the warping factor f satisfies condition (24). Let $D : C^\infty(E) \rightarrow C^\infty(F)$ be a first order elliptic differential operator on M satisfying (1) and (18). Then*

$$(29) \quad L^2\text{-ind } D = \int_M \omega_D + \frac{1}{2}(\eta(S_0) - \dim \ker S_0) + h_0 + h_1.$$

Theorem 3 generalizes in a straightforward way to manifolds with k ends $U_i = (x_{0i}, \infty) \times_f N_i$ all of whose warping factors satisfy (24). Then on each end we have the representation

$$D \simeq \partial_y + \frac{1}{f_i} S_{0i} + \frac{f'_i}{f_i} S_{1i}$$

where (18) is required now for all i . Then we put

$$S_j := \bigoplus_{i=1}^k S_{ji}, \quad j = 0, 1,$$

and

$$(30) \quad h_0 = \sum_{i=1}^k h_{0i} = \sum_{i=1}^k \dim Q_{0i}.$$

h_1 is again defined by (16) and satisfies the estimate analogous to Lemma 3,

$$(31) \quad h_1 \leq \sum_{i=1}^k \dim Q_{1i}.$$

5. We want to explain the various ingredients of Theorem 3 in the case of the Gauß-Bonnet operator D_{GB} . So we assume again that M is a complete Riemannian manifold with finitely many ends $U_i = (y_{0i}, \infty) \times_{f_i} N_i$ such that all warping factors satisfy (24). Denoting by $\Omega(M) = \bigoplus_{j \geq 0} \Omega^j(M)$ the smooth forms on M , by $\Omega^{\text{ev}}(M)$ and $\Omega^{\text{odd}}(M)$ those of even and odd degree, respectively, and by d and d^* the exterior derivative and its adjoint with respect to the natural L^2 -structure on $\Omega(M)$, D_{GB} is defined by

$$D_{GB} := d + d^* : \Omega^{\text{ev}}(M) \rightarrow \Omega^{\text{odd}}(M).$$

It is well known that D_{GB} is a first order elliptic differential operator. If M is compact then it is easily seen that with $H^j(M) := \{\omega \in \Omega^j(M) \mid (d^*d + dd^*)\omega = 0\}$, the space of harmonic j -forms,

$$L^2\text{-ind } D_{GB} = \sum_{j \geq 0} (-1)^j \dim H^j(M).$$

By de Rham's theorem $H^j(M)$ is isomorphic to the j^{th} singular cohomology group of M with real coefficients so

$$L^2\text{-ind } D_{GB} = \chi(M),$$

the Euler characteristic of M . In the noncompact complete case the harmonic forms have to be replaced by the L^2 -harmonic forms i.e. we introduce

$$\mathcal{H}^j(M) := \{\omega \in \Omega^j(M) \mid (dd^* + d^*d)\omega = 0, \\ \int_M \omega \wedge *\omega < \infty\}.$$

It follows from a well known theorem of Andreotti and Vesentini that $\omega \in \mathcal{H}^j(M)$ iff $d\omega = d^*\omega = 0$. Hence we obtain

$$L^2\text{-ind } D_{GB} = \sum_{j \geq 0} (-1)^j \dim \mathcal{H}^j(M) =: \chi_{(2)}(M),$$

the L^2 -Euler characteristic of M . It is natural to ask whether $\chi_{(2)}(M)$ is a topological invariant. That this is not the case can be seen already from the fact that the finiteness of $\chi_{(2)}(M)$ depends on the metric and not on the topology alone, cf. [D]. The L^2 -index theorem above will give a formula for $\chi_{(2)}(M)$ if we can show that D_{GB} satisfies our assumptions. For this purpose we note that any $\omega \in \Omega^j(U_i)$ can be written as

$$\omega = \omega_j(\mathbf{y}) + \omega_{j-1}(\mathbf{y}) \wedge d\mathbf{y}$$

where $\omega_\ell \in C^\infty((y_{0i}, \infty), \Omega^\ell(N_i))$, $\ell = j - 1, j$.

A lengthy but straightforward calculation then gives the following result.

LEMMA 4 *On $\Omega^{\text{ev}}(U_i)$ we have*

$$D_{GB} \simeq \partial_{\mathbf{y}} + \frac{1}{f_i(\mathbf{y})} S_{0i} + \frac{f'_i(\mathbf{y})}{f_i(\mathbf{y})} S_{1i}$$

acting on $C^\infty((y_{0i}, \infty), \Omega(N_i))$. Here

$$(32) \quad S_{0i} \begin{pmatrix} \omega_0 \\ \vdots \\ \omega_{n_i} \end{pmatrix} = \begin{pmatrix} 0 & d_{N_i}^* & & \\ d_{N_i} & \ddots & \ddots & \\ & \ddots & \ddots & d_{N_i}^* \\ & & d_{N_i} & 0 \end{pmatrix} \begin{pmatrix} \omega_0 \\ \vdots \\ \omega_{n_i} \end{pmatrix}$$

where ω_j denotes the component in $\Omega^j(N_i)$ and d_{N_i} , $d_{N_i}^*$ denote the intrinsic operations on N_i . Moreover,

$$(33) \quad S_{1i} \begin{pmatrix} \omega_0 \\ \vdots \\ \omega_{n_i} \end{pmatrix} = \begin{pmatrix} c_0 & & 0 \\ & \ddots & \\ 0 & & c_{n_i} \end{pmatrix} \begin{pmatrix} \omega_0 \\ \vdots \\ \omega_{n_i} \end{pmatrix}$$

where $c_j = (-1)^j(j - \frac{n_i}{2})$.

Note that $n_i =: n = \dim M - 1$ for all i ; as in the compact case we assume from now on that $\dim M$ is even i.e.

$$\dim M = 2k = n + 1, \quad k \geq 1.$$

So D_{GB} satisfies condition (1). Now it is easily checked that

$$(34) \quad \ker S_{0i} = \bigoplus_{j \geq 0} H^j(N_i)$$

and consequently D_{GB} also satisfies (18) for all i . Hence the L^2 -index theorem applies and we obtain

$$(35) \quad \chi_{(2)}(M) = \int_M \omega_{GB} + \frac{1}{2}(\eta(S_0) - \dim \ker S_0) + h_0 + h_1,$$

where S_0 , h_0 , h_1 are defined at the end of §4. We have to investigate the terms on the right hand side of (35) more closely. Clearly, (34) implies that

$$(36) \quad \dim \ker S_0 = \sum_{i,j} \dim H^j(N_i) = \sum_{i,j} b_j(N_i)$$

where b_j is the j^{th} Betti number. Next, the calculations in [B+S,3] Lemma 5.1 prove

LEMMA 5 *If S_0 denotes the operator in (32) on an arbitrary compact Riemannian manifold N then*

$$\eta(S_0) = 0.$$

Using (30) and (33) we also arrive at

$$(37) \quad h_0 = \sum_{\substack{i,j \\ f_i^{-\epsilon_j} \in L^2}} b_j(N_i).$$

Now consider $\int_M \omega_{GB}$, the integral of the Gauß-Bonnet integrand. If M is compact then the Chern-Gauß-Bonnet Theorem asserts that

$$(38) \quad \int_M \omega_{GB} = \chi(M).$$

For a general complete manifold M with finitely many ends we say that the Chern-Gauß-Bonnet theorem holds if (38) is true. This is not true in general as the example $M = \mathbb{R}^n$ shows. On the other hand, the surface case has been studied thoroughly in a classical paper by Cohn-Vossen [CV]; he gives various sufficient conditions for (38) and shows that in great generality the inequality

$$\int_M \omega_{GB} \leq \chi(M)$$

is true. Further work concerns the case of locally symmetric spaces [H] and the case of bounded geometry [Ch-G]. In our situation there seems to apply only the result of Rosenberg [R] Theorem 1.9 which says that (38) holds if

$$\lim_{x \rightarrow \infty} f_i(x) = \lim_{x \rightarrow \infty} f'_i(x) = 0 \quad \text{for all } i.$$

We will show that the Chern-Gauß-Bonnet theorem also holds under our assumptions.

LEMMA 6 *If all warping factors satisfy the condition (24) then the Chern-Gauß-Bonnet theorem holds for M .*

PROOF The function $f_0(y) \equiv 1$ satisfies (24). Then we pick $\varphi \in C_0^\infty(\mathbb{R})$ with $\varphi = 1$ in a sufficiently large neighborhood of 0 and try to deform all warping factors f_i to f_0 near infinity i.e. we put

$$f_{i,\theta} := \varphi f_i + (1 - \varphi)(\theta f_0 + (1 - \theta)f_i), \quad \theta \in [0, 1].$$

It is easily checked that $f_{i,\theta}$ satisfies (24), too. In this way we obtain a family $D_{GB,\theta}$ of elliptic first order differential operators. Now we construct a smooth family of functions g_θ satisfying

$$\begin{aligned} g_\theta^2(y) &= 1, \quad y_0 \leq y \leq y_0 + 1/3, \\ g_\theta^2(y) &= f_\theta(y)e^{F_\theta(y)} \quad \text{if } e^{-F_\theta(y)} \leq 1/3, \\ \int_{y_1}^\infty \frac{du}{g_\theta(u)^2} &= 1, \end{aligned}$$

where $F_\theta(y) = \int_{y_0}^y \frac{du}{f_\theta(u)}$, and we define $s_\theta(y) := \int_y^\infty \frac{du}{g_\theta^2(u)}$. Transforming the square integrable forms on the Riemannian manifold with warping function f_θ using the transformation (10) generated by g_θ and s_θ maps the closure D_θ of the operators $g_\theta D_{GB} g_\theta$ to a family of Fredholm operators with domain independent of θ , varying continuously with θ . Using (27) and (28) we thus conclude that

$$(37) \quad \int_M \omega_D^0 = \int_M \omega_D^1$$

where ω_D^j is the index form of $D_{GB,j}$, $j = 0, 1$. Moreover, it follows easily from the Gauß–Bonnet theorem for manifolds with boundary that

$$(38) \quad \int_M \omega_D^1 = \chi(M).$$

The Lemma follows from (37) and (38). □

Since (24) holds e.g. for $f(y) = y^\beta$ with $\beta < 1$, Lemma 6 applies to warping factors which are not covered in [R]. As pointed out in this paper it is not necessary to control the derivatives of f of order greater than 1; thus it seems likely that the Chern–Gauß–Bonnet theorem will hold if only (4) is satisfied for all i .

Summing up we have proved

THEOREM 4 *Let M be a complete connected Riemannian manifold with finitely many ends U_i , $1 \leq i \leq k$, and assume that each end is a warped product with warping factor f_i satisfying (24). Then*

$$\chi_{(2)}(M) = \chi(M) + \frac{1}{2} \left(\sum_{f_i^{-c_j} \in L^2} b_j(N_i) - \sum_{f_i^{-c_j} \notin L^2} b_j(N_i) \right) + h_1$$

where h_1 is an integer satisfying

$$0 \leq h_1 \leq \sum_{f_i^{-c_j}, f_i^{c_j} \notin L^2} b_j(N_i).$$

We conclude this section with the surface case which allows the explicit calculation of h_1 under much weaker conditions than stated in Theorem 4. In particular, it shows that $h_1 > 0$ in general.

THEOREM 5 *Let M be a complete connected surface with finitely many ends U_i , $1 \leq i \leq k$, and assume that each end is a warped product with warping factor f_i satisfying*

$$(39) \quad \int_{y_{0i}}^{\infty} \frac{du}{f_i(u)} = \infty.$$

Then

$$(40) \quad \chi_{(2)}(M) = \begin{cases} \chi(M) + k & \text{if } \text{vol } M < \infty, \\ \chi(M) + k - 2 & \text{if } \text{vol } M = \infty. \end{cases}$$

This implies that

$$(41) \quad h_1 = \begin{cases} 0 & \text{if } \text{vol } M < \infty, \\ 2[\#\{i \mid f_i \notin L^1\} - 1] & \text{if } \text{vol } M = \infty. \end{cases}$$

PROOF Assume first that $\text{vol } M < \infty$ which is equivalent to $f_i \in L^1$ for $1 \leq i \leq k$; in view of Theorem 4 this yields $h_1 = 0$. By (33) we have $c_j = -1/2$ for $j = 0, 1$, hence we see from (30) that $h_0 = 2k$. Also, $\dim \ker S_0 = 2k$. On each U_i the circles $y = \text{const}$ have constant geodesic curvature equal to $\frac{f'_i(y)}{f_i(y)}$ so it follows from (4) and the Gauß–Bonnet theorem for surfaces with boundary that

$$\int_M \omega_{GB} = \chi(M).$$

Plugging this into (35) and observing Lemma 5 we obtain

$$(42) \quad \chi_{(2)}(M) = \chi(M) - k + 2k = \chi(M) + k.$$

Next, if $\text{vol } M = \infty$ h_1 may be nonzero since

$$\int_{y_{0i}}^{\infty} \frac{du}{f_i(u)} = \infty$$

for all i , by $f_i(y) = o(y)$ as a consequence of (4). Now

$$\chi_{(2)}(M) = \dim \mathcal{H}^0(M) - \dim \mathcal{H}^1(M) + \dim \mathcal{H}^2(M),$$

and we have

$$(43) \quad \dim \mathcal{H}^0(M) = \dim \mathcal{H}^2(M) = \begin{cases} 1 & \text{if } \text{vol } M < \infty, \\ 0 & \text{if } \text{vol } M = \infty, \end{cases}$$

since M is connected and the Hodge $*$ operator induces an isomorphism $\mathcal{H}^0(M) \rightarrow \mathcal{H}^2(M)$. It is also easily checked that $\dim \mathcal{H}^1(M)$ is a conformal invariant of M (cf. [D] for these facts). So (40) follows from (42) and (43) if we can show that under our assumptions M is conformally equivalent to a finite volume surface \tilde{M} with all ends warped products with warping factors f_i satisfying (4). To achieve this we first choose a positive C^∞ function \bar{f} on M such that on U_i $\bar{f}(y) = f_i(y)^{-2}$ if y is sufficiently large. Next we construct a diffeomorphism $\psi : M \rightarrow M$ such that $\psi = \text{id}$ on $y \leq R$ for R sufficiently large and $\psi(y, n) = (\int_{y_0}^y \frac{du}{f_i(u)}, n)$ if y is sufficiently large and $n \in N_i$. Denoting by g the original metric on M we obtain a conformally equivalent metric setting $\bar{g} := (\psi^{-1})^* \bar{f} g$. Clearly, this construction can always be carried out if we have (39), and it gives a conformal equivalence to a manifold with cylindrical ends. But then we can also obtain a conformal equivalence to a manifold all of whose warping factors equal e^{-y} for y sufficiently large which completes the argument.

Finally, (41) follows from (40) and (35) by comparison. □

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