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The Resolvent Expansion for Second Order Regular Singular Operators

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The paper gives a scheme for computing the asymptotics of $\operatorname{tr}(e^{-tL})$ as $t \to 0^+$, where L is an elliptic operator of the form $L = D^2 + x^{-2}A(x)$ and $A(x)$ is a family of operators satisfying appropriate ellipticity and smoothness conditions. A principle example is the Laplace operator for a manifold with an asymptotically conic singularity. The expansion has the usual terms away from the singularity, appropriately regularized at $x = 0$, plus singular contributions determined by the ζ-function of $(A(0) + \frac{1}{4})^{1/2}$. Applications to index theorems are given in a subsequent paper.

1. INTRODUCTION

This paper gives a scheme for computing the asymptotics of tr e^{-tL} as $t \to 0+$, for cerain singular operators L which, near the singularity, have the form

$$L = D^2 + x^{-2}A(x), \qquad 0 < x < \varepsilon, \quad D := -i\frac{d}{dx}. \tag{1.1}$$

Here $A(x)$ is a family of unbounded operators satisfying various conditions which can be expected to hold in the case of elliptic operators L. A principal example is the Laplace operator for a manifold with an asymptotically conical singularity, where the metric has the form

$$g = dr^2 + r^2 g_N(r), \qquad 0 \leqslant r < \varepsilon, \tag{1.2}$$

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with a smooth family of nonsingular metrics $g_N(r)$ on a compact manifold N, the "cross section" of the cone. (N need not be connected, so there may in fact be several singular points corresponding to $r=0$.) An extension of the theory will cover the second order operators arising from Dirac or signature operators on such a manifold. Related expansions go back to Carleman [7]; later, Minakshisundaram and Pleijel [14] used an expansion of the trace of the heat kernel to study the ζ-function $\operatorname{tr}(\Delta^{-s})$. This led to the idea of Atiyah and Bott to obtain index theorems from either the ζ-function or the heat asymptotics [17]. The ζ-function also serves to define determinants $\det(\Delta)$ in quantum theory, and to express the η-invariant in differential geometry [2].

The "regular" case of an elliptic differential operator, perhaps with elliptic boundary conditions, is reasonably well explained by pseudo-differential operator (ψdo) methods (e.g., [17, 18, 22]), and certain degenerate cases have also been treated. Singular cases arising from a group action were treated by Brüning and Heintze [3], and Cheeger [8, 9] treated a basic singular case, the Laplace operator at cone-like singularities where the metric (1.2) has $g_N(r)$ constant for small r. Chou [10] treated the Dirac operator in those cases, and Lue [13] treated (1.2), where $r^2 g_N(r) = f(r)^2 g_N(0)$ for appropriate f.

Our approach to the general case (1.1) is to study an appropriate power of the resolvent, i.e., $\operatorname{tr}(L+\lambda)^{-m}$ for some m. From this we pass to the heat kernel by means of a contour integral.

Operators of the form (1.2) also arise for the Laplace operator on $\mathbb{R}^n$ with appropriate singular potentials, and these have been considered by Callias and Uhlmann [6]. In those cases $A(r)-A(0)$ is of lower order than $A(0)$, which simplifies the problem. On the other hand, they consider the scattering problem, which is more difficult than the heat equation or resolvent.

The Laplacian on k-forms on an asymptotic cone is reduced to (1.1) as follows. Denote by $\Omega^p(N)$ the smooth p-forms on N, $0 \leqslant p \leqslant n = \dim N$, and by $C_{(0,\varepsilon)}N$ the Riemannian manifold $(0, \varepsilon) \times N$ with the metric (1.2). The map

$$\psi_p\colon\ C_0^\infty((0,\varepsilon), \Omega^p(N) \times \Omega^{p-1}(N)) \to \Omega^p(C_{(0,\varepsilon)}N),$$
$$(\omega_p, \omega_{p-1}) \mapsto x^{p-n/2}\pi^*\omega_p(x) + x^{p-1-n/2}\pi^*\omega_{p-1}(x) \wedge dx,$$

where $\pi\colon C_{(0,\varepsilon)}N \to N$ is the projection and x denotes the canonical coordinate in $(0,\varepsilon)$, is bijective onto forms with compact support. Moreover, ψ_p is unitary with respect to the usual L^2 structure on $\Omega^p(C_{(0,\varepsilon)}N)$ and the Hilbert space structure on $C_0^\infty((0,\varepsilon), \Omega^p(N) \times \Omega^{p-1}(N))$ given by

$$\int_0^\varepsilon (\|\omega_p(x)\|^2_{\Omega^p(N), g_N(x)} + \|\omega_{p-1}(x)\|^2_{\Omega^{p-1}(N), g_N(x)})\, dx.$$

Combining ψ_p with the obvious unitary transformation we can replace $g_N(x)$ by $g_N(0)$. A lengthy but straightforward computation then shows that under this transformation, Δ on p-forms with compact support on $C_{(0,\varepsilon)}N$ becomes

$$D^2 + \frac{A(x)}{x^2},$$

where $A(x)$ is a smooth family of second order elliptic operators on $\Omega^p(N) \oplus \Omega^{p-1}(N)$ and symmetric with respect to the Hilbert space structure defined by $g_N(0)$. Moreover,

$$A(0) = \begin{pmatrix} \Delta + \left(\frac{n}{2}-p\right)\left(\frac{n}{2}-p-1\right) & 2(-1)^p d \\ 2(-1)^p \delta & \Delta + \left(\frac{n}{2}+2-p\right)\left(\frac{n}{2}+1-p\right) \end{pmatrix},$$

where Δ and δ are the operators for the metric $g_N(0)$. We treat an example in Section 7.

The simplest case of (1.1) is

$$L_a = D^2 + x^{-2}a \tag{1.3}$$

for constant real a, acting as an unbounded operator in $L^2(\mathbb{R})$. The Mellin transform shows that $a \geqslant -\frac{1}{4}$ is a necessary and sufficient condition that (1.3) be semibounded on $C_0^\infty(\mathbb{R}^*)$ (the C^∞-functions with compact support in $\mathbb{R}^* = (0, \infty)$). We thus assume throughout that $a \geqslant -\frac{1}{4}$ and then we interpret (1.3) as the Friedrichs extension. In this case [5, 4] the resolvent $(L_a + z^2)^{-1}$ has kernel

$$K(x, y; z) = K(y, x; z) = (xy)^{1/2} I_\nu(xz)\, K_\nu(yz) \qquad (x \leqslant y),$$
$$\nu = (a + \tfrac{1}{4})^{1/2},$$

where I_ν and K_ν are Bessel functions. If φ has compact support, then $\varphi(L_a + z^2)^{-1}$ has finite trace given by

$$\int_0^\infty \sigma(x, xz)\, dx, \tag{1.4}$$

where

$$\sigma(x, \zeta) = \varphi(x)\, x I_\nu(\zeta)\, K_\nu(\zeta)$$
$$\sim \varphi(x)\, x[(2\zeta)^{-1} - (16\nu^2 - 4)(4\zeta)^{-3} + \cdots].$$

The individual terms in this expansion lead to integrals which diverge as $x \to 0$, so the asymptotics of (1.4) require special study [12, 21, 6, 4]. We apply the version in the last of these papers, which gives the following:

SINGULAR ASYMPTOTICS LEMMA. *Suppose that $\sigma(x, \zeta)$ is defined on $\mathbb{R}^1 \times C$, where C is the sector $\{|\arg \zeta| < \pi - \varepsilon\}$; σ is C^∞ in x with derivatives analytic in ζ; there are functions $\sigma_{\alpha j}$ in $\mathscr{S}(\mathbb{R}^1)$ with*

$$\left| x^J \partial_x^K \left[\sigma(x, \zeta) - \sum_{\operatorname{Re} \alpha \geqslant -M} \sum_{j=0}^{J_\alpha} \sigma_{\alpha j}(x) \zeta^\alpha \log^j \zeta \right] \right| \leqslant C_{JKM} |\zeta|^{-M}, \qquad |\zeta| \geqslant 1, 0 \leqslant x \leqslant |\zeta|/C_0 \tag{1.5a}$$

and finally, the derivatives $\sigma^{(j)}(x, \zeta) = \partial_x^j \sigma(x, \zeta)$ satisfy

$$\int_0^1 \int_0^1 s^j |\sigma^{(j)}(\theta st, s\xi)| \, ds \, dt \leqslant C_j \tag{1.5b}$$

uniformly for $0 \leqslant \theta \leqslant 1$, $|\xi| = C_0$. Then

$$\int_0^\infty \sigma(x, xz) \, dx \sim \sum_{k \geqslant 0} z^{-k-1} \int_0^\infty \frac{\zeta^k}{k!} \sigma^{(k)}(0, \zeta) \, d\zeta \tag{1.6a}$$

$$+ \sum_{\alpha, j} \int_0^\infty \sigma_{\alpha j}(x)(xz)^\alpha \log^j(xz) \, dx \tag{1.6b}$$

$$+ \sum_{\alpha=-1}^{-\infty} \sum_{j=0}^{J_\alpha} \sigma_{\alpha j}^{(-\alpha-1)}(0) \frac{z^\alpha \log^{j+1} z}{(j+1)(-\alpha-1)!}. \tag{1.6c}$$

The α's may be any sequence of complex numbers with $\operatorname{Re}(\alpha) \to -\infty$; the sum (1.6c) includes only those α which happen to be negative integers. The divergent integrals in (1.6a) and (1.6b) are defined by analytic continuation.

We reduce our general problem to an application of this lemma by making a priori assumptions which are in fact satisfied for the Laplacian at a simple cone-like singularity (where N is smooth and compact without boundary); we hope to establish these assumptions in more general cases, e.g., cross sections with a boundary, and cross sections which themselves have cone-like singularities. We assume an operator L acting on a Hilbert space $\mathscr{H}$ which is a direct sum

$$\mathscr{H} = \mathscr{H}_b \oplus \mathscr{H}_i$$

with $\mathscr{H}_i$ a supposedly simple "interior" part, and $\mathscr{H}_b$ corresponding to a neighborhood of the singularity. $\mathscr{H}$ would typically be L^2 sections of a

bundle E over a manifold M, $\mathscr{H}_i$ would be those sections vanishing identically within some fixed distance ε of the singularity, with $\mathscr{H}_b$ the complementary space. As in the above example of the Laplacian, the operator L acting in $\mathscr{H}_b$ can be written in the form $D^2 + x^{-2}A(x)$, where A acts not on sections of E, but on sections of a related bundle $\tilde{E}$ over N, the cross section of the singularity. These sections form a Hilbert space H with norm $\|\cdot\|_H$, and for each x, $A(x)$ is an unbounded operator on H with fixed domain H_A. Although the above construction gives $A(x)$ only for small x, we assume that it is extended in a convenient way to all $x > 0$, and satisfies the following:

(A1) $A_0 := A(0) \geqslant -\frac{1}{4}$ and $A(x) \geqslant -c + 1$ for some c.

(A2) $\|(A_0 + 1)^{-1}\|_p < \infty$ for some $p < \infty$; here $\|T\|_p = (\Sigma \tau_k^p)^{1/p}$ and τ_k are the eigenvalues of $(T^*T)^{1/2}$.

(A3) $\|A_0[A(x) + c]^{-1}\|_H \leqslant C$.

(A4) $\|A^{(k)}(x)(A_0 + 1)^{-1}\|_H \leqslant C_k$, $k = 0, 1, 2, \ldots$, where $A^{(k)}(x)(A_0 + 1)^{-1}$ denotes the derivative of the operator $A(x)(A_0 + 1)^{-1}$.

(A5) For any monomial $Q(A(x), \ldots, A^{(j)}(x), (A(x) + \lambda)^{-1})$ where the powers of $(A + \lambda)^{-1}$ at least balance the others, we have

$$\sup_{x, \lambda} \|Q(A, \ldots, A^{(j)}, (A + \lambda)^{-1})\|_H < \infty,$$

where the sup is for all $x \geqslant 0$, and all λ in a fixed sector

$$\Gamma = \{\lambda : |\arg(\lambda - 2c)| \leqslant \pi - \varepsilon\}.$$

(A6) For any monomial $Q(A, \ldots, A(j), (A + \lambda)^{-1})$ where the powers of $(A + \lambda)^{-1}$ exceed the others by at least $p + \frac{1}{2}$, Q is for each x a trace class operator on H and

$$\operatorname{tr} Q \sim \sum \sigma_{\alpha j Q}(x) z^\alpha \log^j z, \qquad z \to \infty,$$

where $\operatorname{Re}(\alpha) < -1$ and $z = \lambda^{1/2}$ with λ in Γ.

Remarks. (1) The norms $\|T\|_p$ in (A2) are the Schatten norms for certain ideals generalizing the trace class operators on H [11, XI, 9]. $\|T\|_2$ is the Hilbert–Schmidt norm, and $\|T\|_1 =: \|T\|_{\mathrm{tr}}$ is the trace norm. They satisfy

$$\|TS\|_p \leqslant \|T\|_H \|S\|_p,$$

$$\|TS\|_r \leqslant \|T\|_p \|S\|_q, \qquad \text{where } \frac{1}{p} + \frac{1}{q} = \frac{1}{r},$$

and thus $\|T^m\|_{p/m} \leqslant (\|T\|_p)^m$.

(2) When the cross section N is a compact manifold without boundary and $A(x)$ a family of elliptic differential operators, then (A5) is well known, and (A6) is a slight generalization of the expansion in [17]. It seems likely that these conditions hold for differential boundary problems as well.

As we said, in $\mathscr{H}_b$, L is isomorphic to an operator

$$D^2+x^{-2}A(x);$$

the domain of definition for this operator consists of those functions u such that

$$u \quad \text{is locally an } L^2 \text{ map into } H_A, \tag{1.7a}$$

$$u', u'' \text{ are locally } L^2 \text{ maps into } H, \tag{1.7b}$$

$$\int_0^\varepsilon \| -u''(x)+x^{-2}A(x)\,u(x)\|_H^2\,dx<\infty, \tag{1.7c}$$

$$\lim_{x\to 0+} x^{-1/2}u(x)=\tfrac{1}{2}\lim_{x\to 0+} x^{1/2}u'(x). \tag{1.7d}$$

Condition (1.7d) characterizes the Friedrichs extension for this type of operator (Theorem 6.1). These conditions, together with appropriate assumptions away from the singularity at $x=0$, guarantee that L is self-adjoint and semibounded, and that $(L+z^2)^{-m}$ is trace class for $m\geqslant p+1$. Our "Main Result" (Theorem 5.2) is that, for any smooth function φ supported sufficiently near the singular points, we have

$$\operatorname{tr}[\varphi(L+z^2)^{-m}]=\int_0^\infty \sigma(x, xz)\,dx,$$

where σ satisfies the conditions of the Singular Asymptotics Lemma. The terms $\sigma_{\alpha j}$ in (1.5a) are determined by any good interior parametrix, valid away from the singularity. For the precise statement of the results, the reader can consult Section 5, omitting the proofs.

Our expansion leads directly to one for the heat kernel, which is discussed in Section 7. In that expansion, the coefficient of t^0 is particularly interesting. This consists of three parts: one is the usual "interior" part, which in this case yields one of the regularized terms (1.6b); the other two are singular parts, corresponding to terms in (1.6a) and (1.6c), and these are the same as for the "frozen" operator

$$L_0=D^2+x^{-2}A(0).$$

Thus, in the case of the Laplacian, these singular contributions coincide with the ones computed by Cheeger for the metric cone.

Notation

$\mathbb{R}^*$	is the interval $(0,\infty)$, $\mathbb{R}_+$ is $[0,\infty)$.
$C_0^\infty(Y)$	is C^∞ functions with compact support in Y.
$C^{k+\alpha}$	is functions whose kth derivatives are Hölder continuous with exponent α.
B^∞	is functions all of whose derivatives are bounded.
H	is a fixed Hilbert space.
H^s	is the Sobolev space with exponent s.
$H_A\subset H$	is the domain of the self-adjoint operator A.
Φ	denotes the operator $(\Phi f)(x)=\varphi(x)f(x)$.
X	denotes the operator $(Xf)(x)=xf(x)$.
τ	denotes the differential expression $D^2+x^{-2}A(x)$ applied to any function for which it makes sense pointwise a.e.
C	denotes a "generic constant" which may increase from one appearance to the next, as needed.

Outline

In Section 2, we construct an "interior parametrix" K_i^m for $(D^2+X^{-2}A+z^2)^m$, valid away from $x=0$. This is used to prove a regularity theorem (Theorem 2.1), and to establish the existence of the expansion (1.6a). Section 3 discusses the "frozen" operator

$$L_0=D^2+X^{-2}A(0).$$

We use the eigenvector decomposition of $A(0)$, obtaining estimates in the high eigenspaces by means of a priori estimates, and in the low eigenspaces by means of the Bessel function kernels. This avoids the difficult problem of estimating $I_\nu(xz)\,K_\nu(yz)$ when both ν and z are large. Then in Section 4 we construct a "boundary parametrix" G_b for the variable coefficient operator

$$D^2+X^{-2}A+z^2$$

as a Neumann series with leading term $(D^2+X^{-2}A(0)+z^2)^{-1}$. In Section 5 we patch $(G_b)^m$ together with an assumed interior parametrix G_i^m for $(L+z^2)^m$, and prove that L is self-adjoint, and $\operatorname{tr}(L+z^2)^{-m}$ has an expansion as in (1.6a)–(1.6c). Section 6 shows that our boundary conditions (1.7d) above characterize the Friedrichs extension of L applied to smooth functions vanishing in a neighborhood of the singular point. Section 7 computes some coefficients, giving in particular the expansion for the heat kernel up to the term t^0, where the ζ-function of $A(0)$ enters into the result. Finally, the Appendix proves a Trace Lemma which we use to estimate kernels by means of trace class operator norms.

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2. The Interior Paramatrix

The interior parametrix is constructed by standard pseudodifferential methods. So as to get an operator of trace class, we will now form a parametrix K_i^m for $(D^2+X^{-2}A+z^2)^m$ with an appropriate integer m. Let the function $(x, \xi, z) \mapsto b(x, \xi, z)$ take values in the set of bounded operators on H, and define

$$\operatorname{Op}(b) f(x) := \int_{-\infty}^{\infty} e^{ix\xi} b(x, \xi, z) \hat{f}(\xi)\, \mathbf{d}\xi, \qquad f \in C_0^\infty(\mathbb{R}, H),$$

where $\mathbf{d}\xi := d\xi/2\pi$. We will set $K_i^m = \operatorname{Op}(b)$ with

$$b = \sum_{2m}^{N} b_j, \tag{2.1}$$

where N (large) is chosen later, and b_j is a function in ξ, z, and derivatives of $\mathscr{A} := x^{-2}A$, homogeneous of degree $-j$ in $(\xi, z, \mathscr{A}^{1/2})$; that is, replacing $(\xi, z, \mathscr{A})$ by $(t\xi, tz, t^2\mathscr{A})$ multiplies b_j by t^{-j}. Two remarks: 1) K_i^m is not the m^{th} power of some K_i. 2) The eigenvalue parameter z enters on a par with the Fourier transform variable ξ, since we seek asymptotic behavior as $z \to \infty$. This idea is in Agmon [1].

We have

$$\begin{aligned} &e^{-xi\xi}(D^2+\mathscr{A}+z^2)^m e^{ix\xi} b \\ &= (\xi^2+\mathscr{A}+z^2)^m b + \sum_{j=0}^{2m-1} P_j(\xi, z, \mathscr{A}, \ldots, \mathscr{A}^{(j)}, D)\, b, \end{aligned} \tag{2.2}$$

where P_j is a polynomial in all entries, of degree $2m-j$ in D, of degree $\leqslant m$ in $\mathscr{A}$ and its derivatives, and homogeneous of degree j in $(\xi, z, \mathscr{A}^{1/2})$. To approximate $(D^2+\mathscr{A}+z^2)^{-m}$, set (2.1) in (2.2) and equate the result to the identity I_H, as far as possible. Collecting terms by degree of homogeneity gives

$$b_{2m} = (\xi^2+\mathscr{A}+z^2)^{-m}, \tag{2.3a}$$

$$b_{k+1} = -b_{2m} \sum_{j=0}^{2m-1} P_j b_{k+1+j-2m}. \tag{2.3b}$$

Then the interior parametrix

$$K_i^m := \operatorname{Op}\left(\sum_{2m}^{N} b_j\right) \tag{2.4}$$

satisfies

$$\begin{aligned} (D^2+\mathscr{A}+z^2)^m K_i^m f &= \left(I + \sum_{j=0}^{2m-1} \sum_{k=0}^{2m-j-1} \operatorname{Op}(P_j b_{N-k})\right) f \\ &=: (I-R_i^m) f, \qquad f \in C_0^\infty(\mathbb{R}^*, H). \end{aligned} \tag{2.5}$$

We have

$$b_j = Q_j(\xi, z, \mathscr{A}, \ldots, \mathscr{A}^{(j-2m)}, b_2), \tag{2.6}$$

where $b_2 := (\xi^2+\mathscr{A}+z^2)^{-1}$ and Q_j is a polynomial in all variables, homogeneous of degree $-j$ in $(\xi, z, \mathscr{A}^{1/2})$ if we take into account the dependence of b_2 on these variables. Note that the derivatives $\mathscr{A}^{(k)}$ need not commute with $\mathscr{A}$ and b_2. Conditions (A1)–(A5) in the introduction imply the following:

$$A(x) \quad \text{is self-adjoint and } A(x) \geqslant -c+1 \tag{2.7a}$$

for some constant $c \geqslant 0$;

$$\sup_x \|[A(x)+c]^{-1}\|_p < \infty \tag{2.7b}$$

while for $0 \leqslant j$,

$$\sup_x \|A^{(j)}(x)[A(x)+c]^{-1}\| < \infty. \tag{2.7c}$$

Moreover, $A(x)$ dominates $A(0)$ in the sense that

$$\|A(0)[A(x)+c]^{-1}\| \leqslant C. \tag{2.7d}$$

We have further the ellipticity condition:

$$\sup_{x,\lambda} \| Q(A,..., A^{(j)}, (A+\lambda)^{-1})\|_H < \infty \tag{2.8}$$

for any polynomial Q which, as a function of $(A,..., A^{(j)}, A+\lambda)$ is homogeneous of degree $\leqslant 0$, i.e., the powers of $(A+\lambda)^{-1}$ at least balance the others. The sup is for $x \geqslant 0$ and $\lambda \in \Gamma = \{\lambda : |\arg(\lambda - 2c)| \leqslant \pi - \varepsilon\}$.

LEMMA 2.1. *Given $\varepsilon > 0$, there is a constant C such that for xz in sector Γ,*

$$\| z b_2^{1/2} \|^2 + \| x^{-2} A b_2 \| + \| \xi b_2^{1/2} \|^2 \leqslant C^2, \tag{2.9}$$

$$\| b_2^{1/2} \| \leqslant C(1 + |\xi| + |z|)^{-1}, \tag{2.10}$$

$$\| b_2 \|_p \leqslant C x^2. \tag{2.11}$$

Proof. For $\| \xi b_2^{1/2} \|^2 = \| \xi^2 b_2 \|$ we must estimate

$$\sup \frac{x^2 \xi^2}{|x^2 \xi^2 + x^2 z^2 + \mu|} = \sup \frac{\alpha}{|\alpha + \lambda + \mu|}$$

for $\alpha \geqslant 0$, $|arg(\lambda - 2c)| \leqslant \pi - \varepsilon$, and μ in spec$(A(x))$, thus $\mu \geqslant -c$. This sup is

$$\leqslant \sup \left\{ \frac{\alpha}{|\alpha - \beta|}; \alpha \geqslant 0, \pi \geqslant |\arg(\beta + c)| \geqslant \varepsilon \right\} = (\sin \varepsilon)^{-1}$$

proving the second part of (2.9). We next estiamte $x^{-2} A b_2$. By (2.8)

$$\| x^{-2} A b_2 \| = \| A(x)[x^2 \xi^2 + x^2 z^2 + A(x)]^{-1} \| \leqslant C$$

and

$$\| A[x^2 \xi^2 + x^2 z^2 + A]^{-1} \| \leqslant C \sup[(\mu + c)|\alpha + \lambda + \mu|^{-1}]$$

for $\alpha \geqslant 0$, $|\arg(\lambda - 2c)| \leqslant \pi - \varepsilon$, $\mu \geqslant -c$. This is the same as the previous case, with α and μ replaced, respectively, by $\mu + c$ and $\alpha - c$. Hence, $z^2 b_2 = I - (\xi^2 + x^{-2} A) b_2$ is bounded, since the last term is, proving (2.9); then (2.10) follows. Finally,

$$\| x^{-2} b_2 \|_p \leqslant \|(A + c)^{-1} \|_p \, \|(A + c) \, x^{-2} b_2 \|$$

is uniformly bounded, proving (2.11). ▮

LEMMA 2.2. *Let $Q(\xi, z, \mathscr{A},..., \mathscr{A}^{(j)}, b_2)$ be a monomial homogeneous of degree $h \leqslant 0$ in $(\xi, z, \mathscr{A}^{1/2})$. Then given $\varepsilon > 0$ there is a constant C such that for $x \geqslant \varepsilon$ and $|arg(z^2 - 2c/\varepsilon^2)| \leqslant \pi - \varepsilon$*

$$\| Q \|_H \leqslant C(1 + |\xi| + |z|)^h. \tag{2.12}$$

If $h \leqslant -2m$ then

$$\| Q \|_{p/m} \leqslant C x^{2m} (1 + |\xi| + |z|)^{h + 2m}. \tag{2.13}$$

Proof. The conditions on x and z imply that

$$|\arg(x^2 z^2 - 2c)| = |\arg(z^2 - 2c/x^2)| \leqslant |\arg(z^2 - 2c/\varepsilon^2)| \leqslant \pi - \varepsilon$$

as required in Lemma 2.1. Now Q is a product of factors ξ, z, b_2, and $x^{-2-r} A^{(q)}(x)$ with $r \geqslant 0$; and the number of factors of b_2 exceeds those of ξ^2, z^2, and the $A^{(q)}$ by $-h/2$ (the number of factors of ξ^2 or z^2 may of course be a half integer). The factors of $x^{-2-r} A^{(q)}$ may be combined with factors of b_2 in blocks where each $x^{-2-r} A^{(q)}$ is balanced by a b_2, or each $x^{-r} A^{(q)}$ by a factor

$$x^{-2} b_2 = [x^2 \xi^2 + x^2 z^2 + A(x)]^{-1}.$$

The norm of such a block is bounded, by (2.8), since the left-over factors of x^{-r} are bounded in $x \geqslant \varepsilon$. Next, each factor of ξ or z can be paired with a factor of $b_2^{1/2}$ and estimated by (2.9). This leaves $-h$ factors of $b_2^{1/2}$, so (2.10) implies (2.12). To get (2.13) we pair the $A^{(p)}$, ξ, z with factors of $b_2^{1/2}$ as before, estimate m remaining factors of b_2 by (2.11), and use (2.10) on the remaining $-h - 2m$ factors of $b_2^{1/2}$. ▮

For L^2 estimates on Op(b_j) we use the main step in the proof of the Calderón–Vaillancourt Theorem [23], which can be stated as follows:

LEMMA 2.3. *Let $b(x, \xi)$ be a family of operators on a Hilbert space H with*

$$\| \partial_x^i \partial_\xi^j b(x, \xi) \| \leqslant C$$

for $0 \leqslant i, j \leqslant 3$ and all x and ξ. Then Op(b) *is bounded on $H \otimes L^2(\mathbb{R})$.*

This with Lemma 2.2 gives directly

LEMMA 2.4. *Let $Q = Q(\xi, z, \mathscr{A},..., \mathscr{A}^{(j)}, b_2(\xi, z, \mathscr{A}))$ be a monoial in all entries, which is of degree $l \leqslant 0$ in $(\xi, z, \mathscr{A}^{1/2})$. Then for $|\arg(z^2 - 2c/\varepsilon^2)| \leqslant \pi - \varepsilon$, and for all φ in $B^\infty(0, \infty)$ vanishing for $x \leqslant \varepsilon$,*

$$\| \mathrm{Op}(\varphi Q) \|_{H \otimes L^2} \leqslant C |z|^{-l}.$$

In particular, if $2(h+i)+k+l-j\leqslant 0$ *then*

$$\|\mathscr{A}^i D^k \mathrm{Op}(\varphi b_j)\, D^l \mathscr{A}^h\|_{H\otimes L^2}\leqslant C|z|^{2(h+i)+k+l-j}$$

and the same holds with $\mathrm{Op}(\varphi b_j)$ *replaced by the commutator* $[\mathrm{Op}(\varphi b_j), \partial]$.

Proof. Apply Lemmas 2.2 and 2.3. For the last part, note that $[\mathrm{Op}(\varphi b_j), \partial]=-\mathrm{Op}(\partial_x(\varphi b_j))$. ∎

Next we display the trace class properties of these operators.

LEMMA 2.5. *If* $m\geqslant p$ *in* K_i^m *and if* η, $\psi\in B^\infty(\mathbb{R}^1)$, *one has compact support, their supports are disjoint, and* η *vanishes near* $x=0$, *then*

$$\|\eta\mathscr{A}^i D^k\, \mathrm{Op}(b_s)\, D^j\mathscr{A}^h z^l\psi\|_{\mathrm{tr}}<\infty$$

for all h, i, j, k, l, s.

Proof. Let T be the operator in the lemma. Then for an appropriate symbol β, using standard pseudodifferential methods, we find that

$$Tf(x)=\eta(x)\int\!\!\int e^{i(x-y)\xi}\beta(x,\xi,z)\,\psi(y)f(y)\,dy\,\mathbf{d}\xi.$$

Since η and ψ have disjoint supports, we can integrate by parts to get

$$Tf(x)=\int\gamma(x,y,z)f(y)\,dy$$

with

$$\gamma(x,y,z)=i^M\eta(x)\int\!\!\int e^{i(x-y)\xi}(x-y)^{-M}\,\partial_\xi^M\beta(x,\xi,z)\,\psi(y)f(y)\,dy\,\mathbf{d}\xi.$$

By Lemma 2.2, if M is sufficiently large then

$$\|(\mathscr{A}(x)+c)^m\,\partial_x^k\partial_\xi^M\beta(x,\xi,z)\|_H=x^{-2m}\,\|(A(x)+cx^2)^m\,\partial_x^k\partial_\xi^M\beta(x,\xi,z)\|_H$$
$$\leqslant C(1+|\xi|+|z|)^{-r} \qquad (2.15)$$

for arbitrarily large $r=r(M)$. In view of the compact support of ψ or η, the factor $(x-y)^{-M}$ in (2.15), and the assumption that $m\geqslant p$,

$$\|(A(x)+c)^{-m}\|_{\mathrm{tr}}\leqslant\|(A(x)+c)^{-1}\|_p^m\leqslant C$$

we get

$$\|(1+|z|+x+y)^r\,\partial_x^r\partial_y^r\,\gamma(x,y,z)\|_{\mathrm{tr}}\leqslant C \qquad (2.16)$$

for arbitrarily large r. We now expand γ in Laguerre functions

$$\varphi_j(x)=c_jp_j(x)\,e^{-x/2}\qquad\text{with } p_j(x)=e^x\partial_x^j(x^je^{-x})/j!.$$

Since

$$x\varphi_j''+\varphi_j'+\left(\frac{1}{2}-\frac{x}{4}\right)\varphi_j=j\varphi_j,$$

(2.16) implies that the Fourier coefficients

$$B_{jk}(z)=\int\!\!\int\gamma(x,y,z)\,\varphi_j(x)\,\varphi_k(y)\,dx\,dy$$

satisfy

$$\|B_{jk}(z)\|_{\mathrm{tr}}\leqslant C(1+j^3+k^3)^{-1}\,(1+|z|)^{-l}.$$

Moreover, the operator Φ_{jk} with kernel $\varphi_j(x)\,\varphi_k(y)$ sending

$$\varphi_k\mapsto\varphi_j,\quad \{\varphi_k\}^\perp\mapsto 0$$

has trace norm 1; and tensor products of operators satisfy

$$\|S_1\otimes S_2\|_{\mathrm{tr}}=\|S_1\|_{\mathrm{tr}}\,\|S_2\|_{\mathrm{tr}}.$$

So

$$\|T\|_{\mathrm{tr}}\leqslant\Sigma\,\|B_{jl}\|_{\mathrm{tr}}\,\|\Phi_{jk}\|_{\mathrm{tr}}\leqslant C(1+|z|)^{-1}$$

and Lemma 2.5 is proved. ∎

LEMMA 2.6. *Given h, i, j, k, l then the N in* (2.1) *can be chosen so that*

$$\|\eta A^iD^kR_i^mD^jA^hz^l\|_{\mathrm{tr}}<\infty$$

if η *has compact support disjoint from zero.*

Proof. By Lemma 2.5 we may consider $\eta R_i^m\psi$ where ψ has compact support, $\psi\equiv 1$ in a neighborhood of $\mathrm{supp}(\eta)$. The proof then follows as before except that, in place of a high ξ-derivative, we take many terms in the parametrix, i.e., large N in (2.1) and (2.5).

LEMMA 2.7. *Suppose that* φ *has compact support disjoint from zero. Then, uniformly for* z^2 *in the sector* Γ,

$$\|\varphi K_i^m\|_{(p+1/2)/m}<\infty\qquad\text{if}\quad m<p+1/2,$$
$$\|\varphi K_i^m\|_{\mathrm{tr}}<\infty\qquad\text{if}\quad m\geqslant p+1/2.$$

The same is true for $\varphi[K_i^m, \partial]$.

Proof. By (2.1) and Lemma (2.3), the operator

$$T = (A_0 + c + D^2)^m \varphi K_i^m$$

defines a bounded operator: $L^2(\mathbb{R}^1) \otimes H \to L^2(T^1) \otimes H$, where $T^1 = \mathbb{R}/[0, \alpha]$ and $[0, \alpha] \supset \operatorname{supp} \varphi$. Write

$$\varphi K_i^m = (A_0 + c + D^2)^{-m} T.$$

The operator $A_0 + c + D^2$ on T^1 has eigenvalues

$$\{(2\pi j/\alpha)^2 + \alpha + c : a \in \operatorname{spec}(A_0), j \in \mathbb{Z}\}$$

so

$$\|(A_0 + c + D^2)^{-1}\|_{p+1/2}^{p+1/2} = \sum_{a,j} [(2\pi j/\alpha)^2 + a + c]^{-p-1/2}$$

$$\leqslant C \sum_a \int_{-\infty}^{\infty} (t^2 + a + c)^{-p-1/2}\, dt$$

$$\leqslant C \sum_a (a + c)^{-p} = C' \|(A_0 + c)^{-1}\|_p^p.$$

Hence $\|(A_0 + c + D^2)^{-m}\|_{(p+1/2)/m} < \infty$, and the lemma follows. ∎

We now investigate the regularity properties (away from 0) of weak solutions of the equation $\tau u = f \in L^2(\mathbb{R}^*, H)$. For u in $L^2(\mathbb{R}^*, H)$, we say that $\tau u \in L^2_*(\mathbb{R}^*, H)$ if

$$|(u, \tau\varphi v)| \leqslant C_\varphi \|v\| \tag{2.17}$$

for all v in $H^2(\mathbb{R}^*, H) \cap L^2(\mathbb{R}^*, H_A, (1+x)^{-4}\, dx)$, and all φ in $B^\infty(\mathbb{R})$ with $\varphi \equiv 0$ near 0. The space $L^2_*(\mathbb{R}^*, H)$ is thus "local" near $x = 0$ and "global" near $x = \infty$. Similarly, we say that $\tau u \in H^1_*(\mathbb{R}^*, H)$ if

$$|(u, \tau\varphi v)| + |(u, \tau\partial\varphi v)| \leqslant C_\varphi \|v\| \tag{2.18}$$

for all v in $H^3(\mathbb{R}^*, H) \cap H^1(\mathbb{R}^*, H_A, (1+x)^{-4}\, dx)$ and all φ as above. On the other hand, $\tau u \in H^1_{\text{loc}}(\mathbb{R}^*, H)$ if (2.18) holds just for φ in $C_0^\infty(\mathbb{R}^*)$.

THEOREM 2.1. *Let $u \in L^2(\mathbb{R}^*, H)$ and $\varphi \in B^\infty(\mathbb{R})$ with $\varphi \equiv 0$ near 0.*

(a) *If $\tau u \in L^2_*(\mathbb{R}^*, H)$ then $\varphi u \in H^2(\mathbb{R}^*, H) \cap L^2(\mathbb{R}^*, H_A, (1+x)^{-4} dx)$.*
(b) *If $\tau u \in H^1_*(\mathbb{R}^*, H)$ then $\varphi u \in C^{5/2}(\mathbb{R}^*, H) \cap C^{1/2}(\mathbb{R}^*, H_A)$.*

Proof. For (a), choose v as in (2.17) and φ as in the hypotheses. Pick $\psi \in B^\infty(\mathbb{R})$ vanishing near 0, with $\psi \equiv 1$ in a neighborhood of $\operatorname{supp} \varphi$. Construct the interior parametrix K_i^m as in (2.4) with $m = 1$, N large, and a fixed large positive z. Setting $K_i^m = K$, $R_i^m = R$ we have

$$\varphi v'' = (\tau + z^2)\, \psi K \varphi \partial^2 v + \psi'' K \varphi \partial^2 v + 2\psi' \partial K \varphi \partial^2 v$$
$$+ \psi R \varphi \partial^2 v.$$

By Lemmas 2.4 and 2.5, the operators $z^2 \psi K \varphi \partial^2$, $\psi'' K \varphi \partial^2$, $\psi' \partial K \varphi \partial^2$, and $\psi R \varphi \partial^2$ are bounded in $L^2(\mathbb{R}^*, H)$, so

$$\varphi v'' = \tau \psi K \varphi \partial^2 v + w \quad \text{with} \quad \|w\| \leqslant C_\varphi \|v\|. \tag{2.19}$$

Now choose a further cut-off function $\eta \in B^\infty(\mathbb{R})$ with $\eta\psi \equiv \psi$ and $\eta = 0$ near 0, and find from (2.17)

$$|(u, \tau\psi K \varphi \partial^2 v)| = |(u, \tau\eta\psi K \varphi \partial^2 v)| \leqslant C_\eta \|\psi K \varphi \partial^2 v\|$$
$$\leqslant C_\varphi \|v\| \tag{2.20}$$

since η depends on φ, and $\psi K \varphi \partial^2$ is bounded on L^2.

In applying (2.17) we need

$$\psi K \varphi \partial^2 v \in H^2(\mathbb{R}^*, H) \cap L^2(\mathbb{R}^*, H_A, (1+x)^{-4}\, dx)$$

given that v is in this same space. This function is in H^2 since, by (2.4), $\partial^2 v \in L^2$ and $\partial^2 \psi K_\varphi$ is bounded on L^2. It is in $L^2(\mathbb{R}^*, H_A, (1+x)^{-4}\, dx)$ by (2.7d), since $(1+X)^{-2} (A + c)\, \psi K \varphi$ is bounded on L^2.

Combining (2.20) with (2.19) gives

$$|(u, \varphi v'')| \leqslant c_\varphi \|v\|$$

hence $\varphi u \in H^2(\mathbb{R}^*, H)$. A similar argument shows that

$$|(u, \varphi X^{-2}(A + c)\, v)| \leqslant C_\varphi \|v\| \tag{2.21}$$

hence $\varphi u \in L^2(\mathbb{R}^*, H_A, (1+x)^{-4}\, dx)$, proving (a).

For (b), let $\tau u \in H^1_*$. It follows that $\tau u' \in L^2_*(\mathbb{R}^*, H)$; for

$$|(u \,|\, \partial\tau\varphi v)| = |(u \,|\, \tau\partial\varphi v) - (\mathscr{A}' u \,|\, \varphi v)|$$
$$\leqslant C_\varphi \|v\| + |(u, \varphi \mathscr{A}' v)|$$

by (2.18), and

$$|(u, \varphi \mathscr{A}' v)| = |(u, [\varphi X^{-2}(A + c)]\, \psi X^2 (A + c + 1)^{-1} \mathscr{A}' v)|$$
$$\leqslant C_\varphi \|v\|$$

by (2.21), since $\psi X^2(A+c)^{-1}\mathscr{A}'$ is bounded [ψ vanishes near 0]. Thus by (a), $\varphi u \in H^2$ and $\varphi u' \in H^2$ for all φ in B^∞ vanishing near 0, so in fact $\varphi u \in H^3$, and

$$\|(\varphi u)''(x)-(\varphi u)''(y)\| \leqslant \int_y^x \|(\varphi u)^{(3)}\|$$

$$\leqslant |x-y|^{1/2}\,\|\varphi u\|_{H^3}.$$

Similarly, we have $\varphi X^2(A+c)\,u'$ in $L^2(\mathbb{R}^*, H)$, which gives $\varphi u \in C^{1/2}(\mathbb{R}^*, H_A)$. ∎

3. The Constant Coefficient Parametrix

This is the heart of the problem. We begin by studying the "frozen operator"

$$D^2+X^{-2}A_0$$

through the eigenvalue decomposition of $A_0 := A(0)$. Here X^{-2} is the operator

$$(X^{-2}u)(x) := x^{-2}u(x).$$

For the low eigenvalues we construct the resolvent directly, using Bessel functions; the high eigenvalues are handled by a priori estimates, which obviate the need for estimates of the Bessel functions when both order and argument grow large. We assume that

$$A_0 \geqslant -\tfrac{1}{4} \tag{3.1}$$

and that A_0 has a complete orthonomal set of eigenvectors with eigenvalues a such that

$$\left[\sum_{a\in spA.}(a+1)^{-p}\right]^{1/p} = \|(A_0+1)^{-1}\|_p < \infty, \tag{3.2}$$

where each eigenvalue is counted according to its multiplicity. Thus $(A_0+1)^{-1}$ is in the Schatten class C_p. We define an operator G_0 as a direct sum over the spectral decomposition of A_0,

$$G_0 = \oplus_a (L_a+z^2)^{-1}\otimes \pi_a, \tag{3.3}$$

where π_a is projection on the eigenspace of A_0 with eigenvalue a and L_a is the Friedrichs extension of

$$D^2+X^{-2}a$$

in $L^2(\mathbb{R}_+)$, discussed in [5] and [4].

Lemma 3.1. *For $a \geqslant -\frac{1}{4}$, $D^2+X^{-2}a$ is nonnegative on $C_0^\infty(\mathbb{R}^*)$. Hence so is its Friedrichs extension L_a, and*

$$\|(L_a+z^2)^{-1}\| \leqslant \begin{cases} |z|^{-2} & \text{if } \operatorname{Re}(z^2)\geqslant 0,\\ |\operatorname{Im} z^2|^{-1} & \text{if } \operatorname{Re}(z^2)<0. \end{cases} \tag{3.4}$$

Proof. If $a\geqslant 0$, $D^2+X^{-2}a$ is clearly nonnegative. If $-\frac{1}{4}\leqslant a<0$ then Hardy's inequality for $u \in C_0^\infty(\mathbb{R}^*)$,

$$\|X^{-1}u\| \leqslant 2\,\|u'\|$$

implies that

$$(-u''+aX^{-2}u, u) = \|u'\|^2 + a\,\|X^{-1}u\|^2 \geqslant \|u'\|^2\,(1+4a) \geqslant 0.$$

So this holds for the Friedrichs extension L_a, and (3.4) follows by the spectral theorem. ∎

Remark. One can show using the Mellin transform that $D^2+X^{-2}a$ is *not* bounded below if $a<-\frac{1}{4}$.

The a priori estimates for high eigenvalues are the following:

Lemma 3.2. *Given $\theta>0$ and j real, there are constants c_1, C such that if $|\arg\lambda| \leqslant \pi-\theta$ and $a\geqslant c_1$ then*

$$a\,\|X^{-2-j}(L_a+\lambda)^{-1}X^j\| \leqslant C, \tag{3.5a}$$

$$a^{1/2}\,\|X^{-1}\partial X^{-j}(L_a+\lambda)^{-1}X^j\| \leqslant C, \tag{3.5b}$$

$$\|\partial^2 X^{-j}(L_a+\lambda)^{-1}X^j\| \leqslant C, \tag{3.5c}$$

$$\|X^{-j}(L_a+\lambda)^{-1}X^j\| \leqslant C\,|\lambda|^{-1}, \tag{3.5d}$$

$$\|\partial X^{-j}(L_a+\lambda)^{-1}X^j\| \leqslant C\,|\lambda|^{-1/2}. \tag{3.5e}$$

Proof. Let v be in $C_0^\infty(\mathbb{R}^*)$ and set $u := X^{-j}(L_a+\lambda)^{-1}X^j v$. As noted in the Introduction, the kernel of $(L_a+\lambda)^{-1}$ is

$$(L_a+z^2)^{-1}(x,y) =: \sqrt{x}\,k_\nu(x,y;z)\,\sqrt{y}, \tag{3.6a}$$

where k_ν is a product of Bessel functions of order $\nu := (a+\frac{1}{4})^{1/2}$,

$$k_\nu(x, y; z) = k_\nu(y, x; z) = I_\nu(xz)\, K_\nu(yz), \qquad x \leqslant y. \tag{3.6b}$$

Since v vanishes near 0 and ∞, the kernel (3.6) together with familiar asymptotic properties of I_ν and K_ν show that u decays exponentially as $x \to \infty$ and $u(x) \sim x^{\nu+1/2-j}$ as $x \to 0$, with $\nu = (a+\frac{1}{4})^{1/2}$. Hence if a is large enough, the integrations by parts made below are justified.

The inequalities (3.5a)–(3.5e) are equivalent to

$$c^2 I \geqslant a^2 \|X^{-2}u\|^2 + a\|X^{-1}u'\|^2 + \|u''\|^2 + |\lambda|^2 \|u\|^2, \tag{3.7a}$$

where

$$I = \|X^{-j}(L_a+\lambda)\, X^j u\|^2 \tag{3.7b}$$

and (3.5e) follows from (3.5c) and (3.5d) since $\|\partial u\|^2 = -(\partial^2 u \mid u)$. Integration by parts gives

$$2\,\mathrm{Re}(u'', X^{-2}u) = -2\|X^{-1}u'\|^2 + 6\|X^{-2}u\|^2, \tag{3.8a}$$

$$2\,\mathrm{Re}(u', X^{-1}u) = \|X^{-1}u\|^2, \tag{3.8b}$$

$$2\,\mathrm{Re}(u', X^{-3}u) = 3\|X^{-2}u\|^2. \tag{3.8c}$$

Noting that

$$X^{-j}(L_a+\lambda)\, X^j = L_{a-j(j-1)} - 2jX^{-1}\partial$$

we set $b = a - j(j-1)$, expand I in (3.7b) and find

$$\begin{aligned} I &= \|-u'' + bX^{-2}u - 2jX^{-1}u' + \lambda u\|^2 \\ &= \|u''\|^2 + 2(b+j+2j^2)\|X^{-1}u'\|^2 + (b^2-6b-6jb)\|X^{-2}u\|^2 \\ &\quad + |\lambda|^2\|u\|^2 + 2\,\mathrm{Re}(\lambda)\|u'\|^2 + 2b\,\mathrm{Re}(\lambda)\|X^{-1}u\|^2 - 4j\mathrm{Re}(X^{-1}u', \lambda u). \end{aligned}$$

We use $|2\,\mathrm{Re}(2jX^{-1}u', \lambda u)| \leqslant 4j^2\eta\|X^{-1}u'\|^2 + |\lambda|^2\|u\|^2/\eta$ (with η to be chosen later) and $\mathrm{Re}(\lambda) \geqslant -\gamma|\lambda|$ (with $\gamma = \cos\theta$) to get

$$\begin{aligned} I \geqslant\ & \|u''\|^2 + 2(b+2j^2+j-2j^2\eta)\|X^{-1}u'\|^2 \\ & + (b^2-6b-6jb)\|X^{-2}u\|^2 - 2|\lambda|\gamma\|u'\|^2 \\ & - 2b|\lambda|\gamma\|X^{-1}u\|^2 + |\lambda|^2\left(1-\frac{1}{\eta}\|u\|^2\right). \end{aligned} \tag{3.9}$$

Choose ρ with $\gamma < \rho < 1$. Expanding as before gives

$$\begin{aligned} &\|\rho(-u'' + bX^{-2}u) - (|\lambda|\gamma/\rho)\, u\|^2 \\ &\quad = \rho^2\|u''\|^2 + 2b\rho^2\|X^{-1}u'\|^2 \\ &\qquad + \rho^2(b^2-6b)\|X^{-2}u\|^2 + (|\lambda|\gamma/\rho)^2\|u\|^2 \\ &\qquad - 2|\lambda|\gamma\|u'\|^2 - 2b|\lambda|\gamma\|X^{-1}u\|^2. \end{aligned}$$

Subtract this from (3.9) to get

$$\begin{aligned} I \geqslant\ & (1-\rho^2)[\|u''\|^2 + 2(b + (2j^2+j-2j^2\eta)/(1-\rho^2))\|X^{-1}u'\|^2 \\ & + (b^2-6b-6jb/(1-\rho^2))\|X^{-2}u\|^2] \\ & + |\lambda|^2(1-1/\eta-\gamma^2/\rho^2)\|u\|^2. \end{aligned}$$

Choose $1/\eta = \frac{1}{2}(1-\gamma^2/\rho^2) > 0$, and (3.7a) follows when $b = a - j(j-1)$ is sufficiently large. The lemma is proved. ∎

Now come the estimates for low eigenvalues, using the kernel (3.6). The asymptotic expansions

$$\left.\begin{aligned} I_\nu(\omega) &\sim c\omega^\nu \\ K_\nu(\omega) &\sim c\omega^{-\nu} \\ K_0(\omega) &\sim c\log\omega \end{aligned}\right\} \text{ as } \omega \to 0, \qquad \left.\begin{aligned} I_\nu(\omega) &\sim c\omega^{-1/2}e^{\omega} \\ K_\nu(\omega) &\sim c\omega^{-1/2}e^{-\omega} \end{aligned}\right\} \text{ as } \omega \to \infty$$

are valid uniformly in $|\arg\omega| \leqslant (\pi-\theta)/2$. Hence, estimating separately for z along each ray in this sector and assuming $x \leqslant y$, we find

$$\begin{aligned} &|k_\nu(x, y; z)| \\ &\quad \leqslant Ce^{-c|z||x-y|}\left(\frac{|xz|}{1+|xz|}\,\frac{1+|yz|}{|yz|}\right)^\nu \\ &\qquad \cdot \frac{1-\max(\log_-|xz|, \log_-|yz|)}{(1+|xz|)^{1/2}(1+|yz|)^{1/2}} \end{aligned} \tag{3.10a}$$

$$\leqslant Ce^{-c|z||x-y|} \cdot \frac{1-\max(\log_-|xz|, \log_-|yz|)}{(1+|xz|)^{1/2}(1+|yz|)^{1/2}} \tag{3.10b}$$

since $s/(1+s)$ is an increasing function; we use

$$\log_- t := \min(\log t, 0) \leqslant 0.$$

Note that (3.10b) is valid for all x, y by symmetry. We will also need $\partial_x k_\nu$, for which we use the Bessel derivative formula

$$\partial_x(x^\nu f_\nu(xz)) = \pm zx^\nu f_{\nu-1}(xz)$$

with "+" for $f_\nu = I_\nu$ and "−" for $f_\nu = K_\nu$. Now

$$x\partial_x k_\nu = x\partial_x(x^{-\nu}x^\nu k_\nu) = -\nu k_\nu + x^{1-\nu}\partial_x(x^\nu k_\nu) = -\nu k_\nu + k'_\nu \tag{3.11}$$

with

$$k'_\nu(x, y; z) = \begin{cases} xzI_{\nu-1}(xz)\,K_\nu(yz), & x \leqslant y \\ -xzK_{\nu-1}(xz)\,I_\nu(yz), & x \geqslant y. \end{cases}$$

The first term $-\nu k_\nu$ in (3.11) is controlled by (3.10) above, and the second by an analogous reasoning. The main difference now is that as $\omega \to \infty$

$$\omega I_{\nu-1}(\omega) \sim \omega^{1/2}e^{\omega} \qquad \text{and} \qquad \omega K_{\nu-1}(\omega) \sim \omega^{1/2}e^{-\omega}.$$

This gives

$$|x\partial_x k_\nu| \leqslant \left(\frac{1+|xz|}{1+|yz|}\right)^{1/2} e^{-c|z||x-y|}\,(1-\max(\log_-|xz|, \log_-|yz|). \tag{3.12}$$

In the following lemmas the operator G_0 in (3.3) is split: $G_0 = G_< + G_>$, where $G_>$ is the sum over all sufficiently high eigenvalues and $G_<$ is the sum over the others.

LEMMA 3.3. *For any real k, and uniformly for $|\arg z^2| \leqslant \pi - \varepsilon$, $|z| \geqslant 1$,*

$$\|(1+X)^{-k-1}\,X^{-1/2}(A_0+1)\,G_0X^{-1/2}(1+X)^k\| \leqslant C, \tag{3.13a}$$

$$\|(1+X)^{-k-1}\,X^{1/2}\Lambda\partial G_0X^{-1/2}(1+X)^k\| \leqslant C, \tag{3.13b}$$

$$\|(1+X)^{-k-1}\,X^{3/2}\partial^2G_0X^{-1/2}(1+X)^k\| \leqslant C, \tag{3.13c}$$

$$\|(1+X)^{-k}\,X^\theta z^2G_0X^{-\theta}(1+X)^k\| \leqslant C \tag{3.14}$$

for $|\theta| \leqslant \frac{1}{2}$. Here $\Lambda := (A_0+1)^{1/2}$.

Proof. Consider first $G_>$. In (3.13a) we can take $k \geqslant -\frac{1}{2}$, and pass to the adjoint for $k \leqslant -\frac{1}{2}$. Since $x < (1+x)^{1/2}$ and $(1+x)^{k+1/2} \leqslant C(1+x^{k+1/2})$, we get

$$\begin{aligned}\|(1+X)^{-k-1}\,&X^{-1/2}(A_0+1)\,G_>X^{-1/2}(1+X)^k\| \\ &\leqslant C\,\|(1+X)^{-k-1/2}\,X^{-1}(A_0+1)\,G_>X^{-1}(1+X^{k+1/2})\| \\ &\leqslant C\,\|X^{-1}(A_0+1)\,G_>X^{-1}\| + C\,\|X^{-k-3/2}(A_0+1)\,G_>X^{k-1/2}\| \\ &\leqslant C\end{aligned}$$

by (3.5a). Turning to (3.14), we treat the case $k \geqslant 0$ and take adjoints. For $k \geqslant 0$ it suffices to estimate

$$\begin{aligned}\|(1+X)^{-k}\,&X^\theta z^2G_>X^{-\theta}(1+X^k)\| \\ &\leqslant C\,\|X^{\theta-k}z^2G_>X^{k-\theta}\| + C\,\|X^\theta z^2G_>X^{-\theta}\| \\ &\leqslant C\end{aligned}$$

by (3.5d). Then for (3.13c) we have

$$\begin{aligned}(1+X)^{-k-1}\,X^{3/2}\partial^2G_> &= (1+X)^{-k-1}\,X^{-1/2}A_0G_> \\ &\quad + (1+X)^{-k-1}\,z^2X^{3/2}G_>,\end{aligned}$$

so (3.13c) follows from (3.13a) and (3.14) with $\theta = \frac{1}{2}$. Finally, (3.14b) can be proved like (3.14a), or by interpolation between (3.14a) and (3.14c).

Now take (3.13a) for $G_<$. We have a finite linear combination of operators with kernels

$$K(x, y; z) = (1+x)^{-k-1}\,k_\nu(x, y; z)(1+y)^k.$$

From (3.10) we get for $|z| \geqslant 1$,

$$|K(x, y; z)| \leqslant C\frac{(1+y)^k}{(1+x)^{k+1}}\,e^{-c|z||x-y|}[1-\max(\log_- x, \log_- y)].$$

Apply (3.10) and note that for any k, and any $\varepsilon > 0$,

$$\left(\frac{1+y}{1+x}\right)^k e^{-\varepsilon|x-y|} \leqslant C. \tag{3.15}$$

Thus, changing the constant c in (3.10), for $|z| \geqslant 1$

$$|K(x, y; z) \leqslant Ce^{-c|x-y|}[1-\max(\log_-|x|, \log_-|y|)]$$

and it follows easily that

$$\int |K(x, y; z)|\,dx + \int |K(x, y; z)|\,dy \leqslant C$$

which bounds the norm of the operator in question.

For (3.14) the parameter z plays a bigger role. Using (3.10) and setting $s := x|z|$ and $t := y|z|$ we get kernels $K(x, y; z)$ such that

$$\begin{aligned}|K(x, y; z)| &\leqslant C\,|z|\,s^{1/2+\theta}t^{1/2-\theta}e^{-c|s-t|}\,\frac{1-\max(\log_- s, \log_- t)}{(1+s)^{1/2}\,(1+t)^{1/2}} \\ &=: |z|\,k(s, t).\end{aligned}$$

Since $|\theta| \leqslant \frac{1}{2}$, then $s^{1/2+\theta}t^{1/2-\theta} \leqslant (1+s)^{1/2+\theta}(1+t)^{1/2-\theta}$ and (3.15) gives, for some $c>0$,

$$k(s,t) \leqslant Ce^{-c|s-t|}[1-\max(\log_- s, \log_- t].$$

Hence

$$\int |K(x,y;z)|\, dy + \int |K(x,y;z)|\, dx \leqslant 2\int k(s,t)\, dx \leqslant C.$$

For (3.13a) and (3.14) we get (3.13c) and (3.13b), just as in the case for $G_>$, and this completes the proof of Lemma 3.3. ▮

LEMMA 3.4. *Uniformly for* $|\arg z^2| \leqslant \pi - \theta$, $|z| \geqslant 1$,

$$\| X^{1/2}\partial G_0 X^{-1/2} \| \leqslant C|z|^{-1}, \tag{3.16}$$

$$\| G_0 X^{-1/2}(1+X)^{1/2} \| \leqslant C. \tag{3.17}$$

Proof. We obtain (3.16) first without the factor $|z|^{-1}$, then deduce the z-decay by a scaling argument. Let $T = X^{1/2}\partial G_0 X^{-1/2}$. Then

$$T^*T = [X^{-1/2}G_0X^{-1/2}(1+X)][(1+X)^{-1}X^{3/2}\partial^2 G_0 X^{-1/2}] + [X^{-1/2}G_0X^{-1/2}(1+X)][(1+X)^{-1}X^{1/2}\partial G_0 X^{-1/2}].$$

The two right-hand factors are bounded, by (3.13b, 3.13c), so we must bound the left-hand factor. For $G_>$ we get

$$\| X^{-1/2}G_> X^{-1/2}(1+X) \| \leqslant C \| X^{-1/2}G_>(X^{-3/2}+X^{1/2}) \|$$

which is bounded, by (3.13a) and (3.14). For $G_<$ we have kernels satisfying

$$|k(x,y;z)| = (1+y)|k_v(x,y;z)| \leqslant \left(\frac{1+x}{1+y}\right)^{1/2} e^{-c|z||x-y|}$$

$$\leqslant Ce^{-c'|z||x-y|}$$

as in the proof of Lemma 3.3, which gives (3.16) without the $|z|^{-1}$. To produce this factor, we define a unitary scaling operator

$$U_t f(x) := t^{1/2} f(tx).$$

Then

$$U_t X^\theta = t^\theta X^\theta U_t, \qquad U_t\partial = t^{-1}\partial U_t, \qquad U_t G_0(z) = t^2 G_0(tz)\, U_t$$

so

$$\| X^{1/2}\partial G_0(z) X^{-1/2} \| = \| U_t X^{1/2}\partial G_0(z) X^{-1/2} \| = t \| X^{1/2}\partial G_0(tz) X^{-1/2} \|.$$

Set $t = |z|^{-1}$ and get from (3.13b) with $k=0$

$$\| X^{1/2}\partial G_0(z) X^{-1/2} \| = |z|^{-1} \| X^{1/2}\partial G_0(z/|z|) X^{-1/2} \| \leqslant C|z|^{-1}$$

proving (3.16), since we have already bounded $\| X^{1/2}\partial G_0(z/|z|) X^{-1/2} \|$.
Next, considering behavior as $x \to 0$ and $x \to \infty$, we have for $x > 0$

$$1 \leqslant C[(1+x)^{-3/2}x^{-1/2} + (1+x)^{-1/2}x^{1/2}].$$

Thus (3.17) follows from (3.13a) and (3.14) with $k = \theta = \frac{1}{2}$, and the lemma is proved. ▮

The next results for G_0 concern its relation to trace class.

LEMMA 3.5. *If* $G_> = G_0$ *projected on sufficiently high eigenspaces then*

$$\| (1+X)^{-2} X^{-1/2} G_> X^{-1/2} \|_{p+1/2} \leqslant C. \tag{3.18}$$

Proof. Let $P = -\partial_\theta \theta(\theta - \pi/2)\partial_\theta$ be the Legendre operator on $L^2(0, \pi/2)$, with eigenvalues $j(j+1)$, $j = 0, 1, 2, \ldots$. Setting

$$x = \tan\theta, \qquad g(x) = \cos\theta\, f(\theta)$$

transforms P unitarily to

$$\tilde{P} = -\partial_x[(1+x^2)^2\theta(\theta-\pi/2)]\,\partial_x - [x(2\theta-\pi/2) + \theta(\theta-\pi/2)(2x^2+1)].$$

The coefficient of ∂_x^2 is $O(x^3)$ at ∞; the coefficients of ∂_x and the constant term are, respectively, $O(x^2)$ and $O(x)$. Hence by Lemma 3.2,

$$\| X^{-1/2}G_> X^{-1/2}(1+X)^{-2}(\tilde{P} + A_0 + 1) \| \leqslant C; \tag{3.19}$$

the term with $A_0 + 1$ has more decay than needed, and those with ∂_x^2 and ∂_x have just enough. Further $\tilde{P} + A_0 + 1$ (which is really shorthand for $\tilde{P} \otimes I + I \otimes (A_0 + 1)$) is self-adjoint in $L^2(0,\infty) \otimes H$ with eigenvalues

$$\{ j(j+1) + a + 1 : a \in \operatorname{spec}(A_0), j = 0, 1, 2, \ldots, \}.$$

So

$$\|(\tilde{P}+A_0+1)^{-1}\|_{p+1/2}^{p+1/2}=\sum_{a,j}[j(j+1)+a+1]^{-p-1/2}$$
$$\leqslant C\sum_a\int_0^\infty (a+1+t^2)^{-p-1/2}\,dt$$
$$\leqslant C'\sum_a (a+1)^{-p}=C'\,\|(A_0+1)^{-1}\|_p^p.$$

This, combined with (3.19), proves (3.18).

LEMMA 3.6. *For the low eigenvalue part $G_<$ we have for $|z|\geqslant 2$*

$$\|(1+X)^{-1}X^{-1/2}G_<X^{-1/2}\|_{\mathrm{tr}}\leqslant C\,|z|^{-1}\log|z|. \tag{3.20}$$

Proof. We show first that for z real

$$\|(1+X)^{-1/2}X^{-1/2}G_<X^{-1/2}(1+X)^{-1/2}\|_{\mathrm{tr}}\leqslant C\,|z|^{-1}\log|z|. \tag{3.21}$$

The operator in question is a finite direct sum of one-dimensional positive operators, whose trace norm is just the trace, and this is the integral of the kernel on the diagonal (see the Trace Lemma in the Appendix). The relevant integrals are

$$\int_0^\infty (1+x)^{-1}K_\nu(xz)\,I_\nu(xz)\,dx=z^{-1}\int_0^\infty\left(1+\frac{y}{z}\right)^{-1}K_\nu(y)\,I_\nu(y)\,dy$$
$$\leqslant z^{-1}\int_0^1 K_\nu I_\nu\,dy+C\int_1^\infty\frac{dy}{(1+y)(z+y)}$$
$$=O(z^{-1}\log z),\qquad z\geqslant 2$$

proving (3.21).

For $|z|\geqslant 2$ with $|\arg z^2|\leqslant\pi-\theta$ we have

$$(1+X)^{-1/2}X^{-1/2}G_<(z)\,X^{-1/2}(1+X)^{-1/2}=STS^*, \tag{3.22}$$

where

$$S=X^{-1/2}(1+X)^{-1/2}\,G_<(|z|)^{1/2},\qquad T=G_<(|z|)^{-1/2}\,G_<(z)\,G_<(|z|)^{-1/2}.$$

We find the Hilbert–Schmidt norm

$$\|S\|_2^2=\|S^*\|_2^2=\|SS^*\|_{\mathrm{tr}}\leqslant C\,|z|^{-1}\log|z|$$

by (3.21), and the operator norm

$$\|T\|\leqslant\csc\theta/2$$

by the spectral theorem. Hence for $|z|\geqslant 2$ and $|\arg z^2|\leqslant\pi-\theta$, (3.22) gives (3.21). For (3.20) itself we use the commutator

$$[(1+X)^{1/2},G_<]=\tfrac14 G_<(1+X)^{-3/2}\,G_<-G_<(1+X)^{-1/2}\,\partial G_<$$

which gives

$$\begin{aligned}(1+X)^{-1}X^{-1/2}G_<X^{-1/2}&=[(1+X)^{-1/2}X^{-1/2}\,G_<(1+X)^{-1/2}X^{-1/2}]\cdot I\\&\quad-[\tfrac14(1+X)^{-1}X^{-1/2}G_<X^{-1/2}(1+X)^{-1/2}]\\&\quad\times[(1+X)^{-1}X^{1/2}G_<(1+X)^{-1/2}X^{-1/2}]\\&\quad-[(1+X)^{-1}X^{-1/2}G_<X^{-1/2}(1+X)^{-1/2}]\\&\quad\times[X^{1/2}\partial G(1+X)^{-1}X^{-1/2}].\end{aligned}$$

In each product on the right, the first factor has trace norm $O(|z|^{-1}\log|z|)$ by (3.21), and the second factor has bounded operator norm, by (3.16) and (3.17). This proves (3.20). ∎

The last two lemmas give

LEMMA 3.7. $\|(1+X)^{-2}X^{-1/2}G_0X^{-1/2}\|_{p+1/2}\leqslant C.$

Finally, to identify the range of the parametrix to be constructed in Section 4, we need

LEMMA 3.8. *For f in L^2, the function*

$$u=G_0X^{-1/2}(1+X)^{1/2}f$$

satisfies

$$\lim_{x\to 0+}x^{-1/2}u(x)=2\lim_{x\to 0+}x^{1/2}u'(x). \tag{3.23}$$

Remark. The solutions of $(D^2+X^{-2}a)\,u=0$ have singularities

$$x^{1/2+\nu},\quad x^{1/2-\nu}$$

with $\nu=(a+\frac14)^{1/2}$, if $a>-\frac14$; if $a=-\frac14$ they are

$$x^{1/2},\qquad x^{1/2}\log x.$$

In each case, the existence of $\lim_{x\to 0+} x^{-1/2}u(x)$ picks the weaker of the two singularities, a sort of Dirichlet condition. The two limits in the lemma are zero except when $a=-\frac{1}{4}$, and then they have the relation in (3.23).

Proof of Lemma 3.8. First let $u=G_> X^{-1/2}(1+X)^{1/2}f$; in this case the two limits in question turn out to be zero. From Lemma 3.2,

$$\begin{aligned}\|X^{-3/2}u\| &\leqslant C\,\|X^{-3/2}G_>(X^{-1/2}+X^{3/2})\|\cdot\|f\|\\ &\leqslant C,\\ \|(1+X)^{-1/2}X^{-1/2}u'\| &\leqslant C\,\|X^{-1/2}(1+X)^{-1/2}\,\partial G_> X^{-1/2}(1+X^{1/2})\|\cdot\|f\|\\ &\leqslant C(\|X^{-1/2}\partial G_> X^{-1/2}\|+\|X^{-1}\partial G_>\|)\cdot\|f\|\\ &\leqslant C,\\ \|(1+X)^{-1/2}X^{1/2}u''\| &\leqslant C\,\|(1+X)^{-1/2}X^{1/2}\partial^2 G_> X^{-1/2}(1+X^{1/2})\|\cdot\|f\|\\ &\leqslant C(\|X^{1/2}\partial^2 G_> X^{-1/2}\|+\|\partial^2 G_>\|)\|f\|\\ &\leqslant C.\end{aligned}$$

Hence the function

$$h(x):=x^{-1/2}u(x)$$

has $(1+X)^{-1/2}h'\in L^2$, so h is continuous and

$$h(x)=h(0)+\int_0^x h'.$$

Moreover $X^{-1}h\in L^2$, so $h(0)=0$ and

$$\lim_{x\to 0^+} x^{-1/2}u(x)=\lim_{x\to 0^+} h(x)=0.$$

Similarly $g:=x^{1/2}u'$ has $(1+X)^{-1/2}g'$ and $(1+X)^{-1/2}X^{-1}g$ in L^2, so

$$\lim_{x\to 0^+} x^{1/2}u'(x)=\lim_{x\to 0^+} g(x)=0.$$

Turning to the low eigenvalues, we have

$$x^{-1/2}u(x)=\int_0^\infty k_\nu(x,y;z)(1+y)^{1/2}f(y)\,dy$$

with $k_\nu(x,y;z)=I_\nu(xz)\,K_\nu(yz)$, $x\leqslant y$. By (3.10), when $x\leqslant 1$ and z is fixed, the integrand is dominated by

$$Ce^{-c|z|(y-1)}\,|(1-\log_-|yz|)(1+y)^{1/2}\,|f(y)|.$$

Hence by dominated convergence

$$\lim_{x\to 0+} x^{-1/2}u(x)=\int_0^\infty I_\nu(0)\,K_0(yz)(1+y)^{1/2}f(y)\,dy. \tag{3.24}$$

For the derivative,

$$x^{1/2}u'(x)=\tfrac{1}{2}\int k_\nu f(1+y)^{1/2}\,dy+\int xk_\nu' f(1+y)^{1/2}\,dy.$$

The second integrand has, by (3.12), the same bound as the first, and we get

$$\begin{aligned}\lim_{x\to 0^+} x^{1/2}u'(x)&=\tfrac{1}{2}\int k_\nu(0,y;z)f(y)\,(1+y)^{1/2}\,dy\\ &=\tfrac{1}{2}\int_0^\infty I_\nu(0)\,K_0(yz)f(y)\,(1+y)^{1/2}\,dy\end{aligned}$$

which, with (3.24), proves the lemma. ∎

4. The Boundary Parametrix

The boundary parametrix G_b is constructed from G_0 using a Neumann series. Formally

$$\begin{aligned}(D^2+X^{-2}A+z^2)\,G_0&=I-X^{-2}(A_0-A)\,G_0,\\ G_0(D^2+X^{-2}A+z^2)&=I-G_0X^{-2}(A_0-A),\end{aligned}$$

and we expect a resolvent from the Neumann series

$$\sum_0^\infty (G_0X^{-2}(A_0-A))^j\,G_0=G_0\sum_0^\infty (X^{-2}(A_0-A)\,G_0)^j.$$

However, $X^{-2}(A_0-A)$ has the same order as A_0, so we cannot expect $X^{-2}(A_0-A)\,G_0$ to decay as $z\to\infty$; that could be expected only if we limit ourselves to finitely many eigenvalues of A_0. Thus the series need not converge, even for large z. Further, although $X^{-2}(A_0-A)$ is qualitatively like $X^{-1}A_0$ at $x=0$, this is not enough, for $X^{-1}G_0$ is not bounded; $X^{-1}\,G_>$ is all right, but not $X^{-1}G_<$. To surmount this second problem, we redistribute the powers of x; and to obtain a convergent Neumann series, we redefine $A(x)$ away from $x=0$, obtaining a parametrix G_b such that $G_b(\tau+z^2)f=f$ for f with support near 0.

To begin, assume that

$$\|A^{(k)}(x)(A_0+1)^{-1}\|\leqslant C_k,\qquad k\geqslant 0. \tag{4.1}$$

Thus $A(x)(A_0+1)^{-1}$ is a continuous family of bounded operators. Hence, for any $\delta>0$, we can modify A on some set $\{x\geqslant\varepsilon\}$ to form a new $\tilde{A}$ satisfying (4.1) with the same constants, and also

$$\|[\tilde{A}(x)-A_0](A_0+1)^{-1}\|\leqslant\delta, \tag{4.2}$$

For instance, set $\tilde{A}(x)=A(\gamma(x))$ with γ in $C^\infty(\mathbb{R}^1)$ and, for an appropriate $\varepsilon>0$,

$$\gamma(x)=x,\ x\leqslant\varepsilon;\qquad |\gamma'|\leqslant 1;\qquad 0\leqslant\gamma(x)\leqslant 2\varepsilon;\qquad \gamma(x)=0\quad\text{for } x\geqslant 3\varepsilon.$$

Now define

$$B(x):=x^{-1}(A_0-\tilde{A}(x)) \tag{4.3}$$

so that $(D^2+X^{-2}\tilde{A}+z^2)\,G_0=I-X^{-1}BG_0$. As we noted, $X^{-1}BG_0$ is not bounded, because the singularity at $x=0$ is too great. However, by Lemma 3.3

$$(1+X)^{-1/2}\,X^{-1/2}BG_0X^{-1/2}(1+X)^{1/2}$$

is bounded, and Lemma 4.1 will show that its norm is $\leqslant\frac{1}{2}$ if δ in (4.2) is small and z is large. Hence, using the function

$$\omega(x):=x^{1/2}(1+x)^{-1/2}$$

we write our Neumann series in the form

$$G_b=G_0\Omega^{-1}\sum_{j=0}^{\infty}(X^{-1}\Omega BG_0\Omega^{-1})^j\,\Omega, \tag{4.4}$$

where Ω denotes multiplication by ω. To establish the convergence of (4.4), and of the corresponding series with the Ω factors canceled, we need the inequality (3.13a) with $k=\frac{1}{2}$, which we can write as

$$\|(A_0+1)(1+X)^{-1}\,X^{-1}\Omega G_0\Omega^{-1}\|\leqslant C. \tag{4.5}$$

LEMMA 4.1. *If $\tilde{A}$ satisfies (4.1) and (4.2) with $\delta^{-1}=16C^2C_1$, for the constants C in (4.5) and C_1 in (4.1), then*

$$\|X^{-1}\Omega BG_0\Omega^{-1}\|\leqslant\tfrac{1}{2}, \tag{4.6a}$$

$$\|X^{-1/2}BG_0X^{-1/2}\|\leqslant\tfrac{1}{2} \tag{4.6b}$$

for $|z|\geqslant z_0:=(4CC_1)^2$. Further, for all $|z|\geqslant 1$,

$$\|G_0\Omega^{-1}\|+\|\Omega\|\leqslant C. \tag{4.7}$$

Proof. The inequality (4.7) is just (3.17). To prove (4.6a) we use the scaling operator $U_tf(x)=t^{1/2}f(tx)$. In addition to the formulas after (3.17) we have

$$U_t\Omega=t^{1/2}\Omega_tU_t,\ U_t\Omega^{-1}=t^{-1/2}\Omega_t^{-1}U_t,\ U_tB=t^{-1}B_tU_t,$$

where Ω_t is multiplication by $\omega_t(x):=x^{1/2}(1+tx)^{-1/2}$, and

$$B_t(x):=x^{-1}[A_0-\tilde{A}(tx)].$$

Since U_t is unitary, we have

$$\begin{aligned}\|X^{-1}\Omega BG_0(z)\,\Omega^{-1}\|&=\|X^{-1}\Omega_tB_tG_0(tz)\,\Omega_t^{-1}\|\\&\leqslant\|B_t(A_0+1)^{-1}\,(1+X)\|\\&\quad\cdot\|(1+X)^{-1}\,X^{-1}(A_0+1)\,\Omega_tG_0(tz)\,\Omega_t^{-1}\|.\end{aligned} \tag{4.8}$$

When $t\leqslant 1$ then $\omega_t^{-1}\leqslant\omega^{-1}$ and $\omega_t\leqslant t^{-1/2}\omega$, so the last factor in (4.8) is

$$\leqslant t^{-1/2}\,\|(1+X)^{-1}\,(A_0+1)\,X^{-1}\Omega G_0(tz)\,\Omega^{-1}\|\leqslant t^{-1/2}C \tag{4.9}$$

if $|tz|\geqslant 1$, by (4.5). In the first factor we have from (4.1),

$$\begin{aligned}(1+x)\|\,B_t(x)(A_0+1)^{-1}\,\|_H&=(1+x)\left\|\int_0^t\tilde{A}'(sx)(A_0+1)^{-1}\,ds\right\|_H\\&\leqslant C_1t(1+x)\leqslant t^{1/2}/2C\end{aligned} \tag{4.10}$$

if $x\leqslant 1$ and

$$t:=(4CC_1)^{-2}.$$

If $x\geqslant 1$ then

$$\begin{aligned}(1+x)\|\,B_t(x)(A_0+1)^{-1}\,\|_H&=(1+x^{-1})\|[A_0-\tilde{A}(tx)](A_0+1)^{-1}\,\|_H\\&\leqslant 2\delta=t^{1/2}/2C\end{aligned} \tag{4.11}$$

if $\delta^{-1}:=16C_1C^2$. With this, the factor $\|B_t(A_0+1)^{-1}\,(1+X)\|$ in (4.8) is $\leqslant t^{1/2}/2C$, and this with (4.9) proves (4.6a) for $|tz|\geqslant 1$, thus for

$$|z|\geqslant(4CC_1)^2=z_0.$$

A simpler version of the same argument gives (4.6b), completing the proof. ∎

We will need the conclusion of this lemma for the scaled operator

$$B_t(x)=x^{-1}[A_0-\tilde{A}(tx)],\qquad 0\leqslant t\leqslant 1.$$

This is valid with the same constants z_0, C_1, and C. For if (4.2) holds for all x, then it holds with x replaced by tx; and the same is true for (4.1) when $t \leqslant 1$. Hence we can define for $0 \leqslant t \leqslant 1$ a bounded operator in $L^2 \otimes H$ by

$$G_{b,t} = G_0 \Omega^{-1} \sum_{j=0}^{\infty} (X^{-1} \Omega B_t G_0 \Omega^{-1})^j \Omega. \tag{4.12}$$

From (4.7) and (3.13a, 3.13c) with $k = \frac{1}{2}$, we have

$$\| G_0 \Omega^{-1} \| \leqslant C, \tag{4.13a}$$

$$\|(\Omega^3 X^{-2} A_0 G_0 \Omega^{-1} \| \leqslant C, \tag{4.13b}$$

$$\|(\Omega^3 \partial^2 G_0 \Omega^{-1} \| \leqslant C. \tag{4.13c}$$

Hence we may apply $\Omega^3(D^2 + X^{-2}\tilde{A}_t + z^2)$ to $G_{b,t}$, where $\tilde{A}_t(x) := \tilde{A}(tx)$. We find

$$\Omega^3(D^2 + X^{-2}\tilde{A}_t + z^2)\, G_{b,t} = \Omega^3.$$

Thus, pointwise,

$$(D^2 + X^{-2}\tilde{A}_t + z^2)\, G_{b,t} f = f. \tag{4.14}$$

From (4.12)–(4.14) and (3.23), any function $u = G_{b,t} f$ with f in L^2 satisfies

$$\Omega^3 \partial^2 u \in L^2, \tag{4.15a}$$

$$(1 + X)^{-2} A_0 u \in L^2, \tag{4.15b}$$

$$(D^2 + X^{-2}\tilde{A}_t)\, u \in L^2, \tag{4.15c}$$

$$\lim_{x \to 0^+} x^{-1/2} u(x) = \tfrac{1}{2} \lim_{x \to 0^+} x^{1/2} u'(x). \tag{4.15d}$$

Hence $G_{b,t}$ is a right inverse for the "scaled boundary oprator"

$$L_{b,t} := D^2 + X^{-2}\tilde{A}_t$$

with domain $\mathscr{D}(L_{b,t})$ defined by (4.15). We set $L_b = L_{b,1} = D^2 + X^{-2}\tilde{A}$. To simplify the notation, we state many of our results for L_b, though they hold generally for $L_{b,t}$, $0 \leqslant t \leqslant 1$. In particular the pointwise equation (4.14) gives the operator equation

$$(L_b + z^2)\, G_b = I. \tag{4.16}$$

LEMMA 4.2. *L_b is symmetric.*

Proof. Since $\omega^3(x) \to 1$ as $x \to \infty$, (4.15a) implies that for u in $\mathscr{D}(L_b)$, $\int_\varepsilon^\infty \| u'' \|^2 < \infty$ for all $\varepsilon > 0$, and hence $\int_\varepsilon^\infty \| u' \|^2 < \infty$ as well. It follows that

$$\begin{aligned}(L_b u, v) &= \lim_{\varepsilon \to 0} \int_\varepsilon^\infty (-u'' + X^{-2}\tilde{A}u, v)_H \\ &= \lim_{\varepsilon \to 0} [-(\varepsilon^{-1/2} u(\varepsilon), \varepsilon^{1/2} v'(\varepsilon)) + (\varepsilon^{1/2} u'(\varepsilon), \varepsilon^{-1/2} v(\varepsilon))] \\ &\quad + \lim_{\varepsilon \to 0} \int_\varepsilon^\infty (u, -v'' + X^{-2}\tilde{A}v)_H \\ &= (u, L_b)\end{aligned}$$

by (4.15c, 4.15d) for u and v. The lemma is proved. ∎

In view of the symmetry and (4.16) we have

THEOREM 4.1. *L_b is self-adjoint and semibounded, and G_b is its resolvent. In particular*

$$G_b(z)^* = G_b(\bar{z}), \qquad \| G_b(z) \| = O(|z|^{-2}).$$

Proof. Since $(L_b + z^2)\, G_b = I$ and L_b is symmetric,

$$\begin{aligned}(G_b(z) f, g) &= (G_b(z) f, (L_b + \bar{z}^2)\, G_b(\bar{z})\, g) \\ &= ((L_b + z^2)\, G_b(z) f, G_b(\bar{z})\, g) \\ &= (f, G_b(\bar{z})\, g).\end{aligned}$$

So for z real, $G_b(z)$ is self-adjoint, and (4.16) proves it injective. Thus the spectral resolution of $G_b(z)$ yields a self-adjoint operator $G_b(z)^{-1}$ with domain $G_b(z)\,\mathscr{H}$. By (4.16), $L_b + z^2$ extends $G_b(z)^{-1}$. But a self-adjoint operator has no proper symmetric extensions, so $L_b + z^2 = G_b(z)^{-1}$; and $G_b(z)$ is the resolvent $(L_b + z^2)^{-1}$ of the self-adjoint operator L_b. Since $G_b(z)$ is defined for $z \geqslant z_0$, then L_b is bounded below by $-z_0^2$, completing the proof. ∎

Remark 1. Boundary conditions for the case where H is 1-dimensional, and for more general equations, have been treated in classical studies by Weyl, Stone, and Rellich [20, Chap. X; 16, Sect. 1]. The boundary condition (4.15d) implies that of Weyl and Stone. In fact, according to their theory any self-adjoint extension of the differential expression

$$D^2 + \frac{a(x)}{x^2} =: \tau_a \qquad \text{defined on } C_0^\infty(\mathbb{R}),$$

is given by restricting the adjoint τ_a^* to a linear space

$$\left\{u \in H_*^2(\mathbb{R}^*) \cap L_*^2\left(\mathbb{R}_+, \frac{dx}{1+x^4}\right) \Big| \lim_{x\to 0} [v, a](x) = 0\right\},$$

where $[v, a]$ denotes the Wronskian, and the function $v \in C^1(0, \varepsilon)$, with v' locally absolutely continuous, and

$$\int_0^\varepsilon (|v|^2 + |\tau_a v|^2) < \infty.$$

Now we have a solution v of

$$\left(D^2 + \frac{a(x)}{x^2}\right) v(x) = 0$$

of the form

$$v(x) := f(x)\, x^{1/2+\nu}, \qquad \nu = \sqrt{a(0) + 1/4},$$

with $f \in C^\infty([0, \varepsilon])$. If $u \in \mathscr{D}(L_b)$ then we obtain from (4.15d),

$$\begin{aligned}
&\lim_{x\to 0} [v, u](x) \\
&\quad = \lim_{x\to 0} \{x^{1/2+\nu} f(x)\, u'(x) - [(1/2+\nu)\, x^{-1/2+\nu} f(x) + x^{1/2+\nu} f'(x)]\, u(x)\} \\
&\quad = \lim_{x\to 0} \{x^\nu f(x)(x^{1/2} u'(x) - 1/2 x^{-1/2} u(x)) \\
&\qquad - (\nu f(x)\, x^\nu + x^{1+\nu} f'(x))\, x^{-1/2} u(x)\} \\
&\quad = 0.
\end{aligned}$$

Remark 2. Our proof gives an explicit lower bound for L_b. Thus in our case, we improve the result of Rellich [16, Satz 2a].

The next aim is to show that an appropriate mth power G_b^m is "locally trace class," i.e., φG_b^m is trace class for any φ in $C_0^\infty(\mathbb{R}^1)$ and moreover that the kernel of φG_b^m defines a function

$$\sigma(t, \zeta) := \varphi(t)\, \mathrm{tr}_H\, G_b^m(t, t, \zeta/t)$$

satisfying the conditions of the Singular Asymptotics Lemma. Lemma 4.9 gives

$$\sigma(t, \zeta) = t^{2m-1} \varphi(t)\, \mathrm{tr}_H\, G_{b,t}^m(1, 1, \zeta)$$

which leads us to take derivatives of $G_{b,t}$ with respect to the scaling parameter t.

LEMMA 4.3. *Set $\tilde{A}'_t := \tilde{A}'(tx)$. Then*

$$\partial_t G_{b,t} = -G_{b,t} X^{-1/2} \tilde{A}'_t X^{-1/2} G_{b,t}. \tag{4.17}$$

Proof. In the series (4.12) write

$$X^{-1}\Omega B_t G_0 \Omega^{-1} = [B_t(A_0+1)^{-1}][(A_0+1)\, X^{-1}\Omega G_0 \Omega^{-1}]$$

and note that

$$\begin{aligned}
\partial_t B_t(x)(A_0+1)^{-1} &= -\partial_t \int_0^t \tilde{A}'(sx)(A_0+1)^{-1}\, ds \\
&= -\tilde{A}'_t(x)(A_0+1)^{-1}.
\end{aligned}$$

Hence

$$\begin{aligned}
\partial_t X^{-1}\Omega B_t G_0 \Omega^{-1} &= -[\tilde{A}'_t(A_0+1)^{-1}][(A_0+1)\, X^{-1}\Omega G_0 \Omega^{-1}] \\
&= -X^{-1}\Omega \tilde{A}'_t G_0 \Omega^{-1}.
\end{aligned}$$

This derivative is bounded, by (4.1) and (3.13) with $k = -\frac{1}{2}$, since for each $t > 0$, $\tilde{A}'_t(x) = \tilde{A}'(tx)$ vanishes for large x. (The lack of x-decay that is uniform as $t \to 0$ causes problems elsewhere, which are dealt with below). To simplify the writing of the differentiated series (4.12), let

$$\alpha = G_0 \Omega^{-1} \qquad \text{and} \qquad \beta = X^{-1}\Omega.$$

Then the derivative of the jth term in (4.12) is

$$\begin{aligned}
&-\alpha \sum_{k=0}^{j-1} (\beta B_t \alpha)^k\, (\beta \tilde{A}'_t \alpha)(\beta B_t \alpha)^{j-1-k}\, \beta B_t G_0 \\
&\qquad - \alpha(\beta B_t \alpha)^{j-1}\, \beta \tilde{A}'_t G_0.
\end{aligned}$$

Summed over j, this coincides with the term-by-term product of the right-hand side of (4.17), and the lemma is proved. ∎

Next we establish properties of $G_{b,t}$ that will allow us to verify the necessary integrability conditions (1.5b) on the derivatives of

$$\sigma(t, z) = t^{2m-1} \varphi(t)\, \mathrm{tr}_H\, G_{b,t}^m(1, 1, z). \tag{4.18}$$

From Lemma 4.3, $\partial_t^k G_{b,t}^m$ is a polynomial in

$$G_{b,t},\ X^{-1}\tilde{A}'_t, \ldots, X^{k-2}\tilde{A}_t^{(k)},$$

where the factors of $G_{b,t}$ outnumber the others by m, and each factor $\tilde{A}_t^{(j)}$ is flanked by factors of $G_{b,t}$. Thus we will have to juggle high powers of X in the middle of our terms. This is the purpose of the following lemmas.

LEMMA 4.4. *Suppose that* φ *and* ψ *are* C^∞, *and for fixed* $k \in \mathbb{Z}_+$

$$\psi^{(j)}(x) = O(x^{k-j}), \qquad j = 0, 1, 2, \ldots,$$
$$\varphi(x) = O(x^{-k})$$

as $x \to \infty$. *Then for* $G := G_{b,t}$, $\psi G \varphi$ *is bounded and*

$$\psi G \varphi = G \psi \varphi - G(\psi'' + 2\psi' \partial) G \varphi. \tag{4.19}$$

Similarly $\Omega \psi \partial G \varphi$ *is bounded.*

Proof. Suppose, first, that $k = 0$, and ψ vanishes near 0. Then for any f in L^2, setting $\tau = D^2 + X^{-2}\tilde{A}_t$, we have

$$\psi(\tau + z^2) Gf - (\tau + z^2) \psi Gf = (\psi'' + 2\psi' \partial) Gf.$$

Since ψ vanishes near 0, $\psi' \partial G$ is bounded [see (3.14) and (4.12)], so we can multiply on the left by G, getting

$$G\psi f - \psi G f = G(\psi'' + 2\psi' \partial) Gf. \tag{4.20}$$

Suppose next that $k = 0$ and $\psi(0) = 0$. Define

$$\psi_j(x) = \eta(jx)\, \psi(x),$$

where $1 - \eta \in C_0^\infty(\mathbb{R}^1)$, $\eta \equiv 0$ near 0. Thus $\psi_j \to \psi$ pointwise, and (4.20) holds for ψ_j, so it will follow for ψ if we show the weak convergence

$$(j^2 \eta''(j\cdot)\, \psi Gf, G^*g) \to 0, \qquad (j\eta'(j\cdot)\, \psi' Gf, G^*g) \to 0,$$
$$(j\eta'(j\cdot)\, \psi \partial Gf, G^*g) \to 0.$$

The first integrand $\to 0$ pointwise, vanishes for $x \geqslant x_0$, and is dominated for $0 \leqslant x \leqslant x_0$ by

$$|x^2 j^2 \eta''(jx)|\, |x^{-1}\psi(x)|\, \|x^{-1/2} Gf(x)\|_H\, \|x^{-1/2} G^*g(x)\|_H$$
$$\leqslant C\, \|x^{-1/2} Gf(x)\|\, \|x^{-1/2} G^*g(x)\| \leqslant C$$

by (4.15d) for $u = Gf$ and $v = G(\bar{z})\, g$. For the third integrand we have similarly for $0 \leqslant x \leqslant x_0$, recalling $\omega(x) = x^{1/2}(1+x)^{-1/2}$,

$$|xj\eta'(jx)|\, |x^{-1}\psi(x)|\, \|\omega(x)\, \partial Gf(x)\|_H\, \|\omega^{-1}(x)\, G^*g(x)\|_H$$
$$\leqslant C\, \|x^{1/2} \partial Gf(x)\|_H\, \|\omega^{-1}(x)\, G^*g(x)\|_H \leqslant C.$$

The remaining integrand is likewise dominated, establishing the weak convergence and hence (4.20) in this case. We can now drop the condition $\psi(0) = 0$, since (4.20) holds both for $\psi_0 \equiv \psi(0)$ and for $\psi - \psi_0$. Thus (4.19) is proved for $k = 0$. So is the boundedness of

$$\Omega \partial \psi G \varphi = [\Omega \partial G][\psi \varphi] - [\Omega \partial G X^{-1/2}][X^{1/2}(\psi'' + 2\psi \partial)\, G\varphi]$$

which gives a bound for $\Omega \psi \partial G \varphi$.

Suppose now that the lemma holds for all $k \leqslant l$, and let ψ, φ satisfy the hypotheses with $k = l + 1$. Let

$$\psi_j(x) = \psi(x)(1 + x/j)^{-1} = \psi(x)\, j/(x + j).$$

By the induction hypothesis

$$\|(1+X)^l G (1+X)^{-l}\| + \|X^{1/2}(1+X)^{l-1/2}\, \partial G (1+X)^{-l}\| \leqslant C \tag{4.21}$$

and

$$\psi_j G \varphi = G \psi_j \varphi - G(\psi_j'' + 2\psi_j' \partial)\, G\varphi. \tag{4.22}$$

Since $\varphi(x) = O((1+x)^{-1-l})$, the functions

$$\psi_j \varphi, \qquad \psi_j'(x)(1+x)^{-l}, \qquad \psi_j''(x)(1+x)^{-l}$$

are uniformly bounded, and the weak convergence of (4.22) follows from (4.21). A bound on $\Omega \psi \partial G \varphi$ follows as in the case $k = 0$, and the lemma is proved. ∎

LEMMA 4.5. *For real* $k \geqslant 0$, *and uniformly in* $0 \leqslant t \leqslant 1$,

$$\|(1+X)^{-k}\, X^{1/2} G_{b,t} X^{-1/2} (1+X)^k\| \leqslant C\, |z|^{-2}, \tag{4.23}$$

$$\|(1+X)^{-k}\, X^{1/2} \partial G_{b,t} X^{-1/2} (1+X)^k\| \leqslant C\, |z|^{-1}. \tag{4.24}$$

Proof. Write the series (4.13) in the abbreviated form

$$G = G_{b,t} = G_0 \sum_{j=0}^{\infty} (X^{-1} B_t G_0)^j. \tag{4.25}$$

Take first $k = 0$. Then (4.24) is

$$X^{1/2} \partial G X^{-1/2} = X^{1/2} \partial G_0 X^{-1/2} \sum_0^\infty (X^{-1/2} B G_0 X^{-1/2})^j$$

and (4.24) follows from (3.16) and (4.6b). The proof of (4.23) for $k = 0$ is similar. Assuming (4.23) and (4.24) for a given k, we obtain them for $k+1$ by the commutator $[G, 1+X]=2G\partial G$, which follows from (4.19). Thus

$$\begin{aligned}(1+X)^{-k-1}\, & X^{1/2}\partial GX^{-1/2}(1+X)^{k+1}\\ &=(1+X)^{-k}\, X^{1/2}\partial GX^{-1/2}\,(1+X)^{k}\\ &\quad+(1+X)^{-k-1}\, X^{1/2}GX^{-1/2}(1+X)^{k}\\ &\quad+2[(1+X)^{-k-1}\, X^{1/2}\partial GX^{-1/2}(1+X)^{k}]\\ &\quad\times[X^{1/2}(1+X)^{-k}\,\partial GX^{-1/2}(1+X)^{k}]\\ &=O(|z|^{-1})\end{aligned}$$

by the induction hypothesis. The proof of (4.23) is similar. The result for all real $k\geqslant 0$ follows by interpolation, using the analytic family $(I+X)^{-k-w}\, X^{1/2}\partial GX^{-1/2}(1+X)^{k+w}$, $0\leqslant \mathrm{Re}(w)\leqslant 1$. ∎

LEMMA 4.6. *For real $k\geqslant 0$ and uniformly in $0\leqslant t\leqslant 1$,*

$$\|(1+X)^{-k-2}\, X^{-1/2}G_{b,t}X^{-1/2}(1+X)^{k}\|_{p+1/2}\leqslant C, \tag{4.26}$$

$$\|(1+X)^{-5/2}\, X^{-1/2}G_{b,t}\|_{p+1/2}\leqslant C. \tag{4.27}$$

Proof. For $k=0$, the series (4.25) gives

$$\begin{aligned}(1+X)^{-2}\, & X^{-1/2}G_{b,t}X^{-1/2}\\ &=[(1+X)^{-2}\, X^{-1/2}G_0X^{-1/2}]\sum_0^{\infty}(X^{-1/2}B_tG_0X^{-1/2})^j.\end{aligned}$$

The first term on the right is in $C_{p+1/2}$ by Lemma 3.7, and the series converges uniformly in t, by Lemma 4.1. For $k=1, 2,\ldots$, we use induction and interpolation as in the previous lemma, proving (4.26). Then (4.27) follows, taking $k=\frac{1}{2}$ in (4.26). ∎

LEMMA 4.7. *For $j=1, 2,\ldots$, and uniformly in $0\leqslant t\leqslant 1$,*

$$\|(1+X)^{-k-j}\, X^{j-3/2}\tilde{A}_t^{(j)}G_{b,t}X^{-1/2}(1+X)^{k}\|\leqslant C, \tag{4.28}$$

$$\|(1+X)^{-j-1}\, X^{j-3/2}\,\tilde{A}_t^{(j)}G_{b,t}\|\leqslant C.$$

Proof. The outline is by now familiar, and reduces essentially to the boundedness of

$$\begin{aligned}\|(1+X)^{-j}\, & X^{j-3/2}\tilde{A}_t^{(j)}G_0X^{-1/2}\|\\ &\leqslant\|\tilde{A}_t^{(j)}(A_0+1)^{-1}\|\cdot\|(1+X)^{-1}\, X^{-1/2}(A_0+1)\, G_0X^{-1/2}\|\leqslant C\end{aligned}$$

by (4.1) and (3.13a) with $k=0$. ∎

LEMMA 4.8.

$$\|X^2(1+X)^{-k-3/2}\,\partial^2G_{b,t}X^{-1/2}(1+X)^{k}\|\leqslant C, \tag{4.30}$$

$$\|X(1+X)^{-k-2}\,\partial G_{b,t}X^{-1/2}(1+X)^{k}\|_{2p+1}\leqslant C. \tag{4.31}$$

Proof. For (4.30) with $k=0$, the essential point is

$$\|X^2(1+X)^{-3/2}\,\partial^2G_0X^{-1/2}\|\leqslant C$$

which follows from (3.13c) with $k=0$. The proof proceeds by induction and (4.24). For (4.31), let T be the operator in question; we show that T^* is in $C_{p+1/2}$. Let $\varphi=X(1+X)^{-k-2}$ and $G=G_{b,t}$ and find

$$\begin{aligned}T^*T&=(1+X)^{k}\, X^{-1/2}G\partial\varphi^2\partial GX^{-1/2}(1+X)^{k}\\ &=[(1+X)^{k}\, X^{-1/2}G\varphi' X^{-1/2}][X^{1/2}\varphi\partial GX^{-1/2}(1+X)^{k}]\\ &\quad+[(1+X)^{k}\, X^{-1/2}G\varphi X^{-3/2}][\varphi X^{3/2}\partial^2GX^{-1/2}(1+X)^{k}].\end{aligned}$$

In each term on the right-hand side, the first factor is in $C_{p+1/2}$ by (4.26), since at ∞ the sum of the powers is -3, keeping in mind the definition of φ; and the second factor is bounded, by (4.30) and Lemma 4.5, since at ∞ the sum of the powers is $\leqslant 0$. ∎

LEMMA 4.9. *If $m\geqslant p+1$ then the operator $G_{b,t}^m$ has a kernel $G_{b,t}^m(x, y; z)$ which, for each z, is a continuous map of $\mathbb{R}^*\times\mathbb{R}^*$ into the trace class operators $C_1(H)$, and such that*

$$G_b^m(xt, yt; \zeta/t)=t^{2m-1}G_{b,t}^m(x, y; \zeta), \tag{4.32}$$

where $G_b=G_{b,1}$.

Proof. For a sufficiently large k, the operator

$$\begin{aligned}T&=X(1+X)^{-k}\, G_{b,t}^mX(1+X)^{-k}\\ &=[X(1+X)^{-k}\, G_{b,t}(1+X)^{k-4}][(1+X)^{4-k}\, G_{b,t}(1+X)^{k-7}]\cdots\end{aligned}$$

is in the Schatten class $C_{(p+1/2)/m}$, by Lemma 4.6. Combining that lemma with Lemma 4.8 shows further that ∂T and $T\partial$ are in C_q, where

$$q^{-1}=(2p+1)^{-1}+(m-1)(p+1/2)^{-1}\geqslant 1,$$

given that $m\geqslant p+1$. Hence by the Trace Lemma (Appendix), T has a kernel mapping $\mathbb{R}^*\times\mathbb{R}^*$ continuously into $C_1(H)$, and then so does

$$G_{b,t}^m=X^{-1}(1+X)^{k}\, TX^{-1}(1+X)^{k}.$$

To prove the scaling (4.32), recall from Lemma 4.1 that the unitary operator $U_t f(x) = t^{1/2} f(tx)$ satisfies

$$U_t G_0(z) = t^2 G_0(tz)\, U_t, \qquad U_t B = t^{-1} B_t U_t,$$
$$U_t X^{-1} = t^{-1} X^{-1} U_t.$$

This with (4.25) shows that

$$U_t G_b(z) = t^2 G_{b,t}(tz)\, U_t, \qquad U_t G_b^m(z) = t^{2m} G_{b,t}(tz)\, U_t$$

and hence the kernel $G_b^m(x, y; z)$ has the scaling stated in the Lemma. ∎

THEOREM 4.2. *If $m \geqslant p+1$, the function*

$$\sigma(x, \zeta) = \operatorname{tr}_H G_b^m(x, x; \zeta/x)$$

satisfies the integrability condition (1.5b). *In fact, with $\sigma^{(j)}(x, \zeta) = \partial_x^j \sigma(x, \zeta)$,*

$$\int_0^1 s^j |\sigma^{(j)}(st, s\zeta)|\, ds \leqslant C_j, \qquad 0 \leqslant t \leqslant 1,\ |\zeta| = z_0, \tag{4.33}$$

with the z_0 in Lemma 4.1. *For every φ in $C_0^\infty(\mathbb{R})$,*

$$\operatorname{tr} \varphi G_b^m = \int_0^\infty \varphi(x)\, \sigma(x, xz)\, dx. \tag{4.34}$$

Proof. By (4.32)

$$\sigma(st, s\zeta) = t^{2m-1} \operatorname{tr}_H G_{b,t}^m(s, s; \zeta) \tag{4.35}$$

so

$$s^j \sigma^{(j)}(st, s\zeta) = \partial_t^j [t^{2m-1} \operatorname{tr}_H G_{b,t}^m(s, s; \zeta)].$$

Choose $\varphi \geqslant 0$ in $C_0^\infty(\mathbb{R}^1)$ with $\varphi \equiv 1$ in $[0, 1]$. We have from (4.35) and the Trace Lemma

$$\int_0^1 s^j\, |\sigma^{(j)}(st, s\zeta)|\, ds = \int_0^1 |\partial_t^j [t^{2m-1} \operatorname{tr}_H G_{b,t}^m(s, s; \zeta)]|\, ds$$
$$\leqslant \| \partial_t^j [t^{2m-1} \varphi G_{b,t}^m \|_{\mathrm{tr}} \leqslant C_0. \tag{4.36}$$

For $j = 1$, (4.36) requires us to estimate the trace norm of

$$\varphi \partial_t G_{b,t}^m = -\varphi \sum_{j=1}^m G_{b,t}^j X^{-1/2} \tilde{A}_t' X^{-1/2} G_{b,t}^{m+1-j} \tag{4.37}$$

by Lemma 4.3. The first summand in (4.37) is

$$-[\varphi G X^{-1/2} (1+X)^{3m-1}]\, [(1+X)^{1-3m} X^{-1/2} \tilde{A}_t' G X^{-1/2} (1+X)^{3m-2}]$$
$$\times \cdots \times [(1+X)^{-7} X^{1/2} G X^{-1/2} (1+X)^4]\, [X^{1/2} (1+X)^{-4} G].$$

From Lemma 4.6, m of these factors are in $C_{p+1/2}$, and from Lemma 4.7 the remaining term is bounded, so the product is trace class. The other terms in (4.37) are similarly bounded, and we thus obtain from the Trace Lemma

$$\int_0^1 |\operatorname{tr} \varphi \partial_t G_{b,t}^m(s, s; \zeta)|\, ds \leqslant \| \varphi \partial_t G_{b,t}^m \|_{\mathrm{tr}} \leqslant C_1.$$

This with (4.36) proves (4.33) for $j = 1$. The proof for general j follows the same pattern, using Lemmas 4.6 and 4.7. Finally, again from the Trace Lemma, $\operatorname{tr}(\varphi G_b^m) = \int_0^\infty \varphi(x)\, \sigma(x, xz)\, dx$. ∎

This proof explains the form of the lemmas in Sections 3 and 4. The first point is that when $t \to 0$, $\tilde{A}_t'(x) = \tilde{A}'(tx)$ loses its decay as $x \to \infty$; we compensate for this by introducing a factor $(1+X)^{-2}$ and cancelling it with a $(1+X)^2$ on the next term to the left. Second, G itself is not $C_{p+1/2}$, but $(1+X)^{-3} G$ is; hence we introduce still more negative powers of $(1+X)$. The terms in Lemmas 4.6 and 4.7 are designed to fit together so that all the positive powers are shifted to the left hand factor φG, where they are harmless since φ has compact support.

THEOREM 4.3. *If $m \geqslant p+1$, the function*

$$\sigma(x, \zeta) = \operatorname{tr}_H G_b^m(x, x; \zeta/x)$$

has an expansion in the sense of (1.5a),

$$\sigma(x, \zeta) \sim \sum \sigma_{\alpha j}(x)\, \zeta^\alpha \log^j \zeta.$$

Proof. As in Theorem 4.2,

$$\sigma(t, \zeta) = \operatorname{tr}_H t^{2m-1} G_{b,t}^m(1, 1; \zeta).$$

Let $K_{i,t}^m$ be an interior parametrix as in Section 2, using $\mathscr{A}_t(x) = x^{-2} \tilde{A}(tx)$. Thus

$$K_{i,t}^m = \sum_{j=2m}^{2m+N} \operatorname{Op}(b_j),$$

$$b_j = Q_j(\xi, z, \mathscr{A}_t, \ldots, \mathscr{A}_t^{(j-2m)}, b_2), \qquad b_2 = (\xi^2 + \mathscr{A}_t + z^2)^{-1},$$

$$(D^2 + \mathscr{A}_t + z^2)^m K_{i,t}^m = I - R_{i,t}^m = I - \sum_{k=-N}^{4m-2-N} \operatorname{Op}(R_k(\xi, z, \mathscr{A}_t, \ldots, \mathscr{A}_t^{(N)}, b_2)),$$

where Q_j and R_j are polynomials in all entries, of degree $-\frac{1}{2}j$ in $(\xi^2, z^2, \mathscr{A}_t, ..., \mathscr{A}_t^{(N)})$. Now choose φ, ψ in $C_0^\infty(\mathbb{R}^*)$ with $\varphi \equiv 1$ near $x=1$, $\psi \equiv 1$ on a neighborhood of supp φ. Then

$$(D^2 + \mathscr{A}_t + z^2)^m \psi K_{i,t}^m$$
$$= \psi - \psi R_{i,t}^m - \sum_{j=0}^{m-1} (D^2 + \mathscr{A}_t + z^2)^{m-j-1} (\psi'' + 2\psi'\partial)(D^2 + \mathscr{A}_t + z^2)^j K_{i,t}^m. \tag{4.38}$$

Multiply on the left by $\varphi G_{b,t}^m$ and on the right by φ, and use $\varphi\psi \equiv \varphi$:

$$\varphi G_{b,t}^m \varphi = \varphi K_{i,t}^m \varphi + \left[\varphi G_{b,t}^m \psi R_{i,t}^m \varphi + \sum_{j=0}^{m-1} \varphi G_{b,t}^{j+1} (\psi'' + 2\psi'\partial) \times (D^2 + \mathscr{A}_t + z^2)^j K_{i,t}^m \varphi \right]. \tag{4.39}$$

Let R denote the term in braces. By Lemmas 2.5 and 2.6, R, ∂R, and $R\partial$ all have trace norm $O(|z|^{-M})$, where M can be as large as desired if N is large enough. Hence

$$\sigma(t, \zeta) = t^{2m-1} \operatorname{tr}_H(\varphi G_{b,t}^m \varphi)(1, 1; \zeta)$$
$$= t^{2m-1} \operatorname{tr}_H(\varphi K_{i,t}^m \varphi)(1, 1; \zeta) + O(|\zeta|^{-M}).$$

From the above description of $K_{i,t}^m$ we have

$$K_{i,t}^m(1, 1, \zeta) = \sum_j \frac{1}{2\pi} \int_{-\infty}^{\infty} Q_j(\xi, z, \mathscr{A}_t(1), ..., \mathscr{A}_t^{(j-2m)}(1), (\xi^2 + \mathscr{A}(t) + \zeta^2)^{-1})\, d\xi.$$

Each term has an expansion as required, obtained by integrating the assumed expansion (A6). To check the derivative $\sigma' = \partial_t \sigma$, we differentiate (4.39) with respect to t, using on the right hand side Lemma 4.3:

$$\partial_t G_{b,t} = -G_{b,t} X^{-1/2} \tilde{A}_t' X^{-1/2} G_{b,t}.$$

One sees further that $\partial_t R_{i,t}^m$ and $\partial_t K_{i,t}^m$ have the same structure as $R_{i,t}^m$ and $K_{i,t}^m$, with the same degrees of homogeneity, and it follows that σ' has the same expansion as $\operatorname{tr}_H(\varphi \partial_t K_{i,t}^m \varphi)$ up to $O(|z|^{-M})$. Higher derivatives $\sigma^{(j)}$ are similarly treated, and the proof is complete. ∎

Finally, to blend the boundary parametrix G_b^m with various interior parametrices, we need

LEMMA 4.10. *If $\varphi \in C_0^\infty(\mathbb{R}^1)$, $\eta \in B^\infty(\mathbb{R}^*)$, and φ, η have disjoint support, then for every N, k, j,*

$$\| A^k \partial^j \eta G_b^m \varphi \|_{\mathrm{tr}} = O(|z|^{-N}).$$

Proof. Take (4.38) with $t=1$. Choose $\psi \equiv 1$ on a neighborhood of supp η, $\psi \equiv 0$ on a neighborhood of supp φ. Multiply (4.38) on the left by φG_b^m and on the right by $\eta A^k \partial^j$:

$$\varphi G_b^m \eta A^k \partial^j = \varphi G_b^m \psi R_i^m \eta A^k \partial^j$$
$$+ \sum_{j=0}^{m-1} \varphi G_b^{j+1} (\psi'' + 2\psi'\partial)(D^2 + \mathscr{A}_1 + z^2)^j K_i^m \eta A^k \partial^j.$$

Since supp ψ'' and supp ψ' are disjoint from supp η, Lemma 4.10 now follows from Lemmas 2.5 and 2.6. ∎

5. The Global Parametrix

The global parametrix is a blend of "boundary" and "interior" parametrices. As in the Introduction, we use a Hilbert space structure

$$\mathscr{H} = \mathscr{H}_b \oplus \mathscr{H}_i$$

with the "boundary part"

$$\mathscr{H}_b \cong L^2((0, 2); H),$$

where H is a fixed Hilbert space "fiber." If $f \cong f_b \oplus f_i$ with $f_b \in L^2((0, 2); H)$, and likewise for g, then

$$(f, g) = \int_0^2 (f_b, g_b)_H + (f_i, g_i).$$

(Below, we will often drop the subscripts and write f for either f_b or f_i; the context makes it clear what is intended.) For φ in $C([0, 2])$ we define a "multiplication operator" Φ on $\mathscr{H}$ by

$$\Phi(f_b \oplus f_i) = \varphi(\cdot) f_b(\cdot) \oplus \varphi(2) f_i.$$

By $C^\sim([0, 2])$ we denote those functions in $C^\infty([0, 2])$ which are constant near $x=2$; we can think of them as extending smoothly to functions which are constant on some underlying manifold for $\mathscr{H}_i$.

The set of all measurable f: $(0, 2) \to H$ such that $\int_\delta^2 \|f(x)\|_H^2\, dx < \infty$ for all $\delta > 0$ forms a space $\mathscr{H}_{b*}$, and then we set

$$\mathscr{H}_* :\cong \mathscr{H}_{b*} \oplus \mathscr{H}_i.$$

Those f in $L^2((0, 2), H)$ which vanish in a neighborhood of $x = 0$ form a subspace $\mathscr{H}_{b0}$, and we define the "compact support" subspace of H by

$$\mathscr{H}_0 :\cong \mathscr{H}_{b0} \oplus \mathscr{H}_i.$$

We assume a C^∞ family of operators $A(x)$, $0 \leqslant x \leqslant 2$, with common domain $H_A \subset H$, and satisfying conditions (A1)–(A6) of the Introduction. We then assume an unbounded operator L with domain $\mathscr{D}(L) \subset H$ such that:

(L1) $\mathscr{D}(L)$ is closed under multiplication by functions φ in $C^\sim([0, 2])$. [This is adequate for the Dirichlet boundary condition, but others would require $\varphi'(0) = 0$.]

(L2) $u \in \mathscr{D}(L)$ if and only if:

(a) There is a $\psi \in C^\sim([0, 2])$ vanishing near 0, with $\psi(2) = 1$, such that $\psi u \in \mathscr{D}(L)$.

(b) For all $\varphi \in C_0^\infty(0, 2)$, $\varphi u \in H^2((0, 2), H) \cap L^2((0, 2), H_A)$.

(c) $\lim_{x \to 0+} x^{-1/2} u(x) = 2 \lim_{x \to 0+} x^{1/2} u'(x)$.

(d) $\int_0^2 \| -u''(x) + x^{-2} A(x)\, u(x) \|_H^2\, dx < \infty$.

(L3) For u in $\mathscr{D}(L)$ and for $0 < x < 2$

(a) $Lu(x) = -u''(x) + x^{-2} A(x)\, u(x)$.

(b) For φ, $\psi \in C^\sim([0, 2])$ with disjoint supports and for all $u \in \mathscr{D}(L)$ we have $\varphi L \psi u = 0$.

(L4) If $\psi \in C^\sim([0, 2])$ is real and $\psi \equiv 0$ near 0, then $\Psi L \Psi$ is symmetric on $\mathscr{D}(L)$.

(L5) For $|z| \geqslant z_0$ with $|\arg z^2| \leqslant \pi - \theta$; for $m = 1, 2, \ldots$; and for arbitrarily large N, there is an "interior parametrix" $G_i^m(z)$ such that for all φ, ψ in $C^\sim([0, 2])$ vanishing near 0,

(a) $\Psi G_i^m(z)\colon \mathscr{H} \to \mathscr{D}((L + z^2)^m)$.

(b) If $\psi \equiv 1$ near supp φ, then

$$(L + z^2)^m\, \Psi G_i^m(z)\, \Phi = \Phi - R_i^m(\psi, \varphi)$$

with

$$\|R_i^m(\psi, \varphi)\|_{\mathrm{tr}} = O(|z|^{-N}).$$

(c) If φ, have disjoint supports, and φ vanishes near 2, then

$$\|A^j \partial^k \Phi G_i^m \Psi\|_{\mathrm{tr}} = O(|z|^{-N}), \qquad j + k/2 \leqslant m - 1.$$

(d) If $m \geqslant p + 1$, there are smooth functions $\sigma_{\alpha j}(x)$ and constants $c_{\alpha j}$ such that for φ, ψ as in (b)

$$\operatorname{tr}(\Psi G_i^m \Phi) = \sum_{\substack{\operatorname{Re}(\alpha) \geqslant -N \\ 0 \leqslant j \leqslant J_\alpha}} \left[\int_0^2 \varphi(x)\, \sigma_{\alpha j}(x) (xz)^\alpha (\log xz)^j\, dx + \varphi(2)\, c_{\alpha j} z^\alpha (\log z)^j \right] + O(|z|^{-N}).$$

(The constants $c_{\alpha j}$ account for contributions from $\mathscr{H}_i$.)

LEMMA 5.1. *Assuming* (L1)–(L4), *then L is symmetric.*

Proof. Choose $0 \leqslant \varphi_b$, φ_i in $C^\sim([0, 2])$ with $\varphi_b + \varphi_i \equiv 1$, $\varphi_b \equiv 0$ near 2, and $\varphi_i \equiv 0$ near 0. Choose real functions $\varphi \equiv 1$ near supp φ_b, $\varphi \equiv 0$ near 2, and $\psi \equiv 1$ near supp φ_i, $\psi \equiv 0$ near 0. Then for u, v in $\mathscr{D}(L)$

$$\begin{aligned}
(Lu, v) &= (L\Phi_b u, v) + (L\Phi_i u, v) \\
&= (L\Phi_b u, \Phi v) + (\Psi L \Psi \Phi_i u, v) \qquad \text{[by (L3)]} \\
&= (\Phi_b u, L\Phi v) + (\Phi_i u, \Psi L \Psi v) \\
&\qquad \text{[by (L4) and Lemma 4.2]} \\
&= (\Phi_b u, Lv) + (\Phi_i u, Lv) \\
&= (u, Lv). \quad \blacksquare
\end{aligned}$$

Remark. Lemma 4.2 does not literally apply, but the proof here is exactly the same.

THEOREM 5.1. *Assuming* (L1)–(L5)(c), *L is self-adjoint and semi-bounded, and has the resolvent in* (5.1) *below.*

Proof. Choose the φ_b in the proof of Lemma 5.1 so that in the construction of G_b in Section 4, $\tilde{A}(x) = A(x)$ for x in supp φ_b. Let $G_i = G_i^1$ as in (L5). By (4.15), (4.16), (L5)(a), and (L2), $\Phi G_b \Phi_b + \Psi G_i \Phi_i$ maps into $\mathscr{D}(L)$, and

$$(L + z^2)(\Phi G_b \Phi_b + \Psi G_i \Phi_i) = I - R,$$

where

$$R = (\Phi'' + 2\Phi' \partial)\, G_b \Phi_b + R_i(\psi, \varphi_i)$$

with R_i as in (L5)(b). By (L5)(b),)(c), $\|R_i\| \to 0$ as $z \to \infty$, so for large z we have an operator

$$G = (\Phi G_b \Phi_b + \Psi G_i \Phi_i) \sum_0^\infty R^j \tag{5.1}$$

with

$$(L + z^2)\, G = I.$$

The proof now follows the proof of Theorem 4.1. ∎

Remark. Let τ be the restriction of L obtained by replacing the boundary condition (L2)(c) by

$$u(x) \equiv 0 \qquad \text{for } x \text{ near } 0.$$

Theorem 5.1 shows that τ is symmetric and semibounded. In Section 6 we show that L is the Friedrichs extension of τ.

THEOREM 5.2. *Assuming* (L1)–(L5), *we have for* $\gamma \equiv 0$ *near* 0, *using the notation of* (L5)(d)

$$\operatorname{tr} \gamma(L + z^2)^{-m} = \sum_{\substack{\operatorname{Re}(\alpha) \geq -N \\ 0 \leq j \leq J_\alpha}} \left[\int_0^2 \gamma(x)\, \sigma_{\alpha j}(x) (xz)^\alpha (\log xz)^j \, dx + \gamma(2)\, c_{j\alpha} z^\alpha (\log z)^j \right] + O(|z|^{-N}). \tag{5.2}$$

On the other hand, if β *is supported so near zero that* $L = L_b$ *near* supp β, *then*

$$\operatorname{tr} \beta G^m = \operatorname{tr} \beta G_b^m + O(|z|^{-N}) \tag{5.3}$$

while

$$\operatorname{tr} \beta G_b^m = \int_0^2 \beta(x)\, \sigma(x, xz)\, dx, \tag{5.4}$$

where σ *satisfies the integrability conditions* (1.5b), *and with the notation of* (L5)(d)

$$\sigma(x, \zeta) \sim \sum \sigma_{\alpha j}(x)\, \zeta^\alpha \log^j \zeta \tag{5.5}$$

in the sense of (1.5a). *Throughout, N may be arbitrarily large.*

Remark. The point of (5.5) is that the expansion of the "boundary parametrix" trace in (5.4) can be computed from the interior parametrix assumed in (L5); this is, in practice, generally simpler to analyze than the parametrix constructed in Section 2.

Proof. Proceeding as in Theorem 5.1, we have

$$(L + z^2)^m (\Phi G_b^m \Phi_b + \Psi G_i^m \Phi_i) = I - \sum_{j=0}^{m-1} (L + z^2)^{m-j-1} (\Phi'' + 2\Phi' \partial)(L + z^2)^j G_b^m \Phi_b - R_i^m(\psi, \varphi_i).$$

Multiply on the left by G^m (guaranteed to exist, by Theorem 5.1) and rearrange terms:

$$G^m = \Phi G_b^m \Phi_b + \Psi G_i^m \Phi_i + R, \tag{5.6a}$$

where

$$\|R\|_{\mathrm{tr}} = \left\| \sum_0^{m-1} G^{j+1} (\Phi'' + 2\Phi' \partial)\, G_b^{m-j} \Phi_b + G^m R_i^m \right\|_{\mathrm{tr}} = O(|z|^{-N}) \tag{5.6b}$$

by (L5)(b) and Lemma 4.10. Given $\gamma \equiv 0$ near 0, we can choose φ and φ_b with support where $\gamma = 0$; then $\psi = \varphi_i = 1$ on supp γ and we get

$$\gamma G^m = \gamma G_i^m \varphi_i + \gamma R$$

hence

$$\operatorname{tr}(\gamma G_m) = \operatorname{tr}(\gamma G_i^m \varphi_i) + O(|z|^{-N})$$

which with (L5)(d) proves (5.2). Suppose next that $L = L_b$ near supp β. Choose $\varphi_b \equiv 1$ and $\varphi \equiv 1$ on supp β; then $\varphi_i \equiv 0$ on supp β. Since $\operatorname{tr}(ST) = \operatorname{tr}(TS)$, we get (5.3) from (5.6).

The representation (5.4) with the integrability condition (1.5b) is proved in Theorem 4.1, and Theorem 4.2 guarantees an expansion

$$\sigma(x, \zeta) \sim \sum s_{\alpha j}(x)\, \zeta^\alpha \log^j \zeta. \tag{5.7}$$

It remains only to identify the coefficients $s_{\alpha j}$. Suppose that η vanishes near 0, and $L = L_b$ near supp η. Then by (5.3) and (5.4)

$$\operatorname{tr} \eta G^m = \int_0^2 \eta(x)\, \sigma(x, xz)\, dx + O(|z|^{-N}).$$

By (5.7) and the Singular Asymptotics Lemma, noting that η vanishes near 0,

$$\operatorname{tr} \eta G^m \sim \sum \int_0^2 \eta(x)\, s_{\alpha j}(x)(xz)^\alpha (\log xz)^j\, dx + O(|z|^{-N}).$$

Compare the coefficient of $z^\alpha \log^k z$ here with the one in (5.2) with $\gamma = \eta$, and note that $\eta(2) = 0$, to get

$$\int \eta(x) \sum_{j=k}^{J_\alpha} s_{\alpha j}(x)\, x^\alpha C_k^j (\log x)^{j-k}\, dx = \int \eta(x) \sum_{j=k}^{J_\alpha} \sigma_{\alpha j}(x)\, x^\alpha C_k^j (\log x)^{j-k}\, dx$$

for $\operatorname{Re}(\alpha) > -N$. Since this holds for all η with appropriate support, we get the equality of finite sums

$$\sum_{j=k}^{J_\alpha} s_{\alpha j}(x)\, C_k^j (\log x)^{j-k} = \sum_{j=k}^{J_\alpha} \sigma_{\alpha j}(x)\, C_k^j (\log x)^{j-k}$$

for all $k \leqslant J_\alpha$. For each $x > 0$, this nonsingular triangular system has a unique solution, and the proof is complete.

6. The Friedrichs Extension

Let τ be the operator L in Section 5, but with the boundary condition (L2)(c) replaced by

(L2)(c′) $u \equiv 0$ in some neighborhood of $x = 0$.

Since τ is a restriction of L, Theorem 5.1 shows that it is symmetric and semibounded. We now show:

Theorem 6.1. *L is the Friedrichs extension of τ, and $\mathscr{D}(L)$ equals*

$$\{u \in \mathscr{H} \mid u \text{ satisfies (L2a, b, d), and } \|u(x)\|_H = O(x^{1/2}) \text{ as } x \to 0\}. \quad (6.1)$$

Proof. Denote by $\tilde{L}$ the Friedrichs extension of τ. It is sufficient to show that $\mathscr{D}(\tilde{L})$ contains (6.1), since this contains $\mathscr{D}(L)$ by construction. Recall that

$$\mathscr{D}(\tilde{L}) = \mathscr{D}(\tau^*) \cap \mathscr{D}'(\tau),$$

where $\mathscr{D}'(\tau)$ denotes the closure of $\mathscr{D}(\tau)$ under the norm

$$((\tau + c)\, u \mid u)^{1/2}, \qquad \tau + c \geqslant 1, \qquad (6.2)$$

considered as a subspace of $\mathscr{H}$. Choose u in the set (6.1). By (L2)(a) there is $\psi \in C_0^\infty[0, 2]$ with $\psi = 0$ near 0 such that $\psi u \in \mathscr{D}(L)$, hence $\psi u \in \mathscr{D}(\tau) \subset \mathscr{D}(\tilde{L})$. That $v := (1 - \psi)\, u \in \mathscr{D}(\tau^*)$ is a consequence of (L2)(b), (d). It therefore remains to prove the following: if $v \in \mathscr{D}(\tau^*)$, $\operatorname{supp} v \subset [0, 2)$, and

$$\|v(x)\|_H = O(x^{1/2}), \qquad x \to 0,$$

then $v \in \mathscr{D}'(\tau)$. Choose functions $\varphi, \chi \in C^\infty(\mathbb{R})$ such that

$$0 \leqslant \varphi \leqslant 1, \qquad \varphi(x) = 1 \text{ if } |x| \leqslant 1, \qquad \varphi(x) = 0 \text{ if } |x| \geqslant 2, \quad (6.4a)$$

$$\begin{aligned} 0 < \chi(x) &\leqslant x && \text{if } x > 0, \\ \chi(x) &= x && \text{if } 0 \leqslant x \leqslant 1, \\ \chi(x) &= 1 && \text{if } x \geqslant 2, \end{aligned} \qquad (6.4b)$$

and put

$$\alpha_n := [\log n]^{-1/2}, \qquad n \geqslant 2. \qquad (6.4c)$$

Note that $\alpha_n \to 0$, $n \to \infty$. Put

$$\psi_n(x) := \chi(x)^{\alpha_n} (1 - \varphi(nx)), \qquad (6.5a)$$

$$\psi_{nm}(x) := \psi_n(x) - \psi_m(x), \qquad x \in \mathbb{R},\ n, m \geqslant 2. \qquad (6.5b)$$

Then $\psi_n \in C^\infty(\mathbb{R})$; ψ_n is uniformly bounded; $\psi_n v \in \mathscr{D}(\tau)$; and $\psi_n(x) \to 1$, $n \to \infty$, $\psi_{nm}(x) \to 0$, $n, m \to \infty$, for all $x > 0$. Since $\psi_n v \to v$ in $L^2([0, 2]), H)$ we only have to show that $\{\psi_n v\}$ is a Cauchy sequence with respect to the norm (6.2), i.e.,

$$\lim_{n,m \to \infty} ((\tau + c)\, \psi_{nm} v \mid \psi_{nm} v) = 0.$$

Using (L2)(d) one easily computes

$$(\tau \psi_{nm} v \mid \psi_{nm} v) = (\psi_{nm}'^2 v \mid v) + 2 \operatorname{Re}(\psi_{nm} \tau^* v \mid \psi_{nm} v).$$

From (6.5a) we obtain

$$\begin{aligned} \psi_n'(x)^2 \leqslant C[&\alpha_n^2 \chi(x)^{2\alpha_n - 2} \chi'(x)^2 (1 - \varphi(nx))^2 \\ &+ \chi(x)^{2\alpha_n} n^2 \varphi'(nx)^2] \end{aligned} \qquad (6.6)$$

so by (6.4c) $\psi_n'(x)^2 \to 0$ uniformly in $x \geqslant 1$, and

$$\int_1^2 \psi_{nm}'(x)^2 \|v(x)\|_H^2\, dx \to 0, \qquad n, m \to \infty.$$

For the remaining integral over [0, 1] we use (6.3) and estimate the two terms arising from (6.6). The first can be estimated by

$$C\alpha_n^2 \int_0^1 x^{2\alpha_n - 1}\, dx = C\frac{\alpha_n}{2} \to 0, \qquad n \to \infty,$$

the second by

$$Cn^2 \int_0^{2/n} x^{2\alpha_n + 1}\, dx \leqslant Cn^{-2\alpha_n} = Ce^{-2(\log n)^{1/2}} \to 0, \qquad n \to \infty.$$

The proof is complete. ∎

7. The Heat Asymptotics

The computation of the asymptotics of $\operatorname{tr} e^{-tL}$ is based on the Cauchy integral. Denote by Γ the contour

$$\{|\arg(\lambda + c)| = \pi/4\}$$

traversed upward, with c chosen so that $L \geqslant 1 - c$. Then

$$e^{-tL} = t^{1-m} \frac{(m-1)!}{2\pi i} \int_\Gamma e^{-t\lambda} G^m((-\lambda)^{1/2})\, d\lambda. \tag{7.1}$$

Choose a cut-off function γ such that $L = L_b$ on the support of γ and $\gamma(x) \equiv 1$ for small x. By (5.3)

$$\begin{aligned}\operatorname{tr} \gamma e^{-tL} &= t^{1-m} \frac{(m-1)!}{2\pi i} \int_\Gamma e^{-t\lambda} \operatorname{tr} \gamma G_b^m((-\lambda)^{1/2})\, d\lambda \\ &= t^{1-m} \frac{(m-1)!}{2\pi i} \int_\Gamma e^{-t\lambda} \int_0^\infty \gamma(x) \operatorname{tr}_H G_b^m(x, x; (-\lambda)^{1/2})\, dx\, d\lambda.\end{aligned}$$

Writing as before

$$\begin{aligned}\sigma(x, xz) &= \gamma(x) \operatorname{tr}_H G_b^m(x, x; z) \\ &= \gamma(x)\, x^{2m-1} \operatorname{tr}_H G_{b,x}^m(1, 1; xz)\end{aligned} \tag{7.2}$$

we obtain from the Singular Asymptotics Lemma

Theorem 7.1. *If γ is supported sufficiently near to 0, then as $t \to 0+$,*

$$\operatorname{tr} \gamma e^{-tL} \sim \sum_{k \geqslant 0} t^{(k+1)/2 - m} \int_0^\infty \frac{\zeta^k}{k!} \sigma^{(k)}(0, \zeta)\, d\zeta \times \frac{(m-1)!}{2\pi i} \int_\Gamma e^{-\lambda}(-\lambda)^{-(k+1)/2}\, d\lambda \tag{7.3a}$$

$$+ \sum_{\alpha, j} t^{-m-\alpha/2} \frac{(m-1)!}{2\pi i} \int_\Gamma \int_0^\infty \sigma_{\alpha j}(x)(x\sqrt{-\lambda})^\alpha \times \log^j(x\sqrt{-\lambda/t})\, e^{-\lambda}\, dx\, d\lambda \tag{7.3b}$$

$$+ \sum_{\alpha = -1}^{-\infty} \sum_{j=0}^{J_\alpha} t^{-m-\alpha/2} \sigma_{\alpha j}^{(-\alpha-1)}(0) \frac{(m-1)!}{(j+1)(-\alpha-1)!\, 2\pi i} \times \int_\Gamma e^{-\lambda} \sqrt{-\lambda}^{\,\alpha} \log^{j+1} \sqrt{-\lambda/t}\, d\lambda, \tag{7.3c}$$

where

$$\sigma(x, \zeta) = \gamma(x)\, x^{2m-1} \operatorname{tr}_H G_{b,x}^m(1, 1; \zeta) \sim \sum \sigma_{\alpha j}(x)\, \zeta^\alpha \log^j \zeta. \tag{7.4}$$

We single out for further study the leading term, and the term in t^0. Note first that

$$\sigma(x, \zeta) = \gamma(x)\, x^{2m-1} \tilde{\sigma}(x, \zeta), \tag{7.5}$$

where

$$\tilde{\sigma}(x, \zeta) = \operatorname{tr}_H G_{b,x}^m(1, 1; \zeta) \tag{7.6}$$

has a smooth expansion [Theorem 4.3]

$$\tilde{\sigma}(x, \zeta) \sim \sum \tilde{\sigma}_{\alpha j}(x)\, \zeta^\alpha \log^j \zeta. \tag{7.7}$$

The factor x^{2m-1} in (7.5) means that the first non-vanishing term in (7.3a) is

$$\begin{aligned} & t^0 \frac{(m-1)!}{2\pi i} \int_\Gamma e^{-\lambda}(-\lambda)^{-m}\, d\lambda \int_0^\infty \zeta^{2m-1} \tilde{\sigma}(0, \zeta)\, d\zeta \\ &= t^0 \int_0^\infty \zeta^{2m-1} \operatorname{tr}_H G_0^m(1, 1; \zeta)\, d\zeta \\ &=: t^0 c_{0a}.\end{aligned} \tag{7.8}$$

We recall from [19] the main steps in computing c_{0a}; similar calculations are found in [9]. We have

$$G_0^m(\zeta) = (L_0 + \zeta^2)^{-m} = \oplus_{a \in \operatorname{spec} A_0} (L_a + \zeta^2)^{-m},$$

where L_a denotes the Friedrichs extension of $D^2 + x^{-2}a$ in $L^2(\mathbb{R}_+)$. $(L_a + \zeta^2)^{-1}$ has a kernel k_ν given on the diagonal by

$$k_\nu(x, x; \zeta) = x I_\nu(x\zeta) K_\nu(x\zeta), \qquad \nu = \sqrt{a + 1/4}.$$

Hence, on the diagonal, $(L_a + \zeta^2)^{-m}$ has kernel

$$k_\nu^m(x, x; \zeta) = \frac{1}{(m-1)!} \left(-\frac{1}{2\zeta} \frac{\partial}{\partial \zeta} \right)^{m-1} x I_\nu(x\zeta) K_\nu(x\zeta). \tag{7.9}$$

From [19], or Lemma 4.9, $\gamma G_0^m(\zeta)$ has a kernel which is trace class for $\operatorname{Re}(\zeta) > 0$, and since $L_\alpha \geqslant 0$ for $a \geqslant -\frac{1}{4}$, (7.9) gives

$$\operatorname{tr}_H G_0^m(1, 1; \zeta) = \sum_{a \in \operatorname{spec} A_0} \frac{1}{(m-1)!} \left(-\frac{1}{2\zeta} \frac{\partial}{\partial \zeta} \right)^{m-1} I_\nu(\zeta) K_\nu(\zeta). \tag{7.10}$$

To compute c_{0a} in (7.8) we need the Mellin transform of each term in (7.10)

$$\frac{1}{(m-1)!} \int_0^\infty \zeta^w \left(-\frac{1}{2\zeta} \frac{\partial}{\partial \zeta} \right)^{m-1} I_\nu(\zeta) K_\nu(\zeta)\, d\zeta$$

$$= \frac{1}{4\sqrt{\pi}\,(m-1)!}$$

$$\times \left(\Gamma\left(\nu - m + \frac{w+3}{2} \right) \Gamma\left(m - 1 - \frac{w}{2} \right) \Gamma\left(\frac{w+1}{2} \right) \Big/ \Gamma\left(1 + \nu + m - \frac{w+3}{2} \right) \right)$$

$$= \left(\Gamma\left(\frac{w+1}{2} \right) \Gamma\left(m - 1 - \frac{w}{2} \right) \Big/ 4\sqrt{\pi}\,(m-1)! \right) \frac{\Gamma(\nu + z(w) + 1)}{\Gamma(\nu - z(w))}, \tag{7.11}$$

where $z(w) := (w + 1 - 2m)/2$. This function is analytic if

$$\max\{-1, 2m - 2\nu - 3\} < \operatorname{Re}(w) < 2m - 2$$

and by Stirling's formula it is

$$O(\nu^{\operatorname{Re} w + 2 - 2m}) = O(a^{\operatorname{Re} w/2 + 1 - m}).$$

Thus, except for finitely many simple poles arising from values ν with $2m - 2\nu - 3 > -1$, the integral

$$\int_0^\infty \zeta^w \operatorname{tr}_H G_0^m(1, 1; \zeta)\, d\zeta$$

in (7.8) is the sum of a convergent series of functions analytic in the strip

$$-1 < \operatorname{Re} w < 2m - 2 - 2p$$

which is nonempty when we take $m > p + 1/2$. Set

$$w = 2z + 2m - 1$$

in (7.11). By the prescription of the Singular Asymptotics Lemma, we find from (7.8) that c_{0a} is the regular analytic continuation to $z = 0$ of

$$\int_0^\infty \zeta^{2z + 2m - 1} \operatorname{tr}_H G_0^m(1, 1; \zeta)\, d\zeta$$

$$= \frac{\Gamma(-z - 1/2)\, \Gamma(z + m)}{4\sqrt{\pi}\,(m-1)!} \sum_{a \in \operatorname{spec} A_0} \frac{\Gamma(\nu + z + 1)}{\Gamma(\nu - z)}. \tag{7.12}$$

We introduce the ζ-function of $(A_0 + \frac{1}{4})^{1/2}$,

$$\zeta(s) := \sum_{\substack{a \in \operatorname{spec} A_0 \\ \nu(a) > 0}} \nu(a)^{-s}, \qquad \nu(a) = \sqrt{a + 1/4}. \tag{7.13}$$

Since $\sum (a+1)^{-p} < \infty$, $\zeta(s)$ is holomorphic for $\operatorname{Re}(s) > 2p$. We further assume for the present calculations that $\zeta(s)$ has only simple poles, as is the case when A_0 is a non-singular elliptic operator. We obtain from [15, p. 119] the asymptotic expansion

$$\frac{\Gamma(\nu + z + 1)}{\Gamma(\nu - z)} = \nu^{1 + 2z} \sum_{j=0}^N Q_j(z)\, \nu^{-j} + I_{N+1}(z, \nu),$$

where $Q_0 = 1$, $Q_1 = 0$, $Q_2 = -(z/6)(z+1)(2z+1)$, Q_j is an appropriate polynomial, and

$$I_{N+1}(z, \nu) = \Gamma(-1 - 2z)^{-1} \int_0^\infty e^{-\nu t} \Phi_{N+1}(t, z)\, dt,$$

$$\Phi_{N+1}(t, z) = O(t^{N - 2\operatorname{Re} z - 1}) \quad \text{if} \quad N > -2\operatorname{Re}(z) - 1, \quad |z| \leqslant z_0.$$

In view of the factor $\Gamma(-1-2z)^{-1}$, $I_{N+1}(z, \nu)=O(|z|\,\nu^{2\mathrm{Re}\,z-N})$. Hence for sufficiently large N,

$$\sum \frac{\Gamma(\nu+z+1)}{\Gamma(\nu-z)} = \sum_{j=0}^{N} Q_j(z)\,\zeta(j-1-2z)+R(z), \tag{7.14}$$

where $R(z)$ is analytic in $\mathrm{Re}(z)<1$, and $R(0)=0$. Thus the residue and the regular analytic continuation of the left-hand side of (7.14) can be computed from the finite sum on the right. Moreover, assuming simple poles for ζ, we need only the linear parts of the Q_j. These are given by [9], Lemma 4.2, as

$$Q_j(z)=\begin{cases} O(z^2), & j \text{ odd} \\ (-1)^{j/2-1}\,2zB_{j/2}\,j^{-1}+O(z^2), & j \text{ even} >0. \end{cases}$$

Setting $j=2k$, we find from (7.12) that c_{0a} is the regular analytic continuation to $z=0$ of

$$\frac{\Gamma(-z-1/2)\,\Gamma(z+m)}{4\sqrt{\pi}\,\Gamma(m)}\left[\zeta(-1-2z)-\sum_{k=1}^{N/2}(-1)^k k^{-1}B_k z\zeta(2k-1-2z)\right]. \tag{7.15}$$

For a meromorphic function f we denote by

$$\mathrm{Res}_k f(z_0)$$

the coefficient of $(z-z_0)^{-k}$ in the Laurent expansion of f at z_0. Thus Res_1 is the usual residue, and Res_0 the regular analytic continuation. If f is analytic and g has a simple pole at 0, then

$$\mathrm{Res}_0(fg)(0)=f(0)\,\mathrm{Res}_0\,g(0)+f'(0)\,\mathrm{Res}_1\,g(0).$$

Thus when we continue (7.15), noting that

$$\frac{\Gamma'(m)}{\Gamma(m)}=\sum_{j=1}^{m-1}\frac{1}{j}-\gamma,$$

where γ is Euler's constant, and $\Gamma(-\frac{1}{2})=-2\sqrt{\pi}$, we find

$$\begin{aligned} c_{0a}=&-\frac{1}{2}\,\mathrm{Res}_0\zeta(-1)-\frac{1}{4}\sum_{k\geqslant 1}(-1)^k k^{-1}B_k\,\mathrm{Res}_1\zeta(2k-1) \\ &+\frac{1}{2}\left[\frac{\Gamma'(-1/2)}{4\sqrt{\pi}}+\frac{1}{2}\left(\sum_{j=1}^{m-1}\frac{1}{j}-\gamma\right)\right]\mathrm{Res}_1\zeta(-1). \end{aligned} \tag{7.16}$$

Since $\zeta(s)$ is analytic for $\mathrm{Re}(s)$ large, the sum in (7.16) is finite. There are no contributions with $\mathrm{Res}_2\zeta$, etc., since we assume simple poles.

We now turn to the "interior" contributions (7.3b). If $A(x)$ is a family of non-singular elliptic operators then there are no logarithmic terms $\sigma_{\alpha j}(x)\zeta^\alpha \log^j \zeta$ with $j>0$. We will assume a little less, namely that

$$\sigma_{\alpha j}\equiv 0 \quad \text{if} \quad \mathrm{Re}\,\alpha\geqslant -2m \quad \text{and} \quad j\geqslant 1. \tag{7.17}$$

By the arguments leading to (7.3), the operator γe^{-tL} has an operator kernel $\gamma(x)e^{-tL}(x,y)$ with the pointwise asymption expansion

$$\begin{aligned} \mathrm{tr}_H\gamma(x)\,e^{-tL}(x,x)&=\frac{(m-1)!}{2\pi i}\sum_{\mathrm{Re}\,\alpha\geqslant -2m} t^{-m-\alpha/2}x^\alpha\sigma_{\alpha 0}(x) \\ &\quad\times\int_\Gamma e^{-\lambda}(\sqrt{-\lambda})^\alpha\,d\lambda+o_x(t^0) \\ &=\sum_{\beta\leqslant 0} t^\beta x^{-2\beta-2m}\sigma_{-2\beta-2m,0}(x) \\ &\quad\times\frac{(m-1)!}{2\pi i}\int_\Gamma e^{-\lambda}(-\lambda)^{-\beta-m}\,d\lambda+o_x(t^0) \\ &=:\sum_{\beta\leqslant 0} t^\beta g_\beta(x)+o_x(t^0). \end{aligned} \tag{7.18}$$

Thus the coefficients in (7.3b) with $\mathrm{Re}\,\alpha\geqslant -2m$ are, up to constants, just regularizations of

$$\int_0^\infty g_\beta(x)\,dx.$$

But from (7.18) and (7.5)–(7.7) it is apparent that

$$\begin{aligned} g_\beta(x)&=a_{\beta,m}x^{-2\beta-2m}\sigma_{-2\beta-2m,0}(x) \\ &=a_{\beta,m}\gamma(x)\,x^{-2\beta-1}\tilde\sigma_{-2\beta-2m,0}(x), \end{aligned} \tag{7.19}$$

where

$$a_{\beta,m}=\frac{(m-1)!}{2\pi i}\int_\Gamma e^{-\lambda}(-\lambda)^{-\beta-m}\,d\lambda=\frac{\Gamma(m)}{\Gamma(\beta+m)}$$

and $\tilde\sigma_{-2\beta-2m,0}(x)$ is smooth at 0. Hence no regularization is needed if $\beta<0$; but the constant term

$$c_{0b}=\int_0^\infty g_0(x)\,dx \tag{7.20}$$

will in principle need regularization. Note that by (7.19),

$$\tilde{\sigma}_{-2m,0}(0) = \lim_{x\to 0} x g_0(x). \tag{7.21}$$

Turning at last to the terms (7.3c), we see again from

$$\sigma_{\alpha 0}(x) = \gamma(x)\, x^{2m-1} \tilde{\sigma}_{\alpha 0}(x)$$

that all terms before the one with $\alpha = -2m$ are zero, so no negative powers of t arise from this source. In particular, if the index α_0 with maximal real part has $\operatorname{Re}\alpha_0 > -2m$, then the leading term in the entire expansion (7.3) is of type (7.3a), and

$$\begin{aligned}\operatorname{tr} \gamma e^{-tL} &\sim t^{-m-\alpha_0/2} \int_0^\infty \gamma(x)\, x^{\alpha_0} \sigma_{\alpha_0,0}(x)\, dx \frac{(m-1)!}{2\pi i} \int_\Gamma e^{-\lambda} \sqrt{-\lambda}^{\alpha_0}\, d\lambda \\ &= t^{\beta_0} \int_0^\infty \gamma(x)\, g_{\beta_0}(x)\, dx + o(t^{\beta_0}).\end{aligned}$$

This, together with suitable information from the interior, determines the eigenvalue distribution of L. As for the t^0 term, taking $\alpha = -2m$, $j = 0$ in (7.3c) gives a contribution

$$\begin{aligned}c_{0c} &= \sigma^{(2m-1)}_{-2m,0}(0) \frac{(m-1)!}{(2m-1)!} \frac{1}{4\pi i} \int_\Gamma e^{-\lambda}(-\lambda)^{-m} \log(-\lambda)\, d\lambda \\ &= \tilde{\sigma}_{-2m,0}(0) \frac{(m-1)!}{4\pi i} \int_\Gamma e^{-\lambda}(-\lambda)^{-m} \log(-\lambda)\, d\lambda \\ &= \frac{1}{2} \tilde{\sigma}_{-2m,0}(0) \left[-\gamma + \sum_{j=1}^{m-1} 1/j \right].\end{aligned}$$

Adding to this the contributions from (7.16) and (7.19) gives the coefficient of t^0 as

$$\begin{aligned}c_0 &= c_{0a} + c_{0b} + c_{0c} \\ &= -\frac{1}{2} \operatorname{Res}_0 \zeta(-1) - \frac{1}{4} \sum_{k \geq 1} (-1)^k k^{-1} B_k \operatorname{Res}_1 \zeta(2k-1) \\ &\quad + \frac{\Gamma'(-1/2)}{8\sqrt{\pi}} \operatorname{Res}_1 \zeta(-1) + \int_0^\infty g_0(x)\, dx \\ &\quad + \frac{1}{4} \left(\sum_{j=1}^{m-1} \frac{1}{j} - \gamma \right) [\operatorname{Res}_1 \zeta(-1) + 2\tilde{\sigma}_{-2m,0}(0)].\end{aligned}$$

This holds for each integer m sufficiently large. Moreover, the $g_0(x)$ in (7.18) is independent of m, hence from (7.21), so is $\tilde{\sigma}_{-2m,0}(0)$. Since c_0 itself is independent of m, we find that the last term above drops out:

$$2\tilde{\sigma}_{-2m,0}(0) = -\operatorname{Res}_1 \zeta(-1).$$

Thus, finally,

$$\begin{aligned}c_0 = &-\frac{1}{2} \operatorname{Res}_0 \zeta(-1) - \frac{1}{4} \sum_{k \geq 1} (-1)^k k^{-1} B_k \operatorname{Res}_1 \zeta(2k-1) \\ &+ \frac{\Gamma'(-1/2)}{8\sqrt{\pi}} \operatorname{Res}_1 \zeta(-1) + \int_0^\infty g_0(x)\, dx\end{aligned} \tag{7.22}$$

and

$$\operatorname{Res}_1 \zeta(-1) = -2\tilde{\sigma}_{-2m,0}(0) = -2 \lim_{x\to 0} x g_0(x). \tag{7.23}$$

Thus we have the regularized interior term $\int_0^\infty g_0(x)\, dx$ plus singular terms in the ζ-function of $(A_0 + \frac{1}{4})^{1/2}$. Since these depend only on A_0, they are the same as in the constant coefficient case, including the cases studied by Cheeger [9].

The coefficient of $t^0 \log t$ has some interest. Assuming (7.15), this term comes only from (7.3c). It is

$$\begin{aligned}&\sigma^{(2m-1)}_{-2m,0}(0) \frac{(m-1)!}{(2m-1)!} \frac{-1}{4\pi i} \int_\Gamma e^{-\lambda}(-\lambda)^{-m}\, d\lambda \\ &= -\frac{1}{2} \tilde{\sigma}_{-2m,0}(0) = \frac{1}{4} \operatorname{Res}_1 \zeta(-1) = -\frac{1}{2} \lim_{x\to 0} x g_0(x).\end{aligned} \tag{7.24}$$

We conclude with simple examples. Take a finite curve in the upper half plane, beginning at the origin with positive slope $\tan \varphi$, and rotate it about the x axis to form a surface of revolution M; suppose that M is a smooth surface except at the origin. Let r be arc length on the curve, measured from the origin, and let $y = f(r)$ along the curve. The Laplacian is

$$\Delta = -f^{-1}(\partial_r f \partial_r) - f^{-2} \partial_\theta^2 \quad \text{in } L^2(f\, dr\, d\theta)$$

and the change of dependent variable $u \mapsto f^{1/2} u$ transforms Δ into

$$\begin{aligned}L &= -\partial_r^2 + f^{-2}[-\partial_\theta^2 - \tfrac{1}{4}(f')^2 + \tfrac{1}{2} f f''] \\ &= -\partial_r^2 - r^{-2} A(r) \text{ in } L^2(dr\, d\theta),\end{aligned} \tag{7.25}$$

where

$$A(r) = -\alpha^2[1 + r\kappa \cot \varphi + O(r^2)]\partial_\theta^2 - \tfrac{1}{4}[1 - r\kappa \cot \varphi + O(r^2)]$$

with κ the curvature of the generating curve at the origin and $\alpha = \csc \varphi$, where $\tan \varphi$ is the slope of the generating curve at the tip. The eigenvalues of A_0 are

$$\{-\tfrac{1}{4} + \alpha^2 j^2\}_{j=-\infty}^{\infty}$$

so the ζ function (7.13) is

$$\zeta(s) = 2 \sum_{j=1}^{\infty} (\alpha j)^{-s} = 2\alpha^{-s}\zeta_{\mathrm{R}}(s)$$

with ζ_{R} the Riemann ζ-function. Thus $\mathrm{Res}_1 \zeta(-1) = 0$ and (7.22) gives

$$\begin{aligned} c_0 &= -\alpha\zeta_{\mathrm{R}}(-1) + \frac{1}{2\alpha} B_1 \mathrm{Res}_1 \zeta_{\mathrm{R}}(1) + \int_0^\infty g_0(x)\, dx \\ &= \frac{\alpha}{12} - \frac{1}{12\alpha} + \int_0^\infty g_0(x)\, dx. \end{aligned}$$

From (7.23), the integral is actually convergent, since $\mathrm{Res}_1\, \zeta(-1) = 0$. When there is no singularity then the cone angle $\varphi = \pi/2$ and $\alpha = \csc \varphi = 1$, so the singular contribution is zero.

If we apply the operator (7.25) only to rotation-invariant functions, we get a one-dimension example,

$$\begin{aligned} L &= -\partial_r^2 + r^{-2}a(r), \\ a(r) &= -1/4 + r(\kappa \cot \varphi)/4 + O(r^2). \end{aligned}$$

This is covered in [4]; Eq. (4.8) in that paper gives

$$\begin{aligned} \mathrm{tr}(\gamma(L+z^2)^{-1}) = z^{-1} \int_0^\infty \gamma(r)\, dr/2 - z^{-3} \int_0^\infty \gamma(r)\, r^{-2}a(r)\, dr/4 \\ - z^{-3}\kappa \cot \varphi[c + (\log z)/16] \\ + O(z^{-4} \log z), \end{aligned}$$

where c is a positive constant. Thus we find a nontrivial logarithmic term which, given the angle φ, will determine the curvature κ at the tip of the cone.

As a final example, let $L = \Delta + V$, where Δ is the Laplacian on a 2-manifold and V is a potential function having at one point P a singularity asymptotic to cr^{-2}, $c \geq 0$, $r =$ distance from P. The Laplacian is now given by (7.25) with $f(0) = 0$ and $f'(0) = 1$, so

$$A_0 = -\partial_\theta^2 - \tfrac{1}{4} + c.$$

The eigenvalues are $\{c - \frac{1}{4} + j^2\}_{j=-\infty}^{j=\infty}$ and the ζ-function of $(A_0 + \frac{1}{4})^{1/2}$ is

$$\begin{aligned} \zeta(s) &= \sum_{j=-\infty}^{\infty} (c+j^2)^{-s/2} = c^{-s/2} + 2 \sum_{j=1}^{\infty} j^{-s}(1 + cj^{-2})^{-s/2} \\ &= c^{-s/2} + 2\zeta_{\mathrm{R}}(s) - cs\zeta_{\mathrm{R}}(s+2) + R(s) \end{aligned}$$

with $R(s)$ analytic in $\mathrm{Re}(s) > -3$. Thus $\mathrm{Res}_1 \zeta(-1) = 0$, and (7.24) gives a term $(c/4) \log t$ in the heat expansion. From (7.22), this residue contributes also to the t^0 term; the other singular contribution to this term, involving $\mathrm{Res}_0 \zeta(-1)$, seems harder to compute.

APPENDIX: A TRACE LEMMA

We estimate kernels of operators by their trace norm as follows:

LEMMA 4.1. *Let T be a trace class operator on $L^2(\mathbb{R}^1, H)$ where H is a Hilbert space. Then T has a kernel $t(x, y)$ such that*

$$Tf(x) = \int_{-\infty}^{\infty} t(x, y) f(y)\, dy$$

and

$$h \mapsto t(\cdot, \cdot + h) \tag{1}$$

is a continuous map into L^1 maps of $\mathbb{R}^1$ into $C_1(H)$, the trace class operators on H. Further

$$\int_{-\infty}^{\infty} \| t(x, x)\|_{\mathrm{tr}}\, dx \leq \| T\|_{\mathrm{tr}}, \tag{2}$$

$$\int_{-\infty}^{\infty} \mathrm{tr}[t(x, x)]\, dx = \mathrm{tr}\, T. \tag{3}$$

If $\partial T - T\partial$ is also trace class, then (1) is continuous into the absolutely continuous L^1 maps: $\mathbb{R}^1 \mapsto C^1(H)$, and so T has a continuous kernel $t(x, y)$. Moreover

$$\| t(x, y)\|_{\mathrm{tr}} \leq \| \partial T - T\partial\|_{\mathrm{tr}}. \tag{4}$$

Remarks. The kernel of an integral operator is defined only up to a set of measure zero in (x, y) space, so (2), (3), and (4) are meaningless unless the kernel is normalized in some way. The continuity of (1) normalizes t.

The lemma implies that for any kernel t for T, and any approximate identity $\{\varphi_n\}$,

$$\sup_n \iint \varphi_n(x-y) \| t(x, y)\|_{\mathrm{tr}}\, dy\, dx < \infty$$

$$\lim_{n\to\infty} \iint \varphi_n(x-y) \operatorname{tr} t(x, y)\, dy\, dx = \operatorname{tr} T.$$

This is very similar to [24, Chap. III Corollary 10.2]. However, it is convenient to have kernels for which no approximate identity need intercede.

From (4) follows an obvious fact: A nonzero translation-invariant operator on $\mathbb{R}^1$ cannot be of trace class. For if T is translation-invariant, $\partial T - T\partial = 0$.

Proof. $T = RS$, where R and S are Hilbert–Schmidt with Hilbert–Schmidt norms $\|R\|_2$, $\|S\|_2$, and

$$\|T\|_1 = \|R\|_2\, \|S\|_2.$$

We have $Rf(x) = \int r(x, y) f(y)\, dy$ where for almost all (x, y), $\|r(x, y)\|_2 < \infty$, and

$$\int \|r(x, y)\|_2^2\, dx\, dy = \|R\|_2^2 < \infty;$$

and similarly for S. Thus $T = RS$ has a kernel

$$t(x, y) = \int r(x, u)\, s(u, y)\, du$$

with

$$\int_{-\infty}^{\infty} \|t(x, x)\|_{\mathrm{tr}}\, dx \leqslant \|R\|_2\, \|S\|_2 = \|T\|_{\mathrm{tr}}.$$

The family of translations $\mathscr{U}_h f(x) = f(x-h)$ is strongly continuous, so $S\mathscr{U}_h$ is continuous in the norm $\|\cdot\|_2$, and this implies the continuity of the map (1):

$$\int_{-\infty}^{\infty} \|t(x, x+h) - t(x, x+\bar{h})\|_{\mathrm{tr}}\, dx \leqslant \|R\|_2 \cdot \|S\mathscr{U}_h - S\mathscr{U}_{\bar{h}}\|_2 \to 0 \quad \text{as } h \to \bar{h}.$$

To prove (3), choose a basis $\{\varphi_j\}$ for $L^2(\mathbb{R}^1) \otimes H$. Then

$$Rf(x) = \sum r_{jk} \int (f(y), \varphi_k(y))\, dy \varphi_j(x),$$

$$\sum (r_{jk})^2 = \|R\|_2^2$$

and the kernel of R is

$$r(x, y) = \sum r_{jk}(\cdot, \varphi_k(y))\, \varphi_j(x).$$

Let R_n be the operator where the summation is restricted to $j + k < n$, and define S_n likewise. Then $T_n = R_n S_n$ has kernel

$$t_n(x, y) = \int r_n(x, u)\, s_n(u, y)\, du = \sum r_{jk} s_{km}(\cdot, \varphi_m(y))\, \varphi_j(x)$$

summed for $j + k < n$, $k + m < n$. Since $tr(\cdot, \varphi)\, \psi = (\psi, \varphi)$,

$$\operatorname{tr} T_n = \sum r_{jk} s_{kj} = \int \sum r_{jk} s_{km}(\varphi_j(x), \varphi_m(x))\, dx$$

$$= \int \operatorname{tr} t_n(x, x)\, dx.$$

The inequality (2) allows passing to the limit

$$\operatorname{tr} T = \lim \operatorname{tr} T_n = \lim \int \operatorname{tr} t_n(x, x)\, dx = \int \operatorname{tr} t(x, x)\, dx.$$

Finally, to prove (4), suppose that $\partial T - T\partial$ is trace class. Let t be the above kernel for T, and t' a similar kernel for $\partial T - T\partial$. Then as distributions

$$t'(x, y) = \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right) t(x, y)$$

or

$$t'(x, x+h) = \frac{\partial}{\partial x}\, t(x, x+h).$$

Since $t(\cdot, \cdot + h)$ is in L^1, and $t'(x, y+h)$ is the kernel of $(\partial T - T\partial)\, U_h$,

$$\| t(x, x+h) \|_{\mathrm{tr}} = \left\| \int_{-\infty}^{x} t'(\xi, \xi + h)\, d\xi \right\|_{\mathrm{tr}}$$

$$\leqslant \int_{-\infty}^{\infty} \| t'(\xi, \xi + h) \|_{\mathrm{tr}}\, d\xi \leqslant \| \partial T - T\partial \|_{\mathrm{tr}}$$

by (2). The proof is complete. ∎

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