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ON THE ABSENCE OF LOG TERMS IN THE CONSTANT CURVATURE CASE

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Abstract. The classical result of Minakshisundaram and Pleijel on the asymptotic expansion of the trace of the heat semigroup associated with the Laplacean on a compact Riemannian manifold M has been generalized by Brüning and Heintze to the case that a compact group is acting on M by isometries. They obtained an asymptotic expansion involving logarithmic terms. Here we prove that these terms vanish if M has constant sectional curvature or if M is a warped product $M = [0, \pi] \times_f S^n$ with $n \geqslant 2$ and suitable f.

1. Introduction

Let M be a compact Riemannian manifold and let G be a closed subgroup of $I(M)$, the isometry group of M. Denote by Δ ($\geqslant 0$) the Laplacean on M and by $e^{-t\Delta}$ the heat semigroup with kernel Γ_t. Then

$$\operatorname{tr} e^{-t\Delta} = \int_M \Gamma_t(p, p)\, dp \tag{1.1}$$

where dp denotes the Riemannian measure. A celebrated result of Minakshisundaram and Pleijel [11] implies that this trace has an asymptotic expansion as $t \to 0+$:

$$\operatorname{tr} e^{-t\Delta} \sim (4\pi t)^{-m/2} \sum_{j \geqslant 0} a_j t^j, \quad m = \dim M. \tag{1.2}$$

In the equivariant case, the Hilbert space $L^2(M)$ decomposes according to the irreducible representations of G and, since Δ commutes with G, it makes sense to restrict Δ to these representation spaces. Fixing a character χ and a left invariant measure dg of G, (1.1) generalizes to

$$I(t) := \int_{M \times G} \Gamma_t(p, gp)\overline{\chi(g)}\, dg\, dp, \tag{1.3}$$

and one can ask whether (1.3) has an asymptotic expansion analogous to (1.2). This question has been studied in [4,5]; the main result asserts that there is an asymptotic expansion of the form

$$I(t) \sim (4\pi t)^{-l/2} \sum_{\substack{j \geqslant 0 \\ 0 \leqslant k \leqslant k_0 - 1}} a_{jk} t^{j/2} (\log t)^k. \tag{1.4}$$

Here $l = \dim M/G$ and k_0 is bounded by the number of different dimensions of G-orbits in M. This raises of course the question whether logarithmic terms do actually occur. In [5, Theorems 5 and 7], the following negative result was proved.

Theorem 1.1. *If either M is the standard sphere or G has rank at most* 1, *then no logarithmic terms occur in* (1.4).

Since the logarithmic terms reflect the structure of the fixed point set,

$$\mathscr{L} := \{(p, g) \in M \times G \mid gp = p\}, \tag{1.5}$$

which can be very complicated, even for the standard sphere, the absence of log terms is somewhat surprising in this case. An even more natural test case arises from isometric torus actions on Euclidean space. Then we deal with integrals

$$I(t, \varphi) := \int_{\mathbb{R}^{2n} \times T^k} e^{-|x-\rho(\vartheta)x|^2/t} \varphi(x, \vartheta)\, d\vartheta\, dx \tag{1.6}$$

where T^k is the k torus, $\rho : T^k \to O(2n)$ a representation, and $\varphi \in C_0^\infty(\mathbb{R}^{2n} \times T^k)$. The asymptotic expansion of these integrals turns out to be very complicated. For $k = 2$ and $n = 3$ the first author computed the expansion which seemed to produce a logarithmic term. Then Hans Duistermaat pointed out a simplification of the rather long calculations which allowed to detect an unfortunate error: after the correction, the logarithmic terms vanish again. In this note we partially explain this phenomenon in presenting another vanishing result for a large class of integrals of the type (1.6). Since the underlying manifold in (1.6) is flat, we first investigate the case where M in (1.1) has constant sectional curvature. Using essentially the methods of [5] we show in Section 2 that there are no log terms in this case (Theorem 2.2). The integrals (1.6) are quite different since $\{x \in \mathbb{R}^{2n} \mid |x| \leqslant 1\}$ cannot be isometrically and equivariantly imbedded into a flat T^k-manifold, and since they involve the cut-off function φ. We will show in Section 3 that (1.6) has no logarithmic terms if φ is radially symmetric as a function of $x \in \mathbb{R}^{2n}$. This is a special case of a more general vanishing theorem (Theorem 3.1) for warped products $M = [0, \pi] \times_f S^{m-1}$. The proof uses essentially the methods developed in [6,7] for operator Sturm–Liouville equations and the special structure of isometric actions on the standard sphere.

2. Manifolds with constant sectional curvature

Manifolds with constant sectional curvature do not have logarithmic terms in expansion (1.4), for any closed subgroup G of the isometry group. This will be proved using the methods of [5] and the well known structure of such manifolds. In addition, we need the following lemma (a similar result was used implicitly in the proof of [5, Theorem 6]). To formulate it we need some terminology. Let $\tilde{M}$ be a compact Riemannian manifold and $\Gamma \subset I(\tilde{M})$ a closed subgroup acting freely on $\tilde{M}$. Then $M := \tilde{M}/\Gamma$ is a compact manifold which can be provided with a Riemannian structure in such a way that the projection $\pi : \tilde{M} \to M$ becomes a Riemannian submersion; let Δ_M denote the Laplacean with respect to this metric. The function $h(p) := \operatorname{vol} \pi^{-1}(p)$, $p \in M$, is smooth and positive; therefore,

$$q := h^{-1/2}\, \Delta_M h^{1/2} \in C^\infty(M). \tag{2.1}$$

Lemma 2.1. *Let $\tilde{G} \subset I(\tilde{M})$ be a closed subgroup with*

$$\Gamma \subset \tilde{G} \subset N(\Gamma) = \textit{the normalizer of } \Gamma \textit{ in } I(\tilde{M}) \tag{2.2}$$

and

$$h \circ \pi \text{ is } \tilde{G}\text{-invariant}. \tag{2.3}$$

Then $G := \tilde{G}/\Gamma$ *is naturally isomorphic to a closed subgroup of* $I(M)$, h *is* G*-invariant, and the operators* $\Delta_{\tilde{M}}^{\tilde{G}}$ *and* $\Delta_M^G + q$ *are unitarily equivalent.*

Proof. For $\tilde{g} \in \tilde{G}$, $g := \tilde{g}\Gamma$ acts on M by $g\pi(p) = \pi(\tilde{g}p)$. By (2.2), this is well defined and $\tilde{g}$ maps $\pi^{-1}(p)$ to $\pi^{-1}(\tilde{g}p)$. Thus g is an isometry since π is a Riemannian submersion. Now it has been shown in [3, Theorem 1] that the map

$$\Psi : L^2(M) \ni f \mapsto (h \circ \pi)^{-1/2} f \circ \pi \in L^2(\tilde{M})^{\Gamma}$$

is an isometry and intertwines $\Delta_M + q$ and $\Delta_{\tilde{M}}^{\Gamma}$. But Ψ also restricts to an isometry $L^2(M)^G \to L^2(M)^{\tilde{G}}$. Since, by (2.3), q is G-invariant, the assertion follows from [4, Lemma 2.2]. □

We are ready to prove the main result of this section.

Theorem 2.2. *Let* M *be a compact Riemannian manifold with constant sectional curvature. For any closed subgroup* G *of* $I(M)$, *the asymptotic expansion* (1.4) *does not have logarithmic terms.*

Proof. If the sectional curvature K_M is negative, then it is well-known (cf. [9, p. 55]) that $I(M)$ is a finite group. Thus the theorem follows from [5, Theorem 4].

If K_M is positive, then M is a spherical space form, i.e., $M = S^m/\Gamma$ for some finite group $\Gamma \subset O(m+1)$ acting freely on S^m. It is easily seen that $I(M) \simeq N(\Gamma)/\Gamma$ and that h is constant on M. Thus Lemma 2.1 applies, and the assertion follows from [5, Theorem 5] in this case.

It remains to study the flat case. If $K_M = 0$, then $M = \mathbb{R}^m/\Gamma$ where $\Gamma \subset E(m)$ is a torsion-free crystallographic group. By the Bieberbach Theorem (cf. [13, Theorem 3.2.1]), $\Gamma^* := \Gamma \cap \mathbb{R}^m$ is a free abelian group of rank m which is normal with finite index in Γ. Thus M is covered by the torus $T := \mathbb{R}^m/\Gamma^*$ with $G_0 := \Gamma/\Gamma^*$ as the group of covering transformations. M and T have isometry groups $N(\Gamma)/\Gamma$ and $N(\Gamma^*)/\Gamma^*$ respectively, where $N(\Gamma) \subset N(\Gamma^*)$, and we have a surjective homomorphism $\varphi : N(\Gamma)/\Gamma^* \to N(\Gamma)/\Gamma \simeq N(\Gamma)/\Gamma^*/G_0$. Since $G_0 \subset N(\Gamma)/\Gamma^* \subset N(G_0)$, we can find, for any closed subgroup G of $I(M)$, a closed subgroup $\tilde{G}$ of $I(T)$ with $G_0 \subset \tilde{G} \subset N(G_0)$ such that the assumptions of Lemma 2.1 are satisfied for $\tilde{M} := T$, $\Gamma := G_0$. Thus it is enough to treat the case $M = T$. It is readily seen that $I(T)$ has identity component $I(T)^0 = T$ with the natural action on itself. Thus any closed subgroup G of T has as identity component G^0 a closed subgroup of T. So Lemma 2.1 applies again to $\tilde{M} := T$, $\Gamma := G^0$, and the proof is completed by using [5, Theorem 4] once more. □

3. Warped products

Warped products $M = [0, \pi] \times_f S^n$ with $n \geq 2$ and suitable f do not have logarithmic terms in expansion (1.4). We choose a function $f \in C^\infty[0, \pi]$ with the following properties:

$$\begin{aligned} &f(x) > 0 \quad \text{and} \quad f(x) = f(\pi - x), \qquad x \in (0, \pi), \\ &f^{(2k)}(0) = 0, \quad k \geq 0, \qquad f'(0) = 1. \end{aligned} \tag{3.1}$$

Now let M be the suspension of S^n, i.e., the product $[0, \pi] \times S^n$ with $\{0\} \times S^n$ and $\{\pi\} \times S^n$ identified with a point (the northpole N and the southpole S respectively) equipped with the metric

$$g := \mathrm{d}x \otimes \mathrm{d}x + f(x)^2 g_0 \quad \text{on } (0, \pi) \times S^n, \tag{3.2}$$

where x denotes the canonical coordinate in $(0, \pi)$ and g_0 the standard metric on S^n. It is easily seen that (M, g) is a smooth compact Riemannian manifold, which becomes the standard sphere S^{n+1}, for example, if we take $f(x) = \sin x$; in this case we already know from [5, Theorem 5] that we do not encounter log terms in the expansion.

Now we choose a compact subgroup G of $O(n+1)$ and extend its action on S^n to an isometric action on M. Then we obtain the following theorem, the proof of which will be done in a sequence of lemmas.

Theorem 3.1. *Let $\varphi \in C^\infty(M)$, $\varphi = \tilde{\varphi} \circ \sigma$ where $\sigma : [0, \pi] \times S^n \to [0, \pi]$ denotes the projection and $\tilde{\varphi} \in C^\infty[0, \pi]$, and let ρ be a finite-dimensional irreducible representation of G with character χ. Then*

$$\mathrm{tr}_{L^2(M)^G} \varphi\, \mathrm{e}^{-t\Delta^G} = \int_{M \times G} \varphi(p) \Gamma_t(p, gp) \bar{\chi}(g)\, \mathrm{d}g\, \mathrm{d}p \tag{3.3}$$

has an asymptotic expansion as $t \to 0+$ without logarithmic terms.

We want to apply the expansion of the operator heat kernel derived in [2] to evaluate (3.3). This requires that we can write Δ^G as a second-order ordinary differential operator with operator coefficients. To do so we introduce the map

$$\begin{aligned} &\Phi : C_0^\infty\big((0, \pi), L^2(S^n)\big) \to L^2(M), \\ &\Phi u(x, \omega) := f(x)^{-n/2} u(x)(\omega), \quad x \in (0, \pi), \ \omega \in S^n. \end{aligned} \tag{3.4}$$

In what follows we write $f(x) =: x\tilde{f}(x)$. Then we find, by a straightforward computation, the following lemma.

Lemma 3.2. *Φ is bijective $C_0^\infty((0, \pi), C^\infty(S^n)) \to C_0^\infty(M \setminus \{N, S\})$ and unitary as a map from $L^2((0, \pi), L^2(S^n))$ to $L^2(M)$. Under Φ the Laplacean Δ on $C_0^\infty(M \setminus \{N, S\})$ transforms to*

$$-\partial_x^2 + x^{-2} A(x) \quad \text{on } C_0^\infty((0, \pi), C^\infty(S^n)),$$

where

$$A(x) = \tilde{f}(x)^{-2} \Delta_{S^n} + \tfrac{1}{2}n\big(\tfrac{1}{2}n - 1\big) + \tfrac{1}{2}n^2 x F(x) + x^2\big(\tfrac{1}{4}n^2 F(x)^2 + \tfrac{1}{2}nF'(x)\big)$$

and $F(x) = \partial_x \log \tilde{f}(x)$.

Next we restrict Φ to the space $C_0^\infty((0, \pi), C^\infty(S^n)^G)$ and we obtain a bijection

$$\Phi_G : C_0^\infty\big((0, \pi), C^\infty(S^n)^G\big) \to C_0^\infty(M \setminus \{N, S\})^G.$$

Again, Φ_G extends to a unitary map $L^2([0, \pi], L^2(S^n)^G) \to L^2(M)^G$ which we also denote by Φ_G. It follows from Lemma 3.2 that Δ on $C_0^\infty(M \setminus \{N, S\})^G$ transforms to

$$-\partial_x^2 + x^{-2} A(x)^G \quad \text{on } C_0^\infty\big((0, \pi), C^\infty(S^n)^G\big),$$

where

$$A(x)^G := \tilde{f}(x)^{-2}\,\Delta_{S^n}^G + \tfrac{1}{2}n\left(\tfrac{1}{2}n-1\right) + \tfrac{1}{2}n^2xF(x) + x^2\left(\tfrac{1}{4}n^2F(x)^2 + \tfrac{1}{2}nF'(x)\right)$$
$$=: \tilde{f}(x)^{-2}\,\Delta\delta_{S^n}^G + q_n(x) \tag{3.5}$$

and $\Delta_{S^n}^G$ denotes the restriction of Δ_{S^n} to $C^\infty(S^n)^G$. It is easy to see that the operator family $A(x)^G$ satisfies the assumptions in [2, Section 1]; in particular, it is commutative. Next we extend $A(x)^G$ to an operator function $\tilde{A}(x)$ on $\mathbb{R}_+$ satisfying the same assumptions. We denote by T the Friedrichs extension of the symmetric differential operator

$$\tau := -\partial_x^2 + x^{-2}\tilde{A}(x)$$

with domain $C_0^\infty(\mathbb{R}^*, H_1)$ in $L^2(\mathbb{R}_+, H)$ where $H := L^2(S^n)$ and $H_1 := H^2(S^n)^G$, the space of G-invariant functions in the Sobolev space $H^2(S^n)$. We relate this operator to our problem.

Lemma 3.3. *Let $\varphi \in C^\infty(M)$ with $\varphi = \tilde{\varphi} \circ \sigma$, $\tilde{\varphi} \in C_0^\infty(-\pi, \pi)$. Then, as $t \to 0+$,*

$$\mathrm{tr}_{L^2(M)^G}\varphi\ \mathrm{e}^{-t\Delta^G} \sim \mathrm{tr}_{L^2(\mathbb{R}_+,\,H)}\tilde{\varphi}\ \mathrm{e}^{-tT}.$$

Proof. Suppose that we already know that

$$\tilde{\varphi}\boldsymbol{\Phi}_G^{-1}u \in \mathscr{D}(T) \quad \text{if } u \in \mathscr{D}(\Delta^G). \tag{3.6}$$

Choosing $\tilde{\psi} \in C_0^\infty(-\pi, \pi)$ with $\tilde{\psi} \equiv 1$ in a neighborhood of supp $\tilde{\varphi}$ we try

$$\tilde{\psi}\boldsymbol{\Phi}_G^{-1}\ \mathrm{e}^{-t\Delta^G}\boldsymbol{\Phi}_G\tilde{\varphi}$$

as a parametrix for $\partial_t + T$ near 0. It follows from Duhamel's principle (cf. [10, Theorem 6.1]), and from the rapid decay of $\Gamma_t(p, q)$ and its derivatives off the diagonal that

$$\mathrm{tr}_{L^2}\tilde{\varphi}\ \mathrm{e}^{-tT} - \mathrm{tr}_{L^2}\tilde{\psi}\boldsymbol{\Phi}_G^{-1}\ \mathrm{e}^{-t\Delta^G}\boldsymbol{\Phi}_G\tilde{\varphi} = \mathrm{O}_N(t^N), \quad t \to 0+,$$

for all N (here and in what follows tr_{L^2} stands for $\mathrm{tr}_{L^2(\mathbb{R}_+,H)}$). Since $\tilde{\psi}\tilde{\varphi} = \tilde{\varphi}$ and $\tilde{\varphi}\boldsymbol{\Phi}_G^{-1} = \boldsymbol{\Phi}_G^{-1}\varphi$, and since $\boldsymbol{\Phi}_G$ is unitary, we obtain

$$\mathrm{tr}_{L^2}\tilde{\varphi}\boldsymbol{\Phi}_G^{-1}\ \mathrm{e}^{-t\Delta^G}\boldsymbol{\Phi}_G = \mathrm{tr}_{L^2(M)^G}\varphi\ \mathrm{e}^{-t\Delta^G}.$$

So it only remains to prove (3.6). By [7, Theorem 6.1] it is enough to prove that

$$\left\|\tilde{\varphi}(x)\boldsymbol{\Phi}_G^{-1}u(x)\right\|_{L^2}^2 = \mathrm{O}(x) \quad \text{as } x \to 0, \quad \text{for } u \in \mathscr{D}(\Delta^G).$$

Now it has been shown in [4, Lemma 2.2] that $\mathscr{D}(\Delta^G) = H^2(M)^G$. By (3.4) we have for $u \in C^\infty(M)$

$$\tilde{\varphi}(x)\boldsymbol{\Phi}_G^{-1}u(x) = \tilde{\varphi}(x)\left(x\tilde{f}(x)\right)^{n/2}u(x, \cdot) =: x^{n/2}v(x)$$

with $v(x) = 0$ for x near π. Next we observe that

$$\int_0^\pi x^n \left\|\partial_x v(x)\right\|_H^2 \mathrm{d}x = \int_0^\pi x^n \int_{S^n} \left|\partial_x\left(\tilde{\varphi}(x)\tilde{f}(x)^{n/2}u(x, \omega)\right)\right|^2 \mathrm{d}\omega\ \mathrm{d}x$$
$$\leqslant C\left(\left\|\nabla u\right\|_{L^2(M)}^2 + \left\|u\right\|_{L^2(M)}^2\right) = C\left\|u\right\|_{H^1(M)}^2$$

where $d\omega$ denotes the standard volume element on S^n. For $u \in H^2(M)$ and small x we thus find

$$x^n \| v(x) \|_H^2 = x^n \left\| \int_x^\pi \partial_t v(t)\, dt \right\|_H^2 \leqslant C_u x^n \int_x^\pi t^{-n}\, dt \leqslant C_u x$$

since $n \geqslant 2$. This completes the proof. □

The computation of expansion (3.3) can now be carried out combining [2, Lemma 3.3 and Theorem 2.1]. For $\tilde{\varphi} \in C_0^\infty(\mathbb{R}^*)$ we obtain as in [2, Equation (2.20)]

$$\begin{aligned} \mathrm{tr}_{L^2} \tilde{\varphi}\, e^{-tT} &= \int_0^\infty \tilde{\varphi}(x)\, \mathrm{tr}_H\, e^{-tT}(x, x)\, dx \\ &\sim (4\pi t)^{-1/2} \sum_{j \geqslant 0} t^j \int_0^\infty \tilde{\varphi}(x)\, \mathrm{tr}_H U_j(x)\, e^{-tx^{-2}\tilde{A}(x)}\, dx, \end{aligned} \tag{3.7}$$

where the U_j are universal polynomials in the derivatives of $\tilde{A}$. So the asymptotic expansion of $\mathrm{tr}_{L^2} \tilde{\varphi}\, e^{-tT}$ is reduced to the pointwise expansion of $\mathrm{tr}_H U_j(x)\, e^{-tx^{-2}\tilde{A}(x)}$. For $\tilde{\varphi} \in C_0^\infty(\mathbb{R})$ we can use the scaling property

$$e^{-tT}(x, x) = x^{-1}\, e^{-tx^{-2}T_x}(1, 1)$$

of T (cf. [7, Section 4]) to get

$$\mathrm{tr}_{L^2} \tilde{\varphi}\, e^{-tT} = \int_0^\infty \tilde{\varphi}(x) x^{-1}\, \mathrm{tr}_H\, e^{-tx^{-2}T_x}(1, 1)\, dx. \tag{3.8}$$

Here T_y is the Friedrichs extension of the "scaled operator"

$$\tau_y := -\partial_x^2 + x^{-2}\tilde{A}(yx).$$

As already noted in [2, Section 2] the Singular Asymptotics Lemma of [6] can now be applied to

$$\sigma(x, \zeta) := \tilde{\varphi}(x)\zeta^{-1}\, \mathrm{tr}_H\, e^{-\zeta^{-2}T_x}(1, 1)$$

which by [2, Theorem 2.1] has the following asymptotic expansion as $\zeta \to \infty$:

$$\sigma(x, \zeta) \sim \tilde{\varphi}(x)(4\pi)^{-1/2} \sum_{j \geqslant 0} \zeta^{-2j}\, \mathrm{tr}_H \left[U_j(x)\, e^{-\zeta^{-2}\tilde{A}(x)} \right]. \tag{3.9}$$

By (3.5) we have $\tilde{A}(x) = \tilde{f}(x)^{-2}\, \Delta_{S^n}^G + q_n(x)$ for small x with some smooth function q_n, so we can write in view of $\tilde{f} > 0$

$$\tilde{A}^{(k)}(x) =: a_k(x)\tilde{A}(x) + b_k(x)$$

with smooth functions a_k and b_k. Therefore, $U_j(x)$ is a polynomial in $\tilde{A}(x)$ with smooth coefficients. These coefficients are, in turn, easily seen to be universal polynomials in $\tilde{f}^{-1}$ and the derivatives $\tilde{f}^{(k)}$. Now it follows from the methods used in the proof of Theorem 5 in [5] that $\mathrm{tr}_{L^2(S^n)^G}[(\Delta_{S^n})^k\, e^{-t\Delta_{S^n}^G}]$ has an asymptotic expansion without logarithmic terms for all k; in fact,

$$\mathrm{tr}_{L^2(S^n)^G} \left[\left(\Delta_{S^n}^G\right)^k e^{-t\Delta_{S^n}^G} \right] = (-\partial_t)^k\, \mathrm{tr}_{L^2(S^n)^G}\, e^{-t\Delta_{S^n}^G}.$$

Inserting this in (3.9) we obtain an expansion

$$\sigma(x, \zeta) \sim \sum_{j \geqslant -n} \sigma_j(x)\zeta^{-j} \tag{3.10}$$

with $\sigma_j \in C^\infty(\mathbb{R}_+)$. Then the Singular Asymptotics Lemma gives the logarithmic contributions

$$-\tfrac{1}{2}t^{k/2}\log t\frac{1}{(k-1)!}\partial_x^{k-1}\sigma_k(0) =: -\tfrac{1}{2}t^{k/2}\log t\, a_k, \quad k \geqslant 1. \tag{3.11}$$

To handle the coefficients a_k we need the asymptotic expansion of $\operatorname{tr}_H e^{-t\Delta_{S^n}^G} =: \operatorname{tr}_H e^{-tA}$ more explicitly.

Lemma 3.4. *Let $\rho : G \to \mathrm{Aut}(V)$ be a finite-dimensional irreducible representation with multiplicity* $\dim \mathrm{Hom}_G(V, E_\lambda)$ *in the complexified eigenspace E_λ of Δ_{S^n} with eigenvalue λ. Then*

$$\operatorname{tr}_H e^{-tA} = \sum_{\lambda \geqslant 0} e^{-\lambda t} \dim \mathrm{Hom}_G(V, E_\lambda)$$

has the asymptotic expansion

$$\operatorname{tr}_H e^{-tA} \sim \sum_{i=0}^{(n-1)/2} c_i t^{-i-1/2} e^{(n-1)^2 t/4} + t^{-[n/2]} \sum_{k \geqslant 0} d_k t^k \tag{3.12}$$

with certain coefficients c_i depending on ρ.

Proof. Recall that Δ_{S^n} has eigenvalues $\lambda_k = k(k+n-1)$. By [5, Section 5, Proposition 1] there exists an integer $m \geqslant 1$ and polynomials $P_0, \ldots, P_{m-1} \in \mathbb{Q}[x]$ of degree at most $n-1$, such that $\dim \mathrm{Hom}_G(V, E_\lambda) = P_r(k) =: \sum_{i=0}^{m-1} c_{ir} k^i$ if $k \equiv r \bmod m$ and k is sufficiently large. Hence we can write

$$\begin{aligned}\operatorname{tr}_H e^{-tA} &= \sum_{r=0}^{m-1} \sum_{k \geqslant k_0} e^{-t(mk+r)(mk+r+n-1)} P_r(mk+r) + q(t) \\ &=: \sum_{r=0}^{m-1} \sum_{i=0}^{n-1} \sum_{k \geqslant k_0} \tilde{c}_{ir} e^{-tm^2(k+\alpha_r)^2} (k+\alpha_r)^i e^{(n-1)^2 t/4} + q(t)\end{aligned}$$

where q is an entire function in t and $\alpha_r := (2r+n-1)/2m$, $\tilde{c}_{ir} \in \mathbb{Q}$. Expansion (3.12) now follows by bringing in the asymptotic expansions

$$\sum_{k \geqslant 0} e^{-t(k+\alpha)^2} (k+\alpha)^{2l} \sim :\tfrac{1}{2}\Gamma\left(l+\tfrac{1}{2}\right) t^{-l-1/2} + f_{\alpha,2l}(t) \tag{3.13a}$$

and

$$\sum_{k \geqslant 0} e^{-t(k+\alpha)^2} (k+\alpha)^{2l+1} \sim :\tfrac{1}{2}\Gamma(l+1) t^{-l-1} + f_{\alpha,2l+1}(t) \tag{3.13b}$$

with certain explicitly calculable functions $f_{\alpha,i} \in C^\infty(\mathbb{R}_+)$. These expansions can be derived from [5, Lemma 13] or the representation of e^{-x} as an inverse Mellin transform (cf. [8, p. 50]) and known properties of the Riemann ζ-function (there is a misprint in [5, Lemma 13(2)]: the factor $(-1)^n$ has to be deleted). □

If we put $B := A + \frac{1}{2}n(\frac{1}{2}n-1)$, then expansion (3.12) becomes

$$\operatorname{tr}_H e^{-tB} \sim e^{t/4} \sum_{i=0}^{(n-1)/2} c_i t^{-i-1/2} + t^{-[n/2]} \sum_{k \geqslant 0} d_k t^k. \tag{3.13}$$

Lemma 3.5. *The coefficients a_k of the logarithmic terms in* (3.11) *have the form*

$$a_k = \sum_{j=0}^{n} b_j p_j^k \tag{3.14}$$

where the coefficients b_j depend only on G and ρ, and p_i^k are universal polynomials in the derivatives of $\tilde{\varphi}$ and $\tilde{f}$ at 0, independent of G and ρ.

Proof. By (3.11), (3.10) and (3.9), a_k is the coefficient of $t^{k/2}$ in the asymptotic expansion of

$$(4\pi)^{-1/2} \sum_{j\geqslant 0} t^j \frac{1}{(k-1)!} \partial_x^{k-1} \Big|_{x=0} \left(\tilde{\varphi}(x)\ \mathrm{tr}_H\left[U_j(x)\ \mathrm{e}^{-t\tilde{A}(x)}\right]\right).$$

Since we can assume $\tilde{\varphi}$ to be an even function, and since, by (3.5), U_j and $\tilde{A}$ are even too, it follows that σ_j is even for all j. Thus $\alpha_{2k} = 0$ for $k \in \mathbb{N}$. Now by

$$\tilde{A}(x) = \tilde{f}(x)^{-2} B + q_n(x) - \tfrac{1}{2}n\left(\tfrac{1}{2}n - 1\right)\tilde{f}(x)^{-2}$$

and by [2, Lemma 2.2], we have

$$U_j(x) = \sum_{i=0}^{[2j/3]} c_{ji}(x) B^i,$$

where the c_{ji} are even functions with $c_{ji}^{(l)}(0)$ a universal polynomial in the even derivatives of $\tilde{f}$ at 0. From this we obtain easily by induction

$$\frac{1}{(k-1)!} \partial_x^{k-1}\left(U_j(x)\ \mathrm{e}^{-t\tilde{A}(x)}\right) = \sum_{i=0}^{k-1+[2j/3]} d_{ijk}(x) B^i\ \mathrm{e}^{-t\tilde{A}(x)}$$

where $d_{ijk}(0)$ is again a universal polynomial in the derivatives $\tilde{f}^{(2r)}(0)$. Thus a_{2k+1} equals the coefficient of $t^{k+1/2}$ in the asymptotic expansion of

$$\begin{aligned}(4\pi)^{-1/2} &\sum_{\substack{j\geqslant 0 \\ 0\leqslant i\leqslant k-1+[2j/3]}} t^j d_{ijk}(0)\ \mathrm{tr}_H B^i\ \mathrm{e}^{-tB} \\ &= (4\pi)^{-1/2} \sum_{\substack{j\geqslant 0 \\ 0\leqslant i\leqslant k-1+[2j/3]}} t^j d_{ijk}(0)(-1)^i\ \partial_t^i\ \mathrm{tr}_H\ \mathrm{e}^{-tB}.\end{aligned}$$

Plugging in (3.13) we obtain the desired result. □

Next we show that the constants b_i in (3.14) can be controlled by choosing suitable group actions on S^n. We view S^n as the unit sphere in $T_N M \simeq \mathbb{R}^{n+1}$. Then, for $k = 1, \ldots, n$, $\mathrm{SO}(k)$ acts on S^n by $g(x_1, \ldots, x_{n+1}) = (g(x_1, \ldots, x_k),\ x_{k+1}, \ldots, x_{n+1})$ where on $\mathbb{R}^k$ we take the standard effective action; here $\mathrm{SO}(1) := \mathbb{Z}_2$ acts by $x_1 \mapsto -x_1$. Then $\dim(S^n/\mathrm{SO}(k)) = n + 1 - k$, and from [5, Theorems 4 and 5] we obtain the asymptotic expansion

$$\mathrm{tr}_H\ \mathrm{e}^{-t\Delta_{S^n}^G} \sim (4\pi t)^{-(n+1-k)/2} \sum_{j\geqslant 0} a_j t^{j/2}$$

with $a_0 = \mathrm{vol}(S^n/\mathrm{SO}(k)) \neq 0$. Therefore, for $0 \leqslant l \leqslant [\frac{1}{2}(n-1)]$ there is a closed subgroup $G_l \subset \mathrm{SO}(n+1)$ with

$$\mathrm{tr}_H\, \mathrm{e}^{-t\Delta_{S^n}^{G_l}} \sim \sum_{i=0}^{l} b_i t^{-i-1/2}\, \mathrm{e}^{(n-1)^2 t/4} + t^{-l} \sum_{k\geqslant 0} c_k t^k$$

and $b_l \neq 0$; in fact, we only have to choose $G_l := \mathrm{SO}(n-2l)$. Since the coefficients a_{2k+1} are linear combinations of the b_i's, the proof of Theorem 3.1 will be completed if we can show that the action of these special groups G_l on M produces no log terms. The proof of this fact is similar to the proof of [5, Theorem 7] and is given in the following lemma.

Lemma 3.6. *Let, for $1 \leqslant k \leqslant n$, $\mathrm{SO}(k)$ act on M as above. Then, with $\varphi \in C^\infty(M)^{\mathrm{SO}(k)}$ and χ a character of $\mathrm{SO}(k)$, the integral*

$$I(t) := \int_{M\times \mathrm{SO}(k)} \varphi(p)\Gamma_t(p, gp)\bar{\chi}(g)\, \mathrm{d}g\, \mathrm{d}p \tag{3.15}$$

has an asymptotic expansion as $t \to 0+$ without logarithmic terms.

Proof. Since $\mathrm{SO}(k)$ acts by isometries, the $\mathrm{SO}(k)$-integrand in (3.15) is a class function. Thus we can apply the Weyl integration formula (cf. [1, p. 56]): if T_k denotes a maximal torus of $\mathrm{SO}(k)$, there is an $f \in C^\infty(M \times T_k)$ such that

$$I(t) = \int_{M\times T_k} \Gamma_t(p, \vartheta p) f(p, \vartheta)\, \mathrm{d}\vartheta\, \mathrm{d}p. \tag{3.16}$$

We observe that we may assume $f \in C_0^\infty(U \times T_k)$ for an arbitrary small T_k-invariant neighborhood U of N. In fact, if $f \in C_0^\infty(M\setminus\{N, S\} \times T_k)$, then (3.15) has no logarithmic terms by [5, Theorem 4] since T_k acts freely on a neighborhood of the projection of supp f in M. Moreover, M carries an equivariant isometry mapping S to N. We choose U in such a way that the Minakshisundaram–Pleijel expansion for Γ_t (cf. [5, eq. (2)]) is valid in $\overline{U} \times \overline{U}$. Inserting this expansion in (3.16) and using the exponential map we have reduced the problem to the expansion of

$$\tilde{I}(t) := \int_{\mathbb{R}^{n+1}\times T_k} \mathrm{e}^{-d^2(x,\vartheta x)/4t} \tilde{f}(x, \vartheta)\, \mathrm{d}\vartheta\, \mathrm{d}x \tag{3.17}$$

where $\tilde{f} \in C_0^\infty(\mathbb{R}^{n+1} \times T_k)$ and d is the Riemannian distance with respect to the metric $\exp_N^* g$ on $\mathbb{R}^{n+1}$, g the metric on M. Now recall that T_k has rank $[\frac{1}{2}k] =: \tilde{k}$. Writing

$$x = (x_1, \ldots, x_{n+1}) =: (y_1, z_1, \ldots, y_{\tilde{k}}, z_{\tilde{k}}, x_{2\tilde{k}+1}, \ldots, x_{n+1}) =: (x', x''),$$

the action of $T_k \cong \mathbb{R}^{\tilde{k}}/\mathbb{Z}^{\tilde{k}}$ is given by

$$(\vartheta_1, \ldots, \vartheta_{\tilde{k}})x = (y_1 \cos 2\pi\vartheta_1 - z_1 \sin 2\pi\vartheta_1, \ldots, y_{\tilde{k}} \sin 2\pi\vartheta_{\tilde{k}} + z_{\tilde{k}} \cos 2\pi\vartheta_{\tilde{k}}, x'').$$

Since d is comparable to the Euclidean distance, we obtain the inequality

$$\frac{1}{C}\sum_{i=1}^{\tilde{k}} (y_i^2 + z_i^2)\sin^2\tfrac{1}{2}\vartheta_i \leqslant d^2(x, \vartheta x) \leqslant C\sum_{i=1}^{\tilde{k}} (y_i^2 + z_i^2) \sin^2\tfrac{1}{2}\vartheta_i \tag{3.18}$$

for $(x, \vartheta) \in \operatorname{supp} \tilde{f}$. We write $r_i^2 := y_i^2 + z_i^2$ and $r^2 := \sum_{i=1}^{\tilde{k}} r_i^2 = |x'|^2$. If V is a neighborhood of 0 in T_k, we derive from (3.18) for $(x, \vartheta) \in \operatorname{supp} \tilde{f}$, $\vartheta \notin V$ the inequality

$$\frac{1}{C_V} r^2 \leqslant d^2(x, \vartheta x) \leqslant C_V r^2.$$

Then it follows from Taylor's formula that, with polar coordinates in $\mathbb{R}^{2\tilde{k}}$, $x' = r\omega$ with $(r, \omega) \in \mathbb{R}_+ \times S^{2\tilde{k}-1}$, we have

$$d^2(x, \vartheta x) = a(r, \omega, x'', \vartheta) r^2$$

with $a \geqslant 1/C_V$, where a is a smooth function in all variables for $(x, \vartheta) \in \operatorname{supp} \tilde{f}$, $\vartheta \in V$. Thus, for $\operatorname{supp} \tilde{f} \in \mathbb{R}^{n+1} \times (T_k \setminus V)$, the asymptotic expansion of (3.17) reduces to Gaussian asymptotics which do not involve logarithmic terms. We may, therefore, assume that $\tilde{f} \in C_0^\infty(\mathbb{R}^{n+1} \times V)$ for an arbitrarily small neighborhood V of 0 in T_k. We choose V such that (3.18) implies the estimate

$$\frac{1}{C} \sum_{i=1}^{\tilde{k}} r_i^2 \vartheta_i^2 \leqslant d^2(x, \vartheta x) \leqslant C \sum_{i=1}^{\tilde{k}} r_i^2 \vartheta_i^2 \tag{3.19}$$

in $\overline{U} \times \overline{V}$, and in addition invariant under the reflections in the coordinate hyperplanes. Thus the group $G := T_k \times \mathbb{Z}_2^{\tilde{k}}$ acts orthogonally on $U \times V \subset \mathbb{R}^{n+1} \times \mathbb{R}^{\tilde{k}}$. By construction, $d^2(x, \vartheta x)$ is invariant under this action. The algebra $\mathscr{P}^G$ of G-invariant polynomials is obviously generated by the functions r_i^2, ϑ_j^2, x_l'', $1 \leqslant i, j \leqslant \tilde{k}$, $2\tilde{k} + 1 \leqslant l \leqslant n + 1$. Hence, it follows from a theorem of Schwarz [12] that

$$d^2(x, \vartheta x) = h\left(r_1^2, \ldots, r_{\tilde{k}}^2, \vartheta_1^2, \ldots, \vartheta_{\tilde{k}}^2, x''\right)$$

for some function $h \in C^\infty(\mathbb{R}_+^{\tilde{k}} \times \mathbb{R}_+^{\tilde{k}} \times \mathbb{R}^{n-2\tilde{k}})$. Invoking Lemma 3.7 below, it follows that

$$d^2(x, \vartheta x) = \sum_{i=1}^{\tilde{k}} r_i^2 \vartheta_i^2 h_i(x, \vartheta)$$

where h_i is smooth and w.l.o.g. positive on the support of $\tilde{f}$. Introducing polar coordinates (r_i, ω_i) in the (y_i, z_i)-integral in (3.17) and substituting $\tilde{r}_i := h_i(x, \vartheta)^{1/2} r_i$ we reduce the problem to the situation of [5, Lemma 14]. It follows that (3.17) does not have logarithmic terms in the asymptotic expansion and the proof is complete. □

Finally, we need the following technical result which was used in the proof of Lemma 3.6.

Lemma 3.7. *Let $h \in C^\infty(\mathbb{R}_+^n \times \mathbb{R}_+^n \times \mathbb{R}^m)$ satisfy*

$$|h(x, y, z)| \leqslant C \sum_{i=1}^{n} x_i y_i, \quad (x, y, z) \in \mathbb{R}_+^n \times \mathbb{R}_+^n \times \mathbb{R}^m.$$

Then there are functions $h_i \in C^\infty(\mathbb{R}_+^n \times \mathbb{R}_+^n \times \mathbb{R}^m)$ such that

$$h(x, y, z) = \sum_{i=1}^{n} x_i y_i h_i(x, y, z), \quad (x, y, z) \in \mathbb{R}_+^n \times \mathbb{R}_+^n \times \mathbb{R}^m.$$

Proof. We interpret the sums to be 0 if $n = 0$. Then, for $n = 0$, there is nothing to prove. Suppose we have proved the lemma for some n with $n - 1 \geqslant 0$. From Taylor's formula we have, with $x = (x', x_n)$, $y = (y', y_n)$,

$$h(x, y, z) = h(x', 0, y', 0, z) + x_n h'(x, y, z) + y_n h''(x, y, z)$$

where h', h'' are smooth. By assumption and the induction hypothesis, we have with certain smooth functions $\tilde{h}_i$

$$h(x', 0, y', 0, z) + x_n h'(x, y', 0, z) + y_n h''(x', 0, y, z) = \sum_{i=1}^{n-1} x_i y_i \tilde{h}_i(x, y, z),$$

so the assertion follows from Taylor's formula again. □

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