(a) The conclusion of Theorem 4.5 remains also true if instead of the injectivity of the canonical map $\mathrm{Vect}(X) \to K^0(X)$ we assume that X is a finite CW-complex and that the K_0-groups of the AF-fibres of the continuous fields $\mathscr{E}_L$ and $\mathscr{E}_{L'}$ are with large denominators, in the sense of V. Nistor: On the homotopy group of the automorphisms group of AF-C^*-algebras (to appear in *J. Operator Theory*).

(b) Since the simple AF-algebras have the K_0-groups with large denominators, the conclusion of Theorem 4.6 also holds if instead of the injectivity of the canonical map $\mathrm{Vect}(X) \to K^0(X)$ we assume that X is a finite CW-complex.

In addition to the previous arguments, the proofs of these statements use the stability properties of vector bundles over finite CW-complexes [9].

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On Schrödinger Operators with Discrete Spectrum

JOCHEN BRÜNING

Department of Mathematics, Institut für Mathematik, Memminger Str. 6, D-8900 Augsburg, Federal Republic of Germany

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1. INTRODUCTION

In this note we study the discreteness of operator valued Schrödinger equations. Consider the differential operator

$$\tau := -\partial_x^2 + q \tag{1.1}$$

with a locally integrable potential q acting on $C_0^\infty(\mathbb{R}^*)$, $\mathbb{R}^* := (0, \infty)$. Then τ is symmetric in the Hilbert space $L^2(\mathbb{R}_+)$, $\mathbb{R}_+ := [0, \infty)$; if we also know that τ is bounded from below then the Friedrichs extension T of τ exists and is self-adjoint in $L^2(\mathbb{R}_+)$ with the same lower bound. An extensive literature is devoted to the study of conditions on q which ensure the boundedness below of τ and the discreteness of T (recall that a self-adjoint operator is called discrete if its spectrum consists only of isolated eigenvalues with finite multiplicity). Apparently the subject started with the work of Weyl [W] and Titchmarsh [TI]: a sufficient condition for (boundedness below of τ and) discreteness of T is

$$\lim_{x \to \infty} q(x) = \infty. \tag{1.2}$$

This has been improved considerably by Molchanov [MO]: assuming

$$q(x) \geqslant q_0, \qquad x \in \mathbb{R}_+, \tag{1.3}$$

(which clearly implies the boundedness below of τ) a necessary and sufficient condition for the discreteness of T is

$$\lim_{x \to \infty} \int_x^{x+\varepsilon} q(y)\, dy = \infty \qquad \text{for} \quad 0 < \varepsilon \leqslant 1. \tag{1.4}$$

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Brinck [BRI] showed that the condition

$$\int_x^{x+\varepsilon} q(y)\,dy \geqslant -C^2 \qquad \text{for all} \quad x \geqslant 0 \text{ and } 0 < \varepsilon \leqslant 1 \tag{1.5}$$

for some constant C, independent of x and ε, implies the boundedness below of τ, and that also in this case (1.4) is necessary and sufficient for the discreteness of T. This was further generalized by Ismagilov [I] in the following way. Assume that for some $0 < h \leqslant 1$ and $0 \leqslant x \leqslant t \leqslant x+h$

$$\int_x^t q(y)\,dy \geqslant \alpha(t) - \beta(x), \tag{1.6a}$$

where α and β satisfy

$$\int_x^t [\alpha(y)^2 + \beta(y)^2]\,dy \leqslant C^2. \tag{1.6b}$$

Then τ is bounded below and (1.4) is sufficient but no longer necessary for discreteness (cf. [I, p. 1140]). This is, however, the case with the following condition:

> denote by K_x^ε the square $\{(y,t) | 0 \leqslant t-y \leqslant \varepsilon,\ x \leqslant t+y \leqslant x+\varepsilon\}$ then with μ the Lebesgue measure in $\mathbb{R}^2$ we have
>
> $$\lim_{x\to\infty} \mu\left[\left\{(y,t)\,\middle|\, \int_y^t q(z)\,dz < M\right\} \cap K_x^\varepsilon\right] = 0 \tag{1.7}$$
>
> for all $0 < \varepsilon \leqslant 1$ and $M > 0$.

The most general criterion (as far as we know) has been given by Zelenko [Z]. Assume that $q = q_1 + q_2$ where

> q_1 is locally integrable and satisfies the conditions (1.6) and $q_2(x) \geqslant \gamma'(x) + 2\gamma^2(x)$ for some absolutely continuous function $\gamma\colon \mathbb{R}_+ \to \mathbb{R}$. (1.8)

Then τ is bounded below and (1.7) for q_1 is sufficient for the discreteness of T. On the other hand, if T is *assumed* to be semibounded then (1.7) is necessary for the discreteness of T.

We now turn to the case of an operator potential; i.e., we consider a family $Q(x)$, $x \in \mathbb{R}_+$, of self-adjoint operators in a fixed Hilbert space H with common domain H_1 independent of x. Under suitable assumptions on

Q (to be stated in the next section) we obtain an operator τ on $C_0^\infty(\mathbb{R}^*, H_1)$ by setting

$$\tau u(x) := -u''(x) + Q(x)\,u(x), \qquad x > 0. \tag{1.9}$$

τ is symmetric in the Hilbert space $L^2(\mathbb{R}_+, H)$ and again the Friedrichs extension T is defined once we know that τ is bounded below. One reason to be interested in the spectral properties of these operator valued equations is that they cover as a major application the Laplacean on certain noncompact manifolds; we will turn to this question in Section 4 below. Considerable work has been done by Russian mathematicians in generalizing the results from the scalar case (cf. the article of Maslov [MA] and the references given there). If $Q(x)$ in (1.9) is assumed to be bounded below with lower bound $q(x)$ then all the conditions (1.2) through (1.8) make sense, but so far it seems that with the exception of [KL] only operator potentials Q satisfying (1.3) have been considered. Under this assumption (which implies the boundedness below of τ) it has been shown by Levitan and Suvorchenkova [L+S] that the Molchanov condition (1.4) on the lower bound q is sufficient for the discreteness of T. Guided by various examples of either necessary or sufficient conditions Maslov [MA] gave a condition which is both necessary and sufficient: if the lower bound q of Q satisfies (1.3) we introduce the function

$$q_M(x,\varepsilon) := \inf\left\{\int_x^{x+\varepsilon} \langle Q(y)\,u(y), u(y)\rangle\,dy \,\middle|\, u \in C^\infty([x, x+\varepsilon], H_1),\right.$$
$$\left. \|u(y)\| = 1,\ y \in [x, x+\varepsilon],\ \int_x^{x+\varepsilon} \|u'(y)\|^2\,dy \leqslant \frac{1}{16\varepsilon}\right\}. \tag{1.10}$$

Then Maslov's result is that

$$\lim_{x\to\infty} q_M(x,\varepsilon) = \infty \qquad \text{for all} \quad 0 < \varepsilon \leqslant 1 \tag{1.11}$$

is necessary and sufficient for the discreteness of T if (1.3) is satisfied. As pointed out by Maslov the main difficulty in the operator case consists in the fact that simple necessary and sufficient criteria in terms of $Q(x)$, generalizing the various conditions from the scalar case, are not available and one has to introduce functions like (1.10). Maslov's work has been extended in [KL] where it has been shown that (1.3) can be generalized to

$$\int_x^{x+\varepsilon} q_-(y)\,dy \geqslant -C^2 \qquad \text{for all} \quad x \geqslant 0 \text{ and } 0 < \varepsilon \leqslant 1; \tag{1.12}$$

here q is again the lower bound of Q and $q_-(x) := \min\{q(x), 0\}$.

The purpose of this note is to present results that unify and extend the previous work just described. In the operator case, the main observation is the following. Instead of imposing conditions on the lower bound q of Q we look for a priori estimates for the bilinear form generated by Q in $L^2(\mathbb{R}_+, H)$. If τ is bounded from below we have for $u \in C_0^\infty(\mathbb{R}^*, H_1)$

$$\int_0^\infty \langle Q(x)u(x), u(x)\rangle \, dx \geqslant -\|u'\|_{L^2}^2 - C\|u\|_{L^2}^2.$$

It turns out that the above-mentioned conditions on q, in particular the most general Zelenko condition (1.8), imply the estimate

$$\int_0^\infty \langle Q(x)\,u(x), u(x)\rangle \, dx \geqslant (\delta - 1)\,\|u'\|_{L^2}^2 - C\,\|u\|_{L^2}^2$$

or equivalently

$$(\tau u, u) \geqslant \delta\,\|u'\|_{L^2}^2 - C\,\|u\|_{L^2}^2 \tag{1.13}$$

for $u \in C_0^\infty(\mathbb{R}^*, H_1)$ with $\operatorname{supp} u$ contained in an interval of length h, $0 \leqslant h \leqslant h_0 \leqslant 1$, and some $\delta > 0$. Under this "coerciveness" assumption on τ we give a necessary and sufficient criterion for the discreteness of T of Maslov type (Theorem 3.1 below) from which the results described above can be derived rather easily. In the scalar case, we give a necessary and sufficient condition for the discreteness of T assuming only that q is locally integrable and that τ is bounded below (Theorem 3.4). The structure of the condition is natural if one studies the Laplacean on noncompact complete manifolds with nice ends. In the latter case we also obtain new necessary or sufficient conditions (Theorems 4.1–4.3).

2. General Discreteness

The general discreteness criterion applies to the following situation. For each $x \geqslant 0$ we are given a self-adjoint operator $Q(x)$ in some Hilbert space H, with domain H_1 independent of x. We assume that the map

$$\mathbb{R}_+ \ni x \mapsto Q(x) \in \mathcal{L}(H_1, H) \tag{2.1}$$

is Bochner integrable. Then we introduce the symmetric operator

$$\tau := -\partial_x^2 + Q(x) \tag{2.2}$$

with domain $C_0^\infty(\mathbb{R}^*, H_1)$ in the Hilbert space $L^2(\mathbb{R}_+, H)$. To guarantee

the existence of self-adjoint extensions of τ we further assume that τ is bounded from below; for simplicity we assume lower bound 0, i.e.,

$$(\tau u, u) \geqslant 0 \qquad \text{for} \quad u \in C_0^\infty(\mathbb{R}^*, H_1). \tag{2.3}$$

Then the Friedrichs extension T of τ exists and is self-adjoint in $L^2(\mathbb{R}_+, H)$. For the proof of our first criterion we prepare a simple lemma.

Lemma 2.1. *Let $0 \leqslant a < b < \infty$ and $u \in C^\infty([a, b], H_1)$ with $u(a) = u(b) = 0$. Then there is a sequence $(u_k)_{k \geqslant 1} \subset C_0^\infty((a, b), H_1)$ such that*

$$u_k(x) \to u(x) \qquad \text{in } H_1, \text{ uniformly in } x \in [a, b],$$
$$u_k' \to u' \qquad \text{in } L^2([a, b], H).$$

In particular, if $e: L^2([a, b], H) \to L^2(\mathbb{R}_+, H)$ denotes extension by 0, then $e(u) \in \mathcal{D}(T^{1/2})$ and

$$e(u_k) \to e(u) \qquad \text{in } \mathcal{D}(T^{1/2}).$$

Proof. Choose a sequence $(\chi_k)_{k \geqslant k_0} \subset C_0^\infty(a, b)$ with $0 \leqslant \chi_k \leqslant 1$, $\chi_k(x) = 1$ in $(a + 2/k,\ b - 2/k)$, $\chi_k(x) = 0$ in $(a, a + 1/k) \cup (b - 1/k, b)$, and $|\chi_k^{(j)}(x)| \leqslant Ck^j$ for $j = 1, 2$. We see immediately that $u_k(x) := \chi_k u(x)$ converges to $u(x)$ in H_1, uniformly in $x \in [a, b]$. One computes for $\tilde{u}_k := u - u_k = (1 - \chi_k)u$

$$\int_a^b \|\tilde{u}_k'(x)\|_H^2 \, dx = \int_a^b [(1 - \chi_k(x))^2\,\|u'(x)\|_H^2 - (1 - \chi_k)\,\chi_k''(x)\,u(x)\|_H^2] \, dx,$$

and by the estimate

$$\|u(x)\|_H \leqslant \min\{x - a, b - x\} \max_{y \in [a,b]} \|u'(y)\|_H$$

the second integral is $O(1/k)$.

By dominated convergence it is clear that $(e(u_k))_{k \geqslant 1}$ is a Cauchy sequence in $\mathcal{D}(T^{1/2})$, and since $e(u_k) \to e(u)$ in $L^2(\mathbb{R}_+, H)$ we have $e(u_k) \to e(u)$ in $\mathcal{D}(T^{1/2})$, too. ∎

The following result is the operator analogue of Ismagilov's "localization principle" (cf. [I] or [GL, p. 39]).

Theorem 2.1. *Assume (2.1) and (2.3). The following conditions are necessary and sufficient for T to be discrete.*

(a) *For $x > 0$ denote by T_x the Friedrichs extension of $\tau | C_0^\infty((0, x), H_1)$ in $L^2([0, x], H)$; then T_x is discrete.*

(b) *For* $x \geqslant 0$ *and* $0 < \varepsilon \leqslant 1$ *introduce*

$$t(x, \varepsilon) := \inf\{(\tau u, u) \mid u \in C_0^\infty((x, x+\varepsilon), H_1),\ \|u\|_{L^2(\mathbb{R}_+, H)} = 1\}; \quad (2.4)$$

then

$$\lim_{x \to \infty} t(x, \varepsilon) = \infty.$$

Proof. (1). We show first that our conditions are necessary. As for condition (a) we recall that a self-adjoint operator is discrete iff its domain embeds compactly into the whole space. The discreteness of T is equivalent to the discreteness of $T^{1/2}$ hence it is enough to show that extension by zero maps $\mathcal{D}(T_x^{1/2})$ into $\mathcal{D}(T^{1/2})$. Now it follows from the definition of the Friedrichs extensions and [KA, Chap. V, Theorem 3.35] that to $u \in \mathcal{D}(T_x^{1/2})$ we can find a sequence $(u_n)_{n \in \mathbb{N}} \subseteq C_0^\infty((0, x), H_1)$ such that $u_n \to u$ in $L^2([0, x], H)$ and $\lim_{n,m \to \infty}(\tau(u_n - u_m),\ u_n - u_m) = 0$. Denoting by e the extension map this implies that $e(u_n) \to e(u)$ in $L^2(\mathbb{R}_+, H)$ and $\lim_{n,m \to \infty}(\tau(e(u_n) - e(u_m)),\ e(u_n) - e(u_m)) = 0$ hence $e(u) \in \mathcal{D}(T^{1/2})$.

Assume next that (b) does not hold. Then we can find $\varepsilon_0 > 0$ and sequences $(x_n)_{n \in \mathbb{N}}$, $(u_n)_{n \in \mathbb{N}}$ with $x_{n+1} > x_n + \varepsilon_0$, $u_n \in C_0^\infty((x_n, x_n + \varepsilon_0), H_1)$, $\|u_n\|_{L^2} = 1$, and such that $(\tau u_n, u_n) \leqslant C$ for some constant C and all n. But then the u_n span an infinite-dimensional subspace $\mathcal{H}$ of $\mathcal{D}(T)$ such that

$$(Tu, u) \leqslant C \qquad \text{for all} \quad u \in \mathcal{H}.$$

This implies that T cannot be discrete so (b) must hold.

(2). We turn to the sufficiency of (a) and (b). Fix $\varepsilon > 0$ and define for $u \in C_0^\infty(\mathbb{R}^*, H_1)$ and $N \in \mathbb{N}$

$$\Phi_1 u(x) := \begin{cases} \sin(x/\varepsilon)\, u(x), & 0 \leqslant x \leqslant N\pi\varepsilon, \\ 0, & x > N\pi\varepsilon, \end{cases}$$
$$\Phi_2 u(x) := \begin{cases} 0, & 0 \leqslant x \leqslant N\pi\varepsilon, \\ \sin(x/\varepsilon)\, u(x), & x > N\pi\varepsilon, \end{cases} \quad (2.5)$$

and define $\Psi_1 u$, $\Psi_2 u$ analogously using $\cos(x/\varepsilon)$ and $(2N+1)/2$ instead of $\sin(x/\varepsilon)$ and N. Then we have

$$\sum_{i=1}^{2} (\|\Phi_i u(x)\|_H^2 + \|\Psi_i u(x)\|_H^2) = \|u(x)\|_H^2. \quad (2.6)$$

We claim that for $u \in C_0^\infty \mathbb{R}^*, H_1)$

$$\Phi_i u,\ \Psi_i u \in \mathcal{D}(T^{1/2}) \qquad \text{for} \quad i = 1, 2, \quad (2.7)$$

$$\|T^{1/2} u\|_{L^2}^2 \geqslant \sum_{i=1}^{2} (\|T^{1/2}\Phi_i u\|_{L^2}^2 + \|T^{1/2}\Psi_i u\|_{L^2}^2) - \frac{1}{\varepsilon^2}\|u\|_{L^2}^2, \quad (2.8)$$

and

$$\|T^{1/2}\Phi_2 u\|_{L^2}^2 + T^{1/2}\Psi_2 u\|_{L^2}^2 \geqslant M(\|\Phi_2 u\|_{L^2}^2 + \|\Psi_2 u\|_{L^2}^2), \quad (2.9)$$

where M can be chosen arbitrary large if N is sufficiently large. Granting (2.7), (2.8), and (2.9) for the moment we now assume that T is not discrete. Then we can find $C > 0$ and a subspace $\mathcal{U} \subset \mathcal{D}(T)$, closed in $L^2(\mathbb{R}_+, H)$, of infinite dimension such that

$$\|T^{1/2} u\|_{L^2}^2 \leqslant C\,\|u\|_{L^2}^2 \qquad \text{for all} \quad u \in \mathcal{U}. \quad (2.10)$$

We may assume that $\mathcal{U}$ is an infinite-dimensional subspace of $C_0^\infty(\mathbb{R}^*, H_1)$; in fact, if $(e_i)_{i \geqslant 1}$ is an orthonormal basis of $\mathcal{U}$ in $L^2(\mathbb{R}_+, H)$ we can find $u_i \in C_0^\infty(\mathbb{R}^*, H_1)$ such that

$$\|u_i - e_i\|^2 + \|T^{1/2}(u_i - e_i)\|^2 \leqslant 2^{-i}, \qquad i \in \mathbb{N},$$

since $C_0^\infty(\mathbb{R}^*, H_1)$ is a core of $T^{1/2}$. Then the space spanned by $(u_i)_{i \geqslant 1}$ has infinite dimension and its elements satisfy (2.10), possibly with a different constant.

Combining (2.8), (2.9), and (2.10) we find for $M \geqslant M(C, \varepsilon)$

$$\|\Phi_2 u\|^2 + \|\Psi_2 u\|^2 \leqslant \tfrac{1}{2}\|u\|^2$$

hence by (2.6)

$$\|\Phi_1 u\|^2 + \|\Psi_1 u\|^2 \geqslant \tfrac{1}{2}\|u\|^2. \quad (2.11)$$

This together with (2.8) and (2.10) gives

$$\|T^{1/2}\Phi_1 u\|^2 + \|T^{1/2}\Psi_1 u\|^2 \leqslant C'(\|\Phi_1 u\|^2 + \|\Psi_1 u\|^2). \quad (2.12)$$

Now (2.12) means that for $v \in \Phi_1 \mathcal{U} \oplus \Psi_1 \mathcal{U} =: \mathcal{V}$

$$\|(T_{N\pi\varepsilon}^{1/2} \oplus T_{(2N+1)\pi\varepsilon/2}^{1/2})v\|^2 \leqslant C'\,\|v\|^2.$$

Since by (2.11) $\mathcal{V}$ has infinite dimension this contradicts (a).

It remains to prove (2.7), (2.8), and (2.9). Equation (2.7) follows from Lemma 2.1. Equation (2.8) follows from (2.7) and a straightforward calculation.

To prove (2.9) we fix M and choose N so large that

$$t(x, \varepsilon) \geqslant M \qquad \text{if} \quad x \geqslant N \quad (2.13)$$

which is possible by assumption (b). We put

$$x_0 := N\pi\varepsilon, \qquad x_{n+1} := x_n + \pi\varepsilon \quad \text{if} \quad n \geqslant 0,$$
$$y_0 := (2N+1)\,\pi\varepsilon/2, \qquad y_{n+1} := y_n + \pi\varepsilon \quad \text{if} \quad n \geqslant 0.$$

Using Lemma 2.1 as above it follows from (2.13) that for $n \geqslant 0$

$$\int_{x_n}^{x_{n+1}} [\|(\Phi_2 u)'(x)\|^2 + \langle Q(x)\,\Phi_2 u(x), \Phi_2 u(x)\rangle]\,dx$$
$$\geqslant M \int_{x_n}^{x_{n+1}} \|\Phi_2 u(x)\|^2\,dx, \tag{2.14a}$$

$$\int_{y_n}^{y_{n+1}} [\|(\Psi_2 u)'(x)\|^2 + \langle Q(x)\,\Psi_2 u(x), \Psi_2 u(x)\rangle]\,dx$$
$$\geqslant M \int_{y_n}^{y_{n+1}} \|\Psi_2 u(x)\|^2\,dx. \tag{2.14b}$$

Summing (2.14) over all n proves (2.9) and completes the proof of the theorem. ∎

As an easy corollary we obtain an abstract version of the "decomposition principle" in the discrete case, introduced in [GL] for scalar equations and in [D+L] for the case of the Laplacean.

THEOREM 2.2. *Suppose that T satisfies* (2.3). *Then T is discrete iff*

(a) *T_x is discrete for all $x > 0$;*

($\tilde{\text{b}}$) *writing for $x \geqslant 0$*

$$t(x) := \inf\{(\tau u, u) \mid u \in C_0^\infty((x, \infty), H_1),\ \|u\|_{L_2} = 1\}$$

we have

$$\lim_{x \to \infty} t(x) = \infty.$$

Proof. (1) Assume that (a) and ($\tilde{\text{b}}$) hold. Then (b) is also true and T is discrete by Theorem 2.1.

(2) Let T be discrete and fix $M > 0$. Determine $N = N(M)$ such that (2.9) holds with M replaced by $M+1$ and $\varepsilon = 1$. Choose $x \geqslant (2N+1)\pi/2$ and $u \in C_0^\infty((x, \infty), H_1)$. Then $\Phi_1 u = \Psi_1 u = 0$ and from (2.8), (2.9), and (2.6) we obtain

$$(\tau u, u) \geqslant M\,\|u\|_{L^2}^2. \quad ∎$$

The proof of Theorem 2.1 can be used to obtain the following global versions of (1.13).

THEOREM 2.3. *The following conditions are equivalent to* (1.13).

(α) *There are constants $C_1, C_2 > 0$ such that for all $u \in C_0^\infty(\mathbb{R}^*, H_1)$*

$$(\tau u, u) \geqslant C_1\,\|u'\|_{L^2}^2 - C_2\,\|u\|_{L^2}^2. \tag{2.15}$$

(β) *$\mathcal{D}(T^{1/2}) \subset H^{1/2}(\mathbb{R}^*, H) \simeq H^{1,2}(\mathbb{R}^*) \otimes H$ where $H^{1,2}$ denotes the usual Sobolev space.*

Proof. We prove the implications (1.13) $\Rightarrow$ (α) $\Rightarrow$ (β) $\Rightarrow$ (1.13). Assuming (1.13) for $u \in C_0^\infty((x, x+h), H_1)$ for $x \geqslant 0$ and some h, $0 < h \leqslant 1$, we conclude from Lemma 2.1 that (1.13) also holds for $u \in C^\infty([x, x+h], H_1)$ if $u(x) = u(x+h) = 0$. With $\varepsilon = h/\pi$ in (2.5) we have for $u \in C_0^\infty(\mathbb{R}^*, H_1)$ by (2.8)

$$(\tau u, u) \geqslant \sum_{i=1}^{2} (\|T^{1/2}\Phi_i u\|_{L^2}^2 + \|T^{1/2}\Psi_i u\|^2) - \frac{\pi^2}{h^2}\,\|u\|_{L^2}^2.$$

We write $x_0 := 0$, $x_{n+1} := x_n + \pi\varepsilon$, $y_0 := -\pi\varepsilon/2$, $y_{n+1} := y_n + \pi\varepsilon$ and find from (1.13)

$$\sum_{i=1}^{2} (\|T^{1/2}\Phi_i u\|_{L^2}^2 + \|T^{1/2}\Psi_i u\|_{L^2}^2)$$
$$= \sum_{n \geqslant 0} \Bigg[\int_{x_n}^{x_n + h} (\|\partial_x(\sin(x/\varepsilon)\,u(x))\|^2 + \langle Q(x)\sin(x/\varepsilon)\,u(x), \sin(x/\varepsilon)\,u(x)\rangle)\,dx$$
$$+ \int_{y_n}^{y_n + h} (\|\partial_x(\cos(x/\varepsilon)\,u(x))\|^2 + \langle Q(x)\cos(x/\varepsilon)\,u(x), \cos(x/\varepsilon)\,u(x)\rangle)\,dx \Bigg]$$
$$\geqslant \sum_{n \geqslant 0} \Bigg[\int_{x_n}^{x_n + h} (\delta\,\|\partial_x(\sin(x/\varepsilon)\,u(x))\|^2 - C\sin^2(x/\varepsilon)\,\|u(x)\|^2)\,dx$$
$$+ \int_{y_n}^{y_n + h} (\delta\,\|\partial_x(\cos(x/\varepsilon)\,u(x))\|^2 - C\cos^2(x/\varepsilon)\,\|u(x)\|^2)\,dx \Bigg]$$
$$\geqslant \delta\,\|u'\|_{L^2}^2 - C\,\|u\|_{L^2}^2.$$

This proves the first implication. The second implication follows from the fact that $C_0^\infty(\mathbb{R}^*, H_1)$ is a core for $T^{1/2}$, by definition of the Friedrichs

extension and [KA, Chap. V, Theorem 3.35]. Next, assuming (β) it follows from the closed graph theorem that the inclusion map $\mathcal{D}(T^{1/2}) \subset H^{1,2}(\mathbb{R}^*, H)$ is continuous from which we obtain (2.15) by definition of $H^{1,2}$. Equation (2.15) clearly implies (1.13). ∎

3. Specific Discreteness

More specific discreteness criteria will now be derived from Theorem 2.1. We start by observing that condition (a) in Theorem 2.1 is implied by very natural assumptions on Q.

Lemma 3.1. *If the function* (2.1) *is continuous and if $Q(z)$ is bounded below and discrete for all x then T_x is discrete for all x.*

Proof. Fix $\bar{x} > 0$. It is enough to show that any sequence $(u_n)_{n \geqslant 1} \subset C_0^\infty((0, \bar{x}), H_1)$ satisfying

$$\|T_{\bar{x}}^{1/2} u_n\|^2 + \|u_n\|^2 \leqslant C \tag{3.1}$$

has a subsequence convergent in $L^2([0, \bar{x}], H)$. To see this let $q(x)$ denote the lower bound of $Q(x)$. It follows from our assumptions on Q and standard interpolation techniques that the function

$$(Q(x_0) - q(x_0) + 1)^{-1/2}(Q(x) - q(x_0) + 1)(Q(x_0) - q(x_0) + 1)^{-1/2} =: \tilde{Q}(x)$$

is continuous on $\mathbb{R}_+$ with values in the space of hermitian operators on H. Since $\tilde{Q}(x_0) = I$ this implies the estimate

$$\langle Q(x)v, v\rangle \geqslant (1 - \delta)\langle Q(x_0)v, v\rangle \tag{3.2}$$

for $v \in H_1$, $0 < \delta < 1$, and $|x - x_0| < \varepsilon = \varepsilon(x_0, \delta)$. We cover $[0, \bar{x}]$ with intervals $J_k := (x_k - \varepsilon, x_k + \varepsilon)$, $1 \leqslant k \leqslant K$, such that (3.2) with x_0 replaced by x_k and $\delta = \frac{1}{2}$ holds in J_k. Let $(\psi_k)_{k \geqslant 1}$ be a partition of unity subordinate to this covering. By (3.2) there is a constant $C > 0$ such that

$$\langle Q(x)v, v\rangle \geqslant -C\|v\|^2 \qquad \text{for} \quad x \in [0, \bar{x}] \text{ and } v \in H_1. \tag{3.3}$$

From (3.2) applied to all x_k and (3.3) we now obtain the estimate

$$\int_0^{\bar{x}} [\|(\psi_k u_n)'(x)\|_H^2 + \langle Q(x_k)\psi_k u_n(x), \psi_k u_n(x)\rangle_H]\, dx$$
$$\leqslant C(\|T_{\bar{x}}^{1/2} u_n\|^2 + \|u_n\|^2) \tag{3.4}$$

for $1 \leqslant k \leqslant K$ and all n, with C independent of k and n. Now the Friedrichs extension of the constant coefficient operator

$$\tau_k := -\partial_x^2 + Q(x_k)$$

in $L^2([0, \bar{x}), H) \simeq L^2[0, \bar{x}] \otimes H$ has domain $H^2(0, \bar{x}) \cap H_0^1(0, \bar{x}) \otimes H_1$ which embeds compactly by Rellich's theorem and the discreteness of $Q(x_k)$. Hence it follows from (3.4) and (3.1) that the sequences $(\psi_k u_n)_{n \geqslant 1}$ as well as all their subsequences have convergent subsequences in $L^2([0, \bar{x}], H)$, hence the same is true for $(u_n)_{n \geqslant 1}$. The lemma is proved. ∎

We observe next that assuming the continuity of (2.1), the discreteness and boundedness below of all $Q(x)$ is necessary for the discreteness of T.

Lemma 3.2. *If the function* (2.1) *is continuous and if T is discrete then $Q(x)$ is discrete and bounded below for all $x \geqslant 0$.*

Proof. We start with the proof of discreteness. Let $(v_n)_{n \geqslant 1}$ be a bounded sequence in H_1 and pick $\psi \in C_0^\infty(\mathbb{R}^*)$ with $\|\psi\|_{L^2} = 1$. Then $u_n := \psi v_n \in \mathcal{D}(T)$ and

$$\begin{aligned}
&\|Tu_n\|^2 + \|u_n\|^2 \\
&\quad \leqslant 2\|v_n\|_H^2 \int_0^\infty |\psi''(x)|^2\, dx \\
&\qquad + 2\int_0^\infty |\psi(x)|^2 \|Q(x)v_n\|_H^2\, dx + \int_0^\infty |\psi(x)|^2 \|v_n\|_H^2\, dx \\
&\quad \leqslant C\|v_n\|_{H_1}^2
\end{aligned}$$

by the continuity of $x \mapsto Q(x)$. Hence $(u_n)_{n \geqslant 1}$ is bounded in $\mathcal{D}(T)$ so by discreteness we may assume that $(u_n)_{n \geqslant 1}$ is convergent in $L^2(\mathbb{R}_+, H)$. Since $\|v_n - v_m\|_H^2 = \|u_n - u_m\|_{L^2}^2$ $(v_n)_{n \geqslant 1}$ is convergent in H which proves that H_1 embeds compactly into H.

Assume next that $Q(x_0)$ is not bounded below for some $x_0 \geqslant 0$. Then we can find a sequence $\lambda_n \to -\infty$ and corresponding $v_n \in H_1$ such that

$$\|v_n\|_H^2 = 1, \qquad \langle Q(x_0)\, v_n, v_n\rangle \leqslant \lambda_n, \qquad \|Q(x_0)v_n\|_H \leqslant |\lambda_n| + 1. \tag{3.5}$$

Since $Q(x_0)$ is closed in H with domain H_1 we have for $v \in H_1$

$$\|v\|_{H_1}^2 \leqslant C(x_0)^2(\|Q(x_0)v\|_H^1 + \|v\|_H^2); \tag{3.6}$$

and since (2.1) is continuous we have also

$$\|(Q(x) - Q(x_0))v\|_H \leqslant (2C(x_0))^{-1}\|v\|_{H_1} \tag{3.7}$$

if $|x-x_0|<\delta$ for suitable $\delta=\delta(x_0)$. Choosing $\psi\in C_0^\infty(x_0-\delta, x_0+\delta)$ with $\int\psi^2=1$ and setting $u_n:=\psi v_n$ we obtain

$$\begin{aligned}(Tu_n, u_n)&\leqslant\int_0^\infty [|\psi'(x)|^2+|\psi(x)|^2\langle(Q(x)-Q(x_0))\,v_n, v_n\rangle]\,dx+\lambda_n\\&\leqslant C(\delta)+\tfrac{1}{2}(|\lambda_n|+2)+\lambda_n\\&\leqslant(C+\tfrac{1}{2}\lambda_n)\,\|u_n\|^2\end{aligned}$$

by (3.5), (3.6), and (3.7). This contradicts (2.3). ∎

Our next goal is to replace (b) in Theorem 2.1 by a more practical condition similar to the Maslov condition, adding now the coerciveness assumption (1.13). To formulate our result we put $\alpha_0:=\max\{\pi^2, \delta^{-1}\}$ with δ the constant in (1.13), and introduce

$$q_\alpha(x,\varepsilon):=\inf\left\{\int_x^{x+\varepsilon}\langle Q(y)u(y), u(y)\rangle\,dy\,|\,u\in C_0^\infty((x, x+\varepsilon), H_1),\right.$$

$$\left.\|u\|_{L^2}^2=1,\ \|u'\|_{L^2}^2<\frac{\alpha}{\varepsilon^2}\right\},\tag{3.8}$$

where α is any constant $>\alpha_0$. Note that this choice of α makes sure that the set on the right hand side of (3.8) is nonempty.

LEMMA 3.3. *Suppose that* (2.1), (2.3), *and* (1.13) *hold. Then* (b) *in Theorem* 2.1 *is equivalent to*

$$\lim_{x\to\infty} q_\alpha(x,\varepsilon)=\infty\qquad \textit{for all}\quad 0<\varepsilon\leqslant h_0\ \textit{and all}\ \alpha>\alpha_0.\tag{3.9}$$

Proof. (1) Assume (b). Then from

$$\begin{aligned}t(x,\varepsilon)&\leqslant\inf\{(\tau u, u)\,|\,u\in C_0^\infty((x, x+\varepsilon), H_1),\ \|u\|\,\|_{L^2}^2=1,\ \|u'\|_{L^2}^2<\alpha/\varepsilon^2\}\\&\leqslant\alpha/\varepsilon^2+q_\alpha(x,\varepsilon)\end{aligned}$$

for all $\alpha>\alpha_0$ we deduce that $\lim_{x\to\infty} q_\alpha(x,\varepsilon)=\infty$.

(2) Assume (3.9) for some $\alpha>\alpha_0$ and fix $\varepsilon_0>0$ and $M>0$. Then we have to find $x_0=x_0(M,\varepsilon_0)$ such that $t(x,\varepsilon_0)\geqslant M$ if $x\geqslant x_0$. With N, ε to be determined later we consider $u\in C_0^\infty((x, x+\varepsilon_0), H_1)$ where we require $x>(2N+1)\pi\varepsilon/2$. Using the definition (2.5) we have $\Phi_1 u=\Psi_1 u=0$, and (2.7) and (2.8) give

$$\Phi_2 u,\ \Psi_2 u\in\mathscr{D}(T^{1/2}),\tag{3.10a}$$

$$(\tau u, u)\geqslant\|T^{1/2}\Phi_2 u\|^2+\|T^{1/2}\Psi_2\Psi_2 u\|^2-\frac{1}{\varepsilon^2}\|u\|_{L^2}^2.\tag{3.10b}$$

We put $x_1:=N\pi\varepsilon$, $x_{n+1}:=x_n+\pi\varepsilon$. Then

$$\|T^{1/2}\Phi_2 u\|^2=\sum_{n\geqslant 1}\int_{x_n}^{x_{n+1}}[\|(\Phi_2 u)'(x)\|_H^2+\langle Q(x)\,\Phi_2 u(x), \Phi_2 u(x)\rangle]\,dx.$$

Given $n\in\mathbb{N}$ assume first that

$$\int_{x_n}^{x_{n+1}}\|(\Phi_2 u)'(x)\|_H^2\,dx\geqslant\frac{\alpha}{\varepsilon^2}\int_{x_n}^{x_{n+1}}\|\Phi_2 u(x)\|_H^2\,dx.\tag{3.11}$$

If $\varepsilon\leqslant h_0$ we deduce from (1.13) and Lemma 2.1 the estimate

$$\begin{aligned}&\int_{x_n}^{x_{n+1}}[\|(\Phi_2 u)'(x)\|_H^2+\langle Q(x)\,\Phi_2 u(x), \Phi_2 u(x)\rangle]\,dx\\&\qquad\geqslant\left(\frac{\alpha\delta}{\varepsilon^2}-C\right)\int_{x_n}^{x_{n+1}}\|\Phi_2 u(x)\|_H^2\,dx.\end{aligned}\tag{3.12}$$

If (3.11) is not satisfied we apply again Lemma 2.1 and deduce from (3.8) that

$$\begin{aligned}&\int_{x_n}^{x_{n+1}}[\|(\Phi_2 u)'(x)\|_H^2+\langle Q(x)\,\Phi_2 u(x), \Phi_2 u(x)\rangle]\,dx\\&\qquad\geqslant q_\alpha(x_n,\varepsilon)\int_{x_n}^{x_{n+1}}\|\Phi_2 u(x)\|_H^2\,dx.\end{aligned}\tag{3.13}$$

Combining (3.12) and (3.13) and summing over all $n\in\mathbb{N}$ we obtain

$$\|T^{1/2}\Phi_2 u\|^2\geqslant\min\left\{\frac{\alpha\delta}{\varepsilon^2}-C, \inf_{n\in\mathbb{N}} q_\alpha(x_n,\varepsilon)\right\}\|\Phi_2 u\|^2.$$

Setting $y_1:=(2N+1)\pi\varepsilon/2$, $y_{n+1}:=y_n+\pi\varepsilon$ a completely analogous argument gives

$$\|T^{1/2}\Psi_2 u\|^2\geqslant\min\left\{\frac{\alpha\delta}{\varepsilon^2}-C, \inf_{n\in\mathbb{N}} q_\alpha(y_n,\varepsilon)\right\}\|\Psi_2 u\|^2.$$

Hence it follows from (3.10b) and (2.6) that

$$(\tau u, u)\geqslant\min\left\{\frac{\alpha\delta-1}{\varepsilon^2}-C, \inf_{n\in\mathbb{N}} q_\alpha(x_n,\varepsilon)-1/\varepsilon^2, \inf_{n\in\mathbb{N}} q_\alpha(y_n,\varepsilon)-1/\varepsilon^2\right\}\|u\|^2.$$

Since $\alpha_0\delta - 1 \geqslant 0$ by assumption we can now choose $0 < \varepsilon < h_0$ such that

$$\frac{\alpha\delta - 1}{\varepsilon^2} - C \geqslant M \qquad \text{or} \qquad \varepsilon^2 \leqslant \frac{\alpha\delta - 1}{M + C}.$$

By (3.9) we can determine $N = N(M, \varepsilon)$ such that

$$q_\alpha(x, \varepsilon) \geqslant M + 1/\varepsilon^2 \qquad \text{if} \quad x \geqslant N\pi\varepsilon.$$

The proof is completed with $x_0 = (N+1)\pi\varepsilon$. ∎

Combining the preceding lemmas we obtain

THEOREM 3.1. *T is discrete if and only if*

(a) *T_x is discrete for all $x > 0$;*

(b′) *for $0 < \varepsilon \leqslant h_0$ and all $\alpha > \alpha_0$*

$$\lim_{x \to \infty} q_\alpha(x, \varepsilon) = \infty.$$

We proceed to show how the results mentioned in Section 1 follow from Theorem 3.1. We observe first that a condition stronger than the Zelenko condition (1.8) implies (1.13).

LEMMA 3.4. *Assume* (2.1), (2.3), *and that Q is bounded below a.e. Denote by $q(x)$ the lower bound of $Q(x)$. If $q = q_1 + q_2$ where q_1 is locally integrable and satisfies* (1.6) *and*

$$q_2(x) \geqslant \gamma'(x) + \zeta\gamma(x)^2 \tag{3.14}$$

for some $\zeta > 1$ and some absolutely continuous function $\gamma\colon \mathbb{R}_+ \to \mathbb{R}$, then τ satisfies (1.13).

Proof. Since for $u \in C_0^\infty(\mathbb{R}^*, H_1)$ we have by assumption

$$(\tau u, u) \geqslant \int_0^\infty (q_1(x) + q_2(x)) \, \|u(x)\|_H^2 \, dx + \|u'\|_{L^2}^2,$$

it is enough to prove the following inequalities:

(1) there is $\eta > 0$ such that

$$\int_0^\infty q_2(x) \, \|u(x)\|_H^2 \, dx \geqslant (\eta - 1) \, \|u'\|_{L^2}^2; \tag{3.15}$$

(2) if $\operatorname{supp} u$ is contained in an interval of length at most h then

$$\int_0^\infty q_1(x) \, \|u(x)\|_H^2 \, dx \geqslant -C_1(h) \, \|u'\|_{L^2}^2, \tag{3.16}$$

where $\lim_{h \to 0} C_1(h) = 0$.

For the proof of (3.15) we estimate

$$\begin{aligned}
\int_0^\infty q_2(x) \, \|u(x)\|_H^2 \, dx &\geqslant \int_0^\infty (\gamma'(x) + \zeta\gamma(x)^2) \, \|u(x)\|^2 \, dx \\
&= \int_0^\infty (\zeta\gamma(x)^2 \, \|u(x)\|_H^2 - 2\gamma(x) \operatorname{Re}\langle u(x), u'(x)\rangle) \, dx \\
&\geqslant \int_0^\infty (\zeta\gamma(x)^2 \, \|u(x)\|_H^2 - 2\gamma(x) \, \|u(x)\|_H \, \|u'(x)\|_H) \, dx \\
&\geqslant -\zeta^{-1} \, \|u'\|_{L^2}^2.
\end{aligned}$$

The proof of (3.16) is implicit in the work of Ismagilov [I]. We repeat the argument for convenience of the reader. We write for $0 \leqslant x \leqslant t \leqslant x + h$ with α, β as in (1.6)

$$O(t) := \int_x^t q_1(u) \, du, \qquad P(t) := \sup_{x \leqslant s \leqslant t} (O(s) - \beta(s)),$$

$$R(t) := O(t) - P(t).$$

Then we have

$$\alpha(t) \leqslant R(t) \leqslant \beta(t). \tag{3.17}$$

In fact, the right inequality is obvious from the definition of P whereas the left one follows from

$$O(t) - (O(s) - \beta(s)) = \beta(s) + \int_s^t q_1(u) \, du \geqslant \alpha(t)$$

by (1.6a). From (1.6b) and (3.17) we find

$$\int_x^{x+h} R(t)^2 \, dt \leqslant C^2. \tag{3.18}$$

Since P is increasing and q_1 is locally integrable we can now estimate for $u \in C_0^\infty((x, x+h), H_1)$

$$\int_x^{x+h} q_1(t)\,\|u(t)\|_H^2\,dt$$

$$= \int_x^{x+h} (P'(t)+R'(t))\,\|u(t)\|_H^2\,dt$$

$$\geqslant -\int_x^{x+h} 2R(t)\,\mathrm{Re}\langle u(t), u'(t)\rangle\,dt$$

$$\geqslant -2 \sup_{x\leqslant t\leqslant x+h} \left\|\int_x^t u'(s)\,ds\right\|_H \|u'\|_{L^2} \left(\int_x^{x+h} R(t)^2\,dt\right)^{1/2}$$

$$\geqslant -2h^{1/2}C\,\|u'\|_{L^2}^2 \tag{3.19}$$

using (3.18). The proof is complete. ∎

To deal with the case $\zeta = 1$ in (3.14) which is of interest in applications we need the following recent result of Gurka [GU] on weighted Sobolev estimates. It will also lead us to general coerciveness and discreteness criteria in the scalar case.

LEMMA 3.5. *Let $0 \leqslant a < b \leqslant \infty$ and let $g, h \in C[a, b]$ be positive in (a, b). Then the weighted Sobolev inequality*

$$\int_a^b g(x)\,|u(x)|^2\,dx \leqslant A \int_a^b h(x)\,|u'(x)|^2\,dx \tag{3.20}$$

holds for all $u \in C_0^\infty(a, b)$ iff

$$B := \sup_{a<t<s<b} \min\left\{\int_t^s g(x)\,dx \int_a^t h(x)^{-1}\,dx, \int_t^s g(x)\,dx \int_s^b h(x)^{-1}\,dx\right\} < \infty. \tag{3.21}$$

Moreover, we have

$$B \leqslant 2A \leqslant 8B. \tag{3.22}$$

This leads to the following coerciveness result.

LEMMA 3.6. *Let Q satisfy the hypotheses of Lemma 3.4 with $\zeta = 1$ and put*

$$a(x) := e^{2\int_0^x \gamma(t)\,dt}. \tag{3.23}$$

Then (1.13) holds if for some h, $0 < h \leqslant 1$,

$$\sup_{x>0} \sup_{0<t<s<b} \min \times \left\{\int_t^s \frac{a'^2}{a}(y)\,dy \int_x^t a(y)^{-1}\,dy, \int_t^s \frac{a'^2}{a}(y)\,dy \int_s^{x+h} a(y)^{-1}\,dy\right\} < \infty. \tag{3.24}$$

If $Q = q$ and $q_2 = \gamma' + \gamma^2$, then (3.24) is also necessary for coerciveness.

Proof. By (the proof of) Theorem 2.3 and Lemma 3.4, it is enough to prove the estimate

$$\|u'\|_{L^2}^2 + \int_0^\infty (\gamma'(x)+\gamma(x)^2)\,\|u(x)\|_H^2\,dx \geqslant \eta\,\|u'\|_{L^2}^2 - C\,\|u\|_{L^2}^2$$

for $u \in C_0^\infty((x, x+h), H_1)$, some h with $0 < h \leqslant 1$, and certain constants $\eta, C > 0$. Using an orthonormal basis in H it is easily seen that it is enough to prove coerciveness for the scalar potential

$$\gamma'(x)+\gamma(x)^2 = \frac{1}{2}\frac{a''(x)}{a(x)} - \frac{1}{4}\left(\frac{a'(x)}{a(x)}\right)^2.$$

Then the unitary map

$$\Phi: L^2(\mathbb{R}_+) \ni u \mapsto a^{-1/2}u \in L^2(\mathbb{R}_+, a\,dx) =: L_a^2$$

transforms $\tau = -\partial_x^2 + \gamma'(x) + \gamma^2(x)$ to

$$\tau_1 = \Phi\tau\Phi^* = -\frac{1}{a}\,\partial_x a\,\partial_x.$$

The inequality

$$(\tau u, u) \geqslant \eta\,\|u'\|^2, \qquad u \in C_0^\infty(x, x+h), \tag{3.25}$$

is then equivalent to

$$(\tau_1 v, v)_{L_a^2} \geqslant \eta \left\|v' + \frac{1}{2}\frac{a'}{a}v\right\|_{L_a^2}^2, \qquad v \in C_0^\infty(x, x+h),$$

and this in turn is equivalent to the weighted Sobolev inequality

$$\int_x^{x+h} \frac{a'(t)^2}{a(t)}\,|v(t)|^2\,dt \leqslant C \int_x^{x+h} a(t)\,|v'(t)|^2\,dt,$$

where C is independent of x. Now the assertion follows from Lemma 3.5.

Assume next that in the scalar case $Q(x)=q(x)$, (1.13) holds for $0<h\leqslant h_0\leqslant 1$. Replacing q by $q+C$ gives the decomposition $q+C=q_1+C+q_2$ where q_1+C still satisfies (1.6). Using (3.16) we obtain (3.25) if h is chosen small enough. Since this implies the weighted Sobolev inequality with C independent of x the assertion follows again from Lemma 3.5. ∎

The preceding lemmas show that Theorem 3.1 is applicable in all cases listed in the Introduction. We now derive various discreteness criteria from condition (b′) in Theorem 3.1. We show first that the Ismagilov condition (1.7) on the lower bound of Q is sufficient for discreteness in a large class of potentials, extending the Zelenko class even for scalar potentials.

THEOREM 3.2. *If the lower bound q of Q is locally integrable and satisfies the assumptions of Lemma* 3.4 *or Lemma* 3.6 *then* (1.7) *is a sufficient condition for discreteness.*

Proof. By Theorem 2.1, it is clearly sufficient to show that we have discreteness for the scalar operator $\tilde{\tau}:=-\partial_x^2+q(x)$ on $L^2(\mathbb{R}_+)$. By Lemma 3.4 or Lemma 3.6, Theorem 3.1 applies to $\tilde{\tau}$. Thus a necessary and sufficient condition for discreteness of $\tilde{\tau}$ is

$$\lim_{x\to\infty}\ \inf_{u\in\mathscr{H}_{x,\varepsilon}}\int_x^{x+\varepsilon} q(t)\,|u(t)|^2\,dt=\infty,$$

where we have put $\mathscr{H}_{x,\varepsilon}:=\{u\in C_0^\infty(x,x+\varepsilon)|\ \|u\|_{L^2}^2=1,\ \|u'\|_{L^2}^2<\alpha/\varepsilon^2\}$ for a suitable constant α. Now

$$\begin{aligned}&\int_x^{x+\varepsilon}(\gamma'(t)+\zeta\gamma^2(t))\,|u(t)|^2\,dt\\&=\int_x^{x+\varepsilon}(\zeta\gamma^2(t)\,|u(t)|^2-2\gamma(t)\,\mathrm{Re}\,u(t)\,u'(t))\,dt\\&\geqslant-\zeta^{-1}\,\|u'\|_{L^2}^2>-\zeta^{-1}\frac{\alpha}{\varepsilon^2}.\end{aligned}$$

By (3.14) it is therefore enough to show that

$$\lim_{x\to\infty}\ \inf_{u\in\mathscr{H}_{x,\varepsilon}}\int_x^{x+\varepsilon} q_1(t)\,|u(t)|^2\,dt=\infty,\tag{3.26}$$

where q_1 satisfies (1.6). The proof of (3.26) is due to Ismagilov [I]; we reproduce it here for completeness. Recall that with the notation

introduced in the proof of Lemma 3.4 $q_1(x)=P'(x)+R'(x)$ with $P'(x)\geqslant 0$, and that by the estimate leading to (3.19) for $u\in\mathscr{H}_{x,\varepsilon}$

$$\int_x^{x+\varepsilon} q_1(t)\,|u(t)|^2\,dt\geqslant\int_x^{x+\varepsilon} P'(t)\,|u(t)|^2\,dt-C_\varepsilon.\tag{3.27}$$

For $x\geqslant 0$, $0\leqslant s\leqslant t\leqslant\varepsilon$, we write

$$\tilde{Q}_x(s,t):=\int_{x+s}^{x+t} q_1(y)\,dy,$$

$$\tilde{P}_x(s,t):=P(x+t)-P(x+s),\qquad \tilde{R}_x(s,t):=R(x+t)-R(x+s),$$

so

$$\tilde{Q}_x(s,t)=\tilde{P}_x(s,t)+\tilde{R}_x(s,t).$$

Now (1.7) means that $\tilde{Q}_x\to\infty$ in measure as $x\to\infty$ and by (3.18) this implies that $\tilde{P}_x\to\infty$ in measure, too. But from $P'\geqslant 0$ it follows that $\tilde{P}_x(s,t)\leqslant\tilde{P}_x(s',t')$ whenever $s'\leqslant s$ and $t'\geqslant t$, hence we conclude that $\tilde{P}_x(s,t)\to\infty$ for all $0\leqslant s<t\leqslant\varepsilon$. To prove (3.26) we now observe that for $u\in\mathscr{H}_{x,\varepsilon}$ we can find $y_0\in(x,x+\varepsilon)$ with $|u(y_0)|^2\geqslant\varepsilon^{-1}$ and $|u(y)|^2\geqslant(4\varepsilon)^{-1}$ if $|y-y_0|<\varepsilon/4\alpha$. Any such interval contains $x+j\varepsilon/12\alpha$ and $x+(j+1)(\varepsilon/12\alpha)$ for some j with $0\leqslant j\leqslant[12\alpha]+1$, so (3.26) follows from $P'\geqslant 0$ and $\tilde{P}_x(0,\varepsilon/12\alpha)\to\infty$ as $x\to\infty$. ∎

Next we derive an extension of Maslov's criterion [MA] which also contains the result of Kleine [KL].

THEOREM 3.3. *Assume* (2.1), (2.3), *and that Q is bounded below a.e. If Brinck's condition* (1.5) *holds for the lower bound q of Q then a necessary and sufficient condition for the discreteness of T is*

$$\lim_{x\to\infty} q_M(x;\varepsilon,\zeta)=\infty,\tag{3.28}$$

where for $\varepsilon,\zeta>0$ we put

$$\begin{aligned}q_M(x;\varepsilon,\zeta):=\inf\Big\{&\int_x^{x+\varepsilon}\langle Q(y)\,u(y),u(y)\rangle\,dy\Big|\\&u\in C^\infty(]x,\,+\varepsilon],H_1),\ \|u(y)\|=1\ \textit{for}\ y\in[x,x+\varepsilon],\\&\int_x^{x+\varepsilon}\|u'(y)\|^2\,dy\leqslant\zeta\Big\}.\end{aligned}$$

Proof. By Lemma 3.4, Theorem 3.1 applies and δ in (1.13) can be any number <1. So we have $\lim_{x\to\infty} q_\alpha(x,\varepsilon)=\infty$ for all $\varepsilon>0$ and all $\alpha>\pi^2$ as necessary and sufficient conditions for discreteness. Now assume that T is discrete. Choose $u\in C^\infty([x,x+\varepsilon],H_1)$ with $\|u(y)\|=1$, $y\in[x,x+\varepsilon]$, and

$$\int_x^{x+\varepsilon}\|u'(y)\|^2\,dy\leqslant\zeta.$$

Write $u_1(y):=(2/\varepsilon)^{1/2}u(y)\sin(\pi/\varepsilon)(y-x)$, $u_2(y):=(2/\varepsilon)^{1/2}u(y)\cos(\pi/\varepsilon)(y-x)$; then

$$\frac{2}{\varepsilon}\int_x^{x+\varepsilon}\langle Q(y)\,u(y),u(y)\rangle\,dy$$
$$=\int_x^{x+\varepsilon}[\langle Q(y)\,u_1(y),u_1(y)\rangle+\langle Q(y)\,u_2(y),u_2(y)\rangle]\,dy.$$

To estimate the second term we use $Q(y)\geqslant q(y)$, the assumptions on q and u, and Lemma 3 in [BRI]. This gives

$$\int_x^{x+\varepsilon}\langle Q(y)\,u_2(y),u_2(y)\rangle\,dy$$
$$\geqslant -C_1\int_x^{x+\varepsilon}(\|(y)\|^2+\|u'(y)\|^2)\,dy\geqslant -C_2,$$

where C_2 depends only on ε and ζ. We observe next that $u_1\in C^\infty([x,x+\varepsilon],H_1)$, $u_1(x)=u_1(x+\varepsilon)=0$,

$$\int_x^{x+\varepsilon}\|u_1(y)\|^2\,dy=1,$$

$$\int_x^{x+\varepsilon}\|u_1'(y)\|^2\,dy\leqslant\varepsilon^{-2}(\pi^2+2\varepsilon\zeta)=:\alpha/\varepsilon^2,$$

hence

$$\frac{2}{\varepsilon}\int_x^{x+\varepsilon}\langle Q(y)\,u(y),u(y)\rangle\geqslant q_\alpha(x,\varepsilon)-C_2,$$

and since $q_\alpha(x,\varepsilon)\to\infty$ as $x\to\infty$ we obtain (3.28).

Next assume (3.28). Choose $u\in C_0^\infty((x,x+\varepsilon),H_1)$ with $\|u\|_{L^2}^2=1$ and $\|u'\|_{L^2}^2\leqslant\alpha/\varepsilon^2$ for some $\alpha>\pi^2$. As in the proof of Theorem 3.2 we can decompose $[x,x+\varepsilon]=I_1\cup I_2\cup I_3$ where each I_j is a closed subinterval of length $\geqslant C\varepsilon$ with C independent of x and ε and $\|u(y)\|^2\geqslant 1/4\varepsilon$ for $y\in I_2$.

Applying Lemma 3 of [BRI] as before we find with $\tilde u(y):=u(y)/\|u(y)\|$

$$\int_x^{x+\varepsilon}\langle Q(y)\,u(y),u(y)\rangle\,dy$$
$$\geqslant\int_{I_1\cup I_2}q(y)\,\|u(y)\|^2\,dy$$
$$+\int_{I_2}\left[\frac{1}{8\varepsilon}\langle Q(y)\,\tilde u(y),\tilde u(y)\rangle+q(y)\left(\|u(y)\|^2-\frac{1}{8\varepsilon}\right)\right]dy$$
$$\geqslant -C+\frac{1}{8\varepsilon}\int_{x^*}^{x^*+C\varepsilon}\langle Q(y)\,\tilde u(y),\tilde u(y)\rangle\,dy.$$

Here C is independent of x and $x^*\in[x,x+\varepsilon]$. Now it is easily checked that $\tilde u\in C^\infty([x^*,x^*+C\varepsilon],H_1)$, $\|\tilde u(y)\|=1$ for $y\in[x^*,x^*+C\varepsilon]$, and

$$\int_{x^*}^{x^*+C\varepsilon}\|\tilde u'(y)\|^2\,dy\leqslant\frac{8\alpha}{\varepsilon}.$$

This gives

$$q_\alpha(x,\varepsilon)\geqslant -C+\frac{1}{8\varepsilon}q_M(x^*;C\varepsilon,8\alpha/\varepsilon).$$

Thus (3.28) implies the discreteness of T by Theorem 3.1. ∎

The methods developed so far enable us to give a necessary and sufficient discreteness criterion in the scalar case assuming merely (2.1) and (2.3) but no further conditions on the potential. This criterion is based on the Riccati equation

$$\gamma'(x)+\gamma(x)^2=q(x)$$

which enters naturally if we apply our technique to the Laplacean on complete manifolds (cf. Section 4). In the remainder of this section we will assume $Q=q$.

THEOREM 3.4. *Assume* (2.1) *and in addition*

$$\inf_{u\in C_0^\infty(\mathbb{R}^*)}(\tau u,u)=0. \tag{3.29}$$

Then T is discrete if and only if the Riccati equation

$$\gamma'(x)+\gamma(x)^2=q(x) \tag{3.30}$$

has a solution γ in $\mathbb{R}^*$ with the property that for all $0<\varepsilon<1$

$$\lim_{x\to\infty} \sup_{x<t<s<x+\varepsilon} \min\left\{\int_t^s a(y)\,dy \int_x^t a(y)^{-1}\,dy, \int_t^s a(y)\,dy \int_s^{x+\varepsilon} a(y)^{-1}\,dy\right\}=0,$$

where a is defined by

$$a(x)=e^{2\int_1^x \gamma(y)\,dy}. \tag{3.31}$$

Proof. We start with the proof of necessity. If T is discrete and satisfies (3.29) then 0 is an eigenvalue. If w is a corresponding eigenfunction normalized by $w(1)=1$ then it is well known that $w(x)>0$ if $x>0$. Thus

$$\gamma(x):=\frac{w'(x)}{w(x)}$$

solves (3.30) for $x>0$. Thus with a defined by (3.31) we have

$$\gamma(x)=\frac{1}{2}\frac{a'(x)}{a(x)}, \qquad q(x)=\frac{1}{2}\frac{a''(x)}{a(x)}-\frac{1}{4}\left(\frac{a'(x)}{a(x)}\right)^2.$$

If $\tilde{T}$ denotes the Friedrichs extension of $\tau\,|\,C_0^\infty(1,\infty)$ in $L^2[1,\infty)$ then $\tilde{T}$ is discrete, too, by the max–min principle. As in the proof of Lemma 3.6 we see that $\tilde{T}$ is unitarily equivalent to the Friedrichs extension T_1 in $L^2([1,\infty), a\,dx)=:L_a^2$ of

$$\tau_1:=\Phi_1\tau\Phi_1^*=-\frac{1}{a}\,\partial_x a\,\partial_x$$

with domain $C_0^\infty(1,\infty)$; here $\Phi_1 u(x)=a(x)^{-1/2}u(x)$. Applying Theorem 2.1 to $\tilde{T}$ we see that for $M>0$ and $0<\varepsilon\leqslant 1$ we can find $x_0=x_0(M;\varepsilon)$ such that for $x\geqslant x_0$

$$(u,u)\leqslant M^{-1}(\tau u,u) \qquad \text{if} \quad u\in C_0^\infty(x,x+\varepsilon).$$

Setting $v:=\Phi_1 u$ we obtain

$$(v,v)_{L_a^2}\leqslant M^{-1}(\tau_1 v,v)_{L_a^2}.$$

Since Φ_1 is bijective as a map of $C_0^\infty(x,x+\varepsilon)$ onto itself we thus obtain

$$\int_x^{x+\varepsilon} a(y)\,|v(y)|^2\,dy\leqslant M^{-1}\int_x^{x+\varepsilon} a(y)\,|v'(y)|^2\,dy \tag{3.32}$$

for all $v\in C_0^\infty(x,x+\varepsilon)$. Now the assertion follows from Lemma 3.5.

For the proof of sufficiency we start with the solution γ of (3.30) and define a again by (3.31). As before we obtain that $\tilde{T}$ is unitarily equivalent to T_1, and from Lemma 3.5 we obtain (3.32) for all $M\geqslant 0$ and $0<\varepsilon\leqslant 1$ if $x\geqslant x_0(M,\varepsilon)$. Thus Theorem 2.1 implies that T_1 and hence $\tilde{T}$ is discrete. But then it follows from Theorem 2.2 that T is discrete, too. ∎

The next lemma provides necessary or sufficient conditions for the discreteness of T in terms of γ; this will be important in the next section.

LEMMA 3.7. (a) *If T is discrete then for any solution γ of* (3.30) *in $\mathbb{R}^*$ we have for $0<\varepsilon\leqslant 1$*

$$\lim_{x\to\infty}\int_x^{x+\varepsilon}|\gamma(t)|\,dt=\infty$$

and

$$\lim_{x\to\infty}\sup_{x<z<y<x+\varepsilon}\left|\int_z^y \gamma(t)\,dt\right|=\infty.$$

(b) *If* (3.30) *has a solution γ such that γ is bounded either from below or above and*

$$\lim_{x\to\infty}\left|\int_x^{x+\varepsilon}\gamma(y)\,dy\right|=\infty$$

for $0<\varepsilon\leqslant 1$ then T is discrete.

Proof. (a) Defining again a by (3.31) we obtain from the proof of Theorem 3.4 that the weighted Sobolev estimate (3.32) holds for $x\geqslant x_0(M,\varepsilon)$. We replace ε by 3ε and choose $v(y)=\sin(\pi/3\varepsilon)(y-x)$ which is possible in view of Lemma 2.1. It follows that

$$\frac{\pi^2}{9\varepsilon^2}\int_x^{x+3\varepsilon} a(y)\,dy\geqslant\frac{3}{4}M\int_{x+\varepsilon}^{x+2\varepsilon} a(y)\,dy. \tag{3.33}$$

Put $x_0:=x$ and $x_i:=x_{i-1}+\varepsilon$, $i\in\mathbb{N}$, and conclude from (3.33) that with $\alpha:=27\varepsilon^2/4\pi^2$ for all $N\in\mathbb{N}$

$$\alpha M\int_{x_1}^{x_N} a(y)\,dy\leqslant\left(\int_{x_0}^{x_1}+\int_{x_N}^{x_{N+1}}+3\int_{x_1}^{x_N}\right)a(y)\,dy. \tag{3.34}$$

We have to distinguish two cases. If $\int_1^\infty a(y)\,dy=\infty$ then it follows from (3.34) that for large N

$$\int_{x_N}^{x_{N+1}} a(y)\,dy\geqslant(\alpha M-3)\int_{x_1}^{x_N} a(y)\,dy-\int_{x_0}^{x_1} a(y)\,dy$$

$$\geqslant\frac{\alpha}{2}M\int_{x_{N-1}}^{x_N} a(y)\,dy.$$

This means that

$$\int_x^{x+\varepsilon} a(y)\,dy \geqslant M \int_{x-\varepsilon}^{x} a(y)\,dy \tag{3.35a}$$

if $x \geqslant x(M, \varepsilon)$.

If $\int_1^\infty a(y)\,dy < \infty$ we let $N \to \infty$ in (3.34) and deduce that for $M \geqslant M_0$

$$\int_{x-\varepsilon}^{x} a(y)\,dy \geqslant M \int_x^{x+\varepsilon} a(y)\,dy. \tag{3.35b}$$

Now we observe that with

$$A(x,\varepsilon) := \min\left\{\int_{x-\varepsilon}^{x+\varepsilon}\left|\frac{a'}{a}(t)\right|dt, \sup_{x-\varepsilon<y<z<x+\varepsilon}\left|\int_y^z \frac{a'}{a}(t)\,dt\right|\right\}$$

we have for $z \in [x-\varepsilon, x]$, $y \in [x, x+\varepsilon]$

$$a(y) \leqslant a(z)\, e^{A(x,\varepsilon)}. \tag{3.36}$$

Assume now that (3.35a) holds; we integrate (3.36) over $y \in [x, x+\varepsilon]$ and obtain from (3.35a)

$$M \int_{x-\varepsilon}^{x} a(y)\,dy \leqslant \varepsilon a(z)\, e^{A(x,\varepsilon)}.$$

Integrating over $z \in [x-\varepsilon, x]$ gives

$$e^{A(x,\varepsilon)} \geqslant M \qquad \text{or} \qquad A(x,\varepsilon) \geqslant \log M$$

which is the assertion in this case. If (3.35b) holds instead we just interchange the role of y and z in (3.36).

(b) As in the proof of Theorem 3.4 it is enough to establish (3.32) for all $v \in C_0^\infty(x, x+\varepsilon)$ if $x \geqslant x(M, \varepsilon)$; here a is again defined by (3.31). In view of Lemma 3.5 this will follow if we prove for $t \in [x, x+\varepsilon]$ the estimate

$$\sup_{x<t<x+\varepsilon} \int_x^t \frac{dy}{a(y)} \int_t^{x+\varepsilon} a(y)\,dy =: \sup_{x<t<x+\varepsilon} F_a(t) \leqslant \frac{1}{M^2} \tag{3.37a}$$

or

$$\sup_{x<t<x+\varepsilon} \int_x^t a(y)\,dy \int_t^{x+\varepsilon} \frac{dy}{a(y)} =: \sup_{x<t<x+\varepsilon} F_b(t) \leqslant \frac{1}{M^2}. \tag{3.37b}$$

By assumption, we have for given $M' > 0$, $0 < \varepsilon' \leqslant 1$, and $x \geqslant x(M', \varepsilon')$ either

$$-\int_x^{x+\varepsilon'} \frac{a'}{a}(y)\,dy \geqslant M' \tag{3.38a}$$

or

$$\int_x^{x+\varepsilon'} \frac{a'}{a}(y)\,dy \geqslant M'. \tag{3.38b}$$

Assume (3.38a); since $F_a(x) = F_a(x+\varepsilon) = 0$ and F_a is positive in $(x, x+\varepsilon)$, F_a takes on its maximum in $t_a \in (x, x+\varepsilon)$. Then

$$0 = F_a'(t_a) = \frac{1}{a(t_a)} \int_{t_a}^{x+\varepsilon} a(y)\,dy - a(t_a) \int_x^{t_a} \frac{dy}{a(y)} \tag{3.39}$$

and it is enough to estimate the first integral on the right hand side of (3.39). To do so we write

$$\frac{1}{a(t_a)} \int_{t_a}^{x+\varepsilon} a(y)\,dy = \int_{t_a}^{x+\varepsilon} e^{\int_{t_a}^y (a'/a)(s)\,ds}\,dy.$$

With ε', M' in (3.38a) to be chosen later we put $y_0 := t_a$, $y_i := y_{i-1} + \varepsilon'$ and determine $L \in \mathbb{Z}_+$ by $y_L < x+\varepsilon \leqslant y_{L+1}$. If $y \in [y_i, y_{i+1}]$, $1 \leqslant i \leqslant L$, we estimate

$$\begin{aligned}\int_{t_a}^y \frac{a'}{a}(s)\,ds &= \left(\int_{y_0}^{y_i} + \int_{y_i}^{y}\right)\frac{a'}{a}(s)\,ds \\ &\leqslant -iM' + \varepsilon' C \\ &\leqslant -\frac{i}{i+1}\frac{M'}{\varepsilon'}(y - t_a) + \varepsilon' C \leqslant -\frac{M'}{2\varepsilon'}(y - t_a) + \varepsilon' C\end{aligned}$$

if we have the one-sided bound $(a'/a)(x) \leqslant C$ for $x \geqslant 1$. Thus we obtain

$$\begin{aligned}\int_{t_a}^{x+\varepsilon} e^{\int_{t_a}^y (a'/a)(s)\,ds}\,dy &\leqslant \varepsilon' e^{\varepsilon' C} + \int_{t_a+\varepsilon'}^{x+\varepsilon} e^{\varepsilon' C - (M'/2\varepsilon')(y-t_a)}\,dy \\ &\leqslant e^{\varepsilon' C}\left(\varepsilon' + \int_{t_a}^{x+\varepsilon} e^{-(M'/2\varepsilon')(y-t_a)}\,dy\right) \\ &\leqslant e^{\varepsilon' C}\varepsilon'\left(1 + \frac{2}{M'}\right).\end{aligned}$$

So we may choose $M' = 2$ and ε' so small that

$$2\varepsilon' e^{C\varepsilon'} \leqslant \frac{1}{M}$$

to reach the desired conclusion. If (3.38b) holds the proof is completely analogous. ∎

Finally, we single out the following obvious corollary to Lemma 3.7.

THEOREM 3.5. *Suppose that the Riccati equation* (3.30) *admits a solution* γ *in* $\mathbb{R}^*$ *which is either bounded from below or bounded from above. Then* T *is discrete if and only if for* $0 < \varepsilon \leqslant 1$

$$\lim_{x \to \infty} \left| \int_x^{x+\varepsilon} \gamma(y)\, dy \right| = \infty.$$

4. AN APPLICATION

Our results can be applied to self-adjoint elliptic operators on noncompact Riemannian manifolds. As an example, we treat the Laplacean on functions on a noncompact complete Riemannian manifold M. We make the following assumption on the structure of M at infinity: there is an open subset U of M such that $M \setminus U$ is a compact manifold with boundary and U is diffeomorphic to $\mathbb{R}^* \times N$ for some compact manifold N. Moreover, we assume that the diffeomorphism induces on $\mathbb{R}^* \times N$ the metric

$$ds^2 = dx^2 + ds_N(x)^2, \tag{4.1}$$

where $ds_N(x)^2$ is a smooth family of metrics on N. If not indicated otherwise, however, N will always be provided with the metric $ds_N(0)^2$. Denote by Δ the nonnegative Laplacean on functions; by the decomposition principle $[D+L]$, Δ is discrete on M iff the Friedrichs extension T of Δ on $\mathbb{R}^* \times N$ is. We show first that the methods of Section 2 apply to T.

LEMMA 4.1. *The Friedrichs extension* T *of* Δ *on* $\mathbb{R}^* \times N$ *is unitarily equivalent to the Friedrichs extension of*

$$\tau := -\partial_x^2 + Q(x) \tag{4.2}$$

with domain $C_0^\infty(\mathbb{R}^*, H_1)$ *in* $L^2(\mathbb{R}_+, H)$ *where*

$$H := L^2(N),$$

$$H_1 := H^2(N), \qquad \text{with} \quad \|u\|_{H_1}^2 := \|u\|_H^2 + \|\Delta u\|_H^2,$$

and

$$Q(x) := \theta^{1/2} \Delta_x \theta^{-1/2} + \frac{1}{2}\frac{\theta''}{\theta} - \frac{1}{4}\left(\frac{\theta'}{\theta}\right)^2. \tag{4.3}$$

Here Δ_x *denotes the Laplacean on* N *with respect to the metric* $ds_N(x)^2$ *and* θ *is defined as follows: if* ω_x *and* ω *are the volume forms on* N *for the metric* $ds_N(x)^2$ *and* $ds_N(0)^2$, *respectively, then*

$$\omega_x = \theta\omega. \tag{4.4}$$

Proof. For $u \in C_0^\infty(\mathbb{R}^* \times N)$ we define

$$\Phi u(x) := \theta^{1/2}(x, \cdot)\, u(x, \cdot) \in H^{1,2}(N) \subset L^2(N). \tag{4.5}$$

Fubini's theorem gives

$$\begin{aligned} \|\Phi u\|_{L^2(\mathbb{R}_+, H)}^2 &= \int_0^\infty \|\Phi u(x)\|_{L^2(N)}^2\, dx \\ &= \int_0^\infty \int_N \theta(x, \cdot)\, |u|^2(x, \cdot)\, \omega\, dx \\ &= \int_U |u|^2. \end{aligned}$$

Thus Φ extends to an isometry $L^2(\mathbb{R}^* \times N) \to L^2(\mathbb{R}_+, H)$ mapping $C_0^\infty(\mathbb{R}^* \times N)$ into $C_0^\infty(\mathbb{R}^*, H_1)$. Using local coordinates it is easily computed that for $u \in C_0^\infty(\mathbb{R}^*, H_1)$ we have

$$\Phi \Delta \Phi^{-1} u(x) = -\partial_x^2 u(x) + \left(\left(\frac{1}{2}\frac{\theta''}{\theta} - \frac{1}{4}\left(\frac{\theta'}{\theta}\right)^2\right) u\right)(x) + (\theta^{1/2} \Delta_x \theta^{-1/2} u)(x), \tag{4.6}$$

where $'$ denotes application of ∂_x. The lemma follows from (4.6). ∎

We start with a necessary condition for the discreteness of Δ based on Theorem 2.1 and Lemma 3.7. Various known conditions are then easy corollaries. The main geometric ingredient is the mean curvature function of the family of hypersurfaces

$$N_x := \{x\} \times N.$$

We define

$$H(x, z) := \text{the mean curvature of } N_x \text{ at } (x, z) \in N_x \tag{4.7}$$

and

$$\bar{H}(x) := \sup_{z \in N} |H(x, z)|, \qquad \underline{H}(x) := \inf_{z \in N} |H(x, z)|. \tag{4.8}$$

H is connected with the operator potential Q in (4.3) by the well-known relation (cf., e.g., [GA, Lemma 3.2])

$$H(x, z) = \frac{\theta'(x, z)}{\theta(x, z)}. \tag{4.9}$$

THEOREM 4.1. *Let $\tilde{\omega}_x := \omega_x / \int_N \omega_x$ be the family of normalized volume forms induced by $ds_N(x)^2$. If Δ is discrete then for $0 < \varepsilon \leqslant 1$*

$$\lim_{x \to \infty} \int_x^{x+\varepsilon} \left| \int_N H \tilde{\omega}_y \right| dy = \infty. \tag{4.10}$$

In particular,

$$\lim_{x \to \infty} \int_x^{x+\varepsilon} \bar{H}(y)\, dy = \infty. \tag{4.11}$$

Proof. Combining Lemma 4.1 and Theorem 2.1 we find that

$$(\Delta u, u) \geqslant M \|u\|^2$$

if $u \in C_0^\infty((x, x+\varepsilon) \times N)$ and $x \geqslant x(M, \varepsilon)$. For $u \in C_0^\infty(x, x+\varepsilon)$ this gives

$$\int_x^{x+\varepsilon} \int_N |u(y)|^2 \omega_y\, dy \leqslant M^{-1} \int_x^{x+\varepsilon} \int_N |u'(y)|^2 \omega_y\, dy \tag{4.12}$$

which is the weighted Sobolev estimate (3.32) with

$$a(y) := \int_N \omega_y.$$

But then by (4.4)

$$\gamma(x) := \frac{1}{2} \frac{a'(x)}{a(x)} = \int_N H \omega_x \Big/ \int_N \omega_x$$

$$= \int_N H \tilde{\omega}_x,$$

so the theorem follows from (the proof of) Lemma 3.7. ∎

LEMMA 4.2. *Assume that Δ is discrete.*

(1) *Let*

$$B(x) := \begin{cases} \operatorname{vol}\{(y, z) \in U \mid y \leqslant x\} & \text{if } \operatorname{vol} U = \infty, \\ [\operatorname{vol}\{(y, z) \in U \mid y \geqslant x\}]^{-1} & \text{if } \operatorname{vol} U < \infty; \end{cases}$$

then

$$\lim_{x \to \infty} \frac{1}{x} \log B(x) = \infty.$$

(2) *$|H|$ cannot be bounded on U.*

(3) *The Ricci curvature cannot be bounded below on U.*

Remark. (1) is essentially proved in [BRO1, 2] by a quite different argument; (2) and (3) are contained in [KL].

Proof. (1) Assume first that with $a(y) = \int_N \omega_y$ we have

$$\infty = \int_1^\infty a(y)\, dy$$

or equivalently $\operatorname{vol} M = \infty$. Then we conclude from (4.12) as in the proof of Lemma 3.7 that for $x \geqslant x(M)$

$$\int_x^{x+1} \int_N \omega_y\, dy \geqslant M \int_{x-1}^x \int_N \omega_y\, dy, \tag{4.13}$$

where M can be choosen arbitrarily large. Iterating (4.13) we obtain for $N \in \mathbb{N}$, $x \geqslant x(M)$,

$$\log B(x+N) \geqslant N \log M + C \tag{4.14}$$

which implies the assertion in this case. If $\operatorname{vol} M < \infty$ we obtain in place of (4.13)

$$\int_x^{x+1} \int_N \omega_y\, dy \geqslant M^{-1} \int_{x-1}^x \int_N \omega_y\, dy$$

which by iteration again gives (4.14).

(2) This is obvious from Theorem 4.1.

(3) The volume growth comparison theorems (cf. [GA, Lemma 4.2]) imply that a lower bound on the Ricci curvature gives a bound on $|H|$ which is impossible by (2). ∎

We turn to sufficient conditions for discreteness which we deduce either from Lemma 3.7 or from Theorem 3.2. In the first case we have to assume a one-sided bound for H, and the coerciveness condition (1.13) in the second.

THEOREM 4.2. (a) *Assume that the mean curvature function H is bounded from below or above on U. Then*

$$\lim_{x\to\infty}\int_x^{x+\varepsilon} \underline{H}(y)\,dy=\infty \tag{4.15}$$

implies the discreteness of T.

(b) *If τ in (4.2) is coercive then*

$$\lim_{x\to\infty}\int_x^{x+\varepsilon} \underline{H}(y)^2\,dy=\infty \tag{4.16}$$

is a sufficient condition for discreteness of T.

Proof. (a) By Lemma 4.2 and Theorem 2.1 the discreteness of Δ follows if for given M, ε, and $x\geqslant x(M,\varepsilon)$ we have for $u\in C_0^\infty((x,x+\varepsilon)\times N)$

$$\int_N\int_x^{x+\varepsilon}|u'(y,z)|^2\,\theta(y,z)\,dy\,\omega(z)\geqslant M\int_N\int_x^{x+\varepsilon}|u(y,z)|^2\,\theta(y,z)\,dy\,\omega(z)$$

which is implied by

$$\int_x^{x+\varepsilon}|v'(y)|^2\,\theta(y,z)\,dy\geqslant M\int_x^{x+\varepsilon}|v(y)|^2\,\theta(y,z)\,dy \tag{4.17}$$

uniformly in $v\in C_0^\infty(x,x+\varepsilon)$ and $z\in N$. It follows from the proof of Lemma 3.7(b) that (4.17) is a consequence of the semiboundedness of H and (4.15).

(b) Assuming the coerciveness of (4.2) a (necessary and) sufficient condition for discreteness of T is

$$\lim_{x\to\infty}\inf_{u\in\mathscr{H}(x;\varepsilon,\alpha)}\int_x^{x+\varepsilon}\langle Q(y)\,u(y),u(y)\rangle_H\,dy=\infty,$$

where

$$\mathscr{H}(x;\varepsilon,\alpha):=\{u\in C_0^\infty((x,x+\varepsilon),H_1)\mid \|u\|_{L^2}^2=1,\ \|u'\|_{L^2}^2\leqslant\alpha/\varepsilon^2\}.$$

Let us denote by $H(x)\in\mathscr{L}(H)$ the multiplication operator defined by the mean curvature function on $H=L^2(N)$. Then we obtain from (4.3) for $u\in\mathscr{H}(x;\varepsilon,\alpha)$

$$\begin{aligned}
&\int_x^{x+\varepsilon}\langle Q(y)\,u(y),u(y)\rangle_H\,dy\\
&\geqslant\int_x^{x+\varepsilon}\left\langle\left(\frac{1}{2}A'(y)+\frac{1}{4}A(y)^2\right)u(y),u(y)\right\rangle_H dy\\
&=\int_x^{x+\varepsilon}\left[-\operatorname{Re}\langle A(y)\,u(y),u'(y)\rangle_H+\frac{1}{4}\|A(y)\,u(y)\|_H^2\right]dy\\
&\geqslant\frac{1}{8}\int_x^{x+\varepsilon}\|A(y)\,u(y)\|_H^2\,dy-2\,\|u'\|_{L^2}^2\\
&\geqslant\frac{1}{8}\int_x^{x+\varepsilon}\underline{H}(y)^2\,\|u(y)\|_H^2\,dy-2\alpha/\varepsilon^2.
\end{aligned}$$

Arguing as in the proof of Theorem 3.2 we find $x^*\in[x,x+\varepsilon]$ such that

$$\|u(y)\|^2\geqslant 1/4\varepsilon\qquad\text{if}\quad x^*\leqslant y\leqslant x^*+\varepsilon/4\alpha,$$

hence

$$\int_x^{x+\varepsilon}\langle Q(y)\,u(y),u(y)\rangle_H\,dy\geqslant\frac{1}{32\varepsilon}\int_{x^*}^{x^*+\varepsilon/4\alpha}\underline{H}(y)^2\,dy-2\alpha/\varepsilon^2.$$

The proof is complete. ∎

COROLLARY ("Basic technical criterion" of [D+L]). *If $\lim_{x\to\infty}\underline{H}(x)=\infty$ then Δ is discrete.*

The assumptions of Theorem 4.2 are generally easy to check in concrete cases. They do not seem to be implied, however, by more familiar assumptions on the geometry of U unless we have good control over the second fundamental form of the hypersurfaces N_x as $x\to\infty$. Thus we obtain the most complete answer if each connected component of U is a warped product.

THEOREM 4.3. *Assume that each connected component of U is a warped product with warping function f_i, $1\leqslant i\leqslant N$.*

(a) *Δ is discrete iff the scalar operators*

$$\tau_i:=-\partial_x^2+\frac{n}{2}\frac{f_i''(x)}{f_i(x)}+\left(\frac{n^2}{4}-\frac{n}{2}\right)\frac{f_i'(x)^2}{f_i(x)^2}$$

have discrete Friedrichs extension in $L^2(\mathbb{R}_+)$ for all i; here $n = \dim N$. Thus a necessary and sufficient condition for discreteness follows from Theorem 3.4.

(b) *If $n \geq 2$ and the operator*

$$\tau := -\partial_x^2 - \mathrm{Ricc}(\partial_x, \partial_x)$$

satisfies the coerciveness assumption (1.13) *then Δ is discrete iff*

$$\lim_{x \to \infty} \int_x^{x+\varepsilon} |H(y)|\, dy = \infty.$$

Proof. (a) By the decomposition principle it is enough to treat each end separately, so assume that $U = R^* \times_f N$. Then $ds_N(x)^2 = f(x)^2 ds_N(0)^2$, so by Lemma 4.1 we obtain that T is unitarily equivalent to the Friedrichs extension $\tilde{T}$ of

$$\begin{aligned}\tau &:= -\partial_x^2 + f(x)^{-2n} \Delta_N + \frac{n}{2}\frac{f''(x)}{f(x)} + \left(\frac{n^2}{4} - \frac{n}{2}\right)\frac{f'(x)^2}{f(x)^2} \\ &=: -\partial_x^2 + f(x)^{-2n} \Delta_N + q(x)\end{aligned}$$

with domain $C_0^\infty(\mathbb{R}^*, H^2(N))$ in $L^2(\mathbb{R}_+, L^2(N))$. Denote by $0 = \lambda_1 < \lambda_2 \leq \cdots \leq \lambda_k \to \infty$ the eigenvalues of Δ_N and decompose $H = L^2(N)$ into the eigenspaces. Then it is easily seen that $\tilde{T}$ is unitarily equivalent to $\bigoplus_{i \geq 1} \tilde{T}_i$ with $\tilde{T}_i$ the Friedrichs extension of

$$\tau_i := -\partial_x^2 + q(x) + \lambda_i f(x)^{-2n}$$

in $L^2(\mathbb{R}_+)$. Clearly, the discreteness of $\tilde{T}$ implies the discreteness of $\tilde{T}_1$. Conversely, if $\tilde{T}_1$ is discrete so is $\tilde{T}_i$ for each i by the max–min principle. Thus the discreteness of $\tilde{T}$ will follow if we show that $\lim_{i \to \infty} \lambda_1(\tilde{T}_i) = \infty$ where $\lambda_1(\tilde{T}_i)$ is the smallest eigenvalue. To see this we use the terminology introduced in Theorem 2.1: with $\varepsilon = 1$ and N sufficiently large in (2.5) we obtain from (2.8) for $u \in C_0^\infty(\mathbb{R}^*)$

$$(\tilde{T}_i u, u) \geq \sum_{j=1}^{2} ((\tilde{T}_i \Phi_j u, \Phi_j u) + (\tilde{T}_i \Psi_j u, \Psi_j u)) - \|u\|_{L^2}^2. \tag{4.18}$$

Fixing $M > 0$ we can make $N = N(M)$ so large that

$$\begin{aligned}(\tilde{T}_i \Phi_2 u, \Phi_2 u) + (\tilde{T}_i \Psi_2 u, \Psi_2 u) &\geq (\tilde{T}_1 \Phi_2 u, \Phi_2 u) + (\tilde{T}_1 \Psi_2 u, \Psi_2 u) \\ &\geq (M+1)(\|\Phi_2 u\|_{L^2}^2 + \|\Psi_2 u\|_{L^2}^2),\end{aligned} \tag{4.19}$$

by Theorem 2.2. Since f is positive there is constant $C(M) > 0$ such that

$$\begin{aligned}&(\tilde{T}_i \Phi_1 u, \Phi_1 u) + (\tilde{T}_i \Psi_1 u, \Psi_1 u) \\ &\quad \geq C(M)^{-2} \lambda_i (\|\Phi_1 u\|_{L^2}^2 + \|\Psi_1 u\|_{L^2}^2).\end{aligned} \tag{4.20}$$

Now choose i so large that $C(M)^{-2}\lambda_i \geq M + 1$; then it follows from (4.18), (4.19), (4.20), and (2.6) that

$$(\tilde{T}_i u, u) \geq M \|u\|_{L^2}^2,$$

hence $\lambda_1(\tilde{T}_i) \geq M$.

Remark. The result just proved follows also from the more general Theorem 3.3 in [BA].

(b) The necessity follows from Theorem 4.1. To prove sufficiency we only have to observe that

$$\frac{n^2}{4} - \frac{n}{2} = \frac{n}{2}\left(\frac{n}{2} - 1\right) \geq 0 \qquad \text{if} \quad n \geq 2$$

and

$$-\mathrm{Ricc}(\partial_x, \partial_x) = n\frac{f''(x)}{f(x)}$$

(cf. [ON, p. 211]). The assertion now follows from Theorem 4.2(b). ∎

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Schur Products and Matrix Completions

VERN I. PAULSEN*

Department of Mathematics, University of Houston, Houston, Texas 77004

STEPHEN C. POWER†

Department of Mathematics, University of Lancaster, Lancaster, England

AND

ROGER R. SMITH

Department of Mathematics, Texas A&M University, College Station, Texas 77843

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We prove that a necessary and sufficient condition for a given partially positive matrix to have a positive completion is that a certain Schur product map defined on a certain subspace of matrices is a positive map. By analyzing the positive elements of this subspace, we obtain new proofs of the results of H. Dym and I. Gohberg and Grone, Johnson, Sa, and Wolkowitz (*Linear Algebra Appl.* **58** (1984), 109–124). (*Linear Algebra Appl.* **36** (1981), 1–24). We also obtain a new proof of a result of U. Haagerup (Decomposition of completely bounded maps on operation algebras, preprint), characterizing the norm of Schur product maps, and a new Hahn–Banach type extension theorem for these maps. Finally, we obtain generalizations of many of these results to matrices of operators, which we apply to the study of representations of certain subalgebras of the $n \times n$ matrices.

1. INTRODUCTION

An $n \times n$ complex matrix is *partially defined* if only some of its entries are specified with the unspecified entries treated as complex variables. A

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