Commun. Math. Phys. 129, 525–534 (1990)

Communications in Mathematical Physics

The $\bar{\partial}$-Operator on Algebraic Curves

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Abstract. For a singular algebraic curve we show that all closed extensions of $\bar{\partial}$ are Fredholm, and we give a general index formula. In particular, we prove a modified version of a conjecture due to MacPherson.

1. Introduction

Let M be a Kähler manifold of complex dimension m and denote by $\Omega^{p,q}(M)$ and $\Omega_0^{p,q}(M)$ the space of smooth complex valued forms of type (p, q) on M and the subspace of forms with compact support, respectively. The Dolbeault complex

$$0 \to \Omega_0^{0,0}(M) \xrightarrow{\bar{\partial}} \cdots \xrightarrow{\bar{\partial}} \Omega_0^{0,m}(M) \to 0 \tag{1.1}$$

is well known to be elliptic. If M is compact the cohomology, $H^{0,*}(M)$, is finite and the index,

$$\chi(M) := \sum_{q \geq 0} (-1)^q \dim H^{0,q}(M), \tag{1.2}$$

is called the arithmetic genus of M (cf. [H]). If M is not compact one can use the Hilbert space structure induced by the metric to define

$$\Omega_{(2)}^{p,q}(M) := \left\{ \omega \in \Omega^{p,q}(M) \mid \int_M \omega \wedge *\omega < \infty, \int_M \bar{\partial}\omega \wedge *\bar{\partial}\omega < \infty \right\}. \tag{1.3}$$

Here the Hodge $*$ operator on real forms is extended as an antilinear map. This leads to another complex

$$0 \to \Omega_{(2)}^{0,0}(M) \xrightarrow{\bar{\partial}} \cdots \xrightarrow{\bar{\partial}} \Omega_{(2)}^{0,m}(M) \to 0, \tag{1.4}$$

the cohomology of which is called the L^2-$\bar{\partial}$-cohomology, $H_{(2)}^{0,*}(M)$. It is natural to ask for conditions on M which ensure the finiteness of $H_{(2)}^{0,*}(M)$. If this is guaranteed one wants to compute the cohomology groups or at least the index of the complex (1.4),

$$\chi_{(2)}(M) := \sum_{q\geq 0}(-1)^q \dim H^{0,q}_{(2)}(M), \tag{1.5}$$

which we refer to as the L^2-arithmetic genus of M. A particularly interesting case arises if M is the nonsingular locus of an algebraic variety equipped with the Fubini-Study metric. In this case MacPherson conjectured [McP] that for a suitable resolution $\tilde{M}$ of M

$$\chi_{(2)}(M) = \chi(\tilde{M}). \tag{1.6}$$

To attack this problem it is convenient to reformulate it as an index problem for a single elliptic operator. If $\bar\partial'$ denotes the formal adjoint of $\bar\partial$ then we define

$$D := \bar\partial + \bar\partial' : \bigoplus_{p\geq 0} \Omega_0^{0,2p}(M) \to \bigoplus_{p\geq 0} \Omega_0^{0,2p+1}(M). \tag{1.7}$$

It is easy to see that D is elliptic. We define a closed extension of D as follows: denote by $A^{0,p}(M)$ the completion of $\Omega_{(2)}^{0,p}(M)$ under the Hilbert norm and by D_p the maximal closed extension of $\bar\partial$ as an operator in $A^{0,p}(M)$ with values in $A^{0,p+1}(M)$. Then we put

$$\begin{aligned}\tilde{D} : \bigoplus_{p\geq 0} \mathscr{D}(D_{2p}) \cap \mathscr{D}(D^*_{2p-1}) &\to \bigoplus_{p\geq 0} A^{0,2p+1}(M),\\ (\omega_0, \omega_2, \ldots) &\mapsto (D_0\omega_0 + D_1^*\omega_2, \ldots).\end{aligned} \tag{1.8}$$

It is then easy to see that $\tilde{D}$ is a Fredholm operator if and only if the L^2-$\bar\partial$-cohomology is finite, and in this case

$$\operatorname{ind}\tilde{D} = \chi_{(2)}(M).$$

It is reasonable to expect that an index formula for $\tilde{D}$ will have an interior contribution (suitably regularized) as in the compact case plus certain contributions from the singularities. Therefore, there is no a priori reason why (1.6) should be true. On the other hand, since D will in general possess many closed extensions there may be a closed extension different from $\tilde{D}$ with index equal to $\chi(\tilde{M})$ for suitable $\tilde{M}$. The purpose of this note is to illustrate this phenomenon in the simplest case when M is an algebraic curve. Our main results (Theorems 3.1, 4.1, 4.2 below) will show that for curves all closed extensions of D are Fredholm operators and that the index of the minimal extension, $\bar{D}$, equals in fact $\chi(\tilde{M})$, where $\tilde{M}$ is the resolution of M. $\tilde{D}$, however, is the maximal extension of D which has a larger index if M is singular. Thus (1.6) is false as stated and it remains to be seen what a correct substitute could be in general. In the curve case at least we should replace the left-hand side of (1.6) by $\operatorname{ind}\bar{D} = \operatorname{ind}\bar\partial_{\min}$.

The incorrectness of MacPherson's conjecture has been observed before by Pardon [P]. He also gave a sheaf theoretic formula for $\operatorname{ind}\tilde{D} = \operatorname{ind}\bar\partial_{\max}$ which is equivalent to our Theorem 4.2.

Our approach does not use sheaf theoretic methods at all and is purely analytic. It relies entirely on the analysis of regular singular operators developed by R. Seeley and the first author, in the general form given in [B]. It is conceivable that a suitable extension of our methods will be able to handle more general cases.

2. Algebraic Curves

An algebraic curve M is a one-dimensional subvariety of $\mathbb{C}P^n$. We assume that M is irreducible. Via the resolution process we can always think of M as the image of a nonconstant holomorphic map

$$\pi : \tilde{M} \to \mathbb{C}P^n, \tag{2.1}$$

where $\tilde{M}$ is a compact Riemann surface. If Σ denotes the singular locus of M then $\Sigma = \pi\{q \in \tilde{M} \mid d\pi(q) = 0\} \cup \{p \in M \mid \sharp\pi^{-1}(p) > 1\}$, and Σ is a finite set. For $p \in \Sigma$ let $\pi^{-1}(p) = \{q_1, \ldots, q_{r(p)}\}$. Then we can find neighborhoods U of p and $\tilde{U}_i$ of q_i such that

$$\pi^{-1}(U) = \bigcup_i \tilde{U}_i,$$

and for $i \neq j$

$$\tilde{U}_i \cap \tilde{U}_j = \emptyset, \qquad \pi(\tilde{U}_i) \cap \pi(\tilde{U}_j) = \{p\}.$$

It follows that π maps $\tilde{U}_i^* := \tilde{U}_i \backslash \{q_i\}$ biholomorphically to $U_i^* := \pi(\tilde{U}_i^*)$ and

$$U \backslash \{p\} = \bigcup_i U_i^*.$$

Thus M decomposes near p into $r(p)$ branches. We introduce a convenient parametrization of each branch as follows. Near $q \in \pi^{-1}(\Sigma)$ choose a holomorphic chart $(\tilde{U}, \tilde{\varphi})$ with $\tilde{\varphi}(\tilde{U}) = D_\varepsilon := \{z \in \mathbb{C} \mid |z| < \varepsilon\}$ for some $\varepsilon > 0$, and near p choose homogeneous coordinates $[Z_0, \ldots, Z_n]$ in $\mathbb{C}P^n$ such that $p = [1, 0, \ldots, 0]$. Then we have

$$\pi \circ \tilde{\varphi}^{-1}(z) = [1, P_1(z), \ldots, P_n(z)], \qquad z \in D_\varepsilon, \tag{2.2}$$

with P_i holomorphic in D_ε. Moreover, by change of coordinates we may assume that for some k, $1 \leq k \leq n$,

$$\begin{aligned} &P_i(z) \equiv 0, \qquad 1 \leq i \leq k-1, \\ &P_k(z) = z^{N_k}, \\ &P_i(z) = \sum_{j \geq N_i} a_{ij} z^j, \qquad k+1 \leq i \leq n, \end{aligned} \tag{2.3}$$

where $1 \leq N_k < N_{k+1} < \cdots < N_n$. Put $N(q) := N_k$ and call

$$N(p) := \sum_{q \in \pi^{-1}(p)} N(q) \tag{2.4}$$

the multiplicity of p. As an example, consider the zero set, M, of the homogeneous polynomial

$$F(Z_0, Z_1, Z_2) = Z_0 Z_2^2 - Z_1^3$$

in $\mathbb{C}P^2$. The only singular point is $p = [1, 0, 0]$ and the resolution is given by

$$\tilde{M} = \mathbb{C}P^1, \qquad \pi([Z_0, Z_1]) = [Z_0^3, Z_0 Z_1^2, Z_1^3].$$

Thus $r(p) = 1$, $N(p) = 2$.

The Fubini-Study metric on $\mathbb{C}P^n$ induces a Kähler metric on $M\backslash\Sigma$, hence a Kähler metric on $D_\varepsilon^* := D_\varepsilon\backslash\{0\}$ via $\pi \circ \tilde{\psi}^{-1}$. If $w_i = Z_i/Z_0$, $1 \leq i \leq n$, are local holomorphic coordinates for $\mathbb{C}P^n$ near $p = [1, 0, \ldots, 0]$ then the Fubini-Study metric is given by the positive $(1, 1)$-form

$$\omega = \frac{\sqrt{-1}}{2}\left[\left(1 + \sum_i |w_i|^2\right)^{-1} \sum_i dw_i \wedge d\bar{w}_i - \left(1 + \sum_i |w_i|^2\right)^{-2} \sum_{i,j} \bar{w}_i w_j \, dw_i \wedge d\bar{w}_i\right]. \tag{2.5}$$

For simplicity, denote the induced $(1, 1)$-form on $M\backslash\Sigma$ also by ω; note that ω is the volume form. Writing

$$\tilde{\omega} := (\pi \circ \tilde{\psi}^{-1})^* \omega =: \frac{\sqrt{-1}}{2} h \, dz \wedge d\bar{z}, \tag{2.6}$$

one calculates that

$$h(z, \bar{z}) = N(q)^2 |z|^{2N(q)-2} + O(|z|^{2N(q)}). \tag{2.7}$$

3. The $\bar{\partial}$-Operator

For $m = 1$ the Dolbeault complex (1.1) becomes

$$0 \to \Omega_0^{0,0}(M\backslash\Sigma) \xrightarrow{\bar{\partial}} \Omega_0^{0,1}(M\backslash\Sigma) \to 0 \tag{3.1}$$

and $\bar{\partial}$ coincides with the operator D in (1.7). To investigate the Fredholm properties of $\bar{\partial}$ using the methods of [B] we have to choose a suitable normal form of $\bar{\partial}$ near Σ. To do so we consider a local parametrization

$$\psi = \pi \circ \tilde{\psi}^{-1} : D_\varepsilon^* \to U^*, \qquad \tilde{\psi}^{-1}(0) = q \in \pi^{-1}(\Sigma), \tag{3.2}$$

of some branch of M near $p \in \Sigma$, as in Sect. 2, and we want a suitable unitary transformation for

$$\bar{\partial} : \Omega_0^{0,0}(U^*) \to \Omega_0^{0,1}(U^*).$$

We transform first to the space $L^2(D_\varepsilon^*, dx \wedge dy)$ using the maps

$$\Phi_0 : \Omega_0^{0,0}(D_\varepsilon^*) \ni f \mapsto (h^{-1/2} f) \circ \psi^{-1} \in \Omega_0^{0,0}(U^*) \tag{3.3a}$$

[with h defined in (2.7)] and

$$\Phi_1 : \Omega_0^{0,1}(U^*) \ni f \, d\bar{z} \mapsto f \circ \psi \in \Omega_0^{0,0}(D_\varepsilon^*). \tag{3.3b}$$

We compute

$$\|\Phi_0 f\|^2_{\Omega^{0,0}_{(2)}(U^*)} = \int_{U^*} (h^{-1/2}f)\circ\psi^{-1} \wedge *(h^{-1/2}f)\circ\psi^{-1}$$
$$= \int_{\psi(D_\varepsilon^*)} (h^{-1}|f|^2)\circ\psi^{-1}\omega = \int_{D_\varepsilon^*} h^{-1}|f|^2\psi^*\omega$$
$$= \int_{D_\varepsilon^*} |f|^2 dx\wedge dy$$

and

$$\|f\,d\bar z\|^2_{\Omega^{0,1}_{(2)}(U^*)} = \int_{U^*} f\,d\bar z \wedge * f\,d\bar z = \int_{U^*} |f|^2|d\bar z|^2\omega$$
$$= \int_{D_\varepsilon^*} |f\circ\psi|^2 h^{-1}\tilde\omega = \int_{D_\varepsilon^*} |f\circ\psi|^2 dx\wedge dy\,.$$

Thus Φ_0 and Φ_1 are unitary, and for $f\in\Omega_0^{0,0}(D_\varepsilon^*) = C_0^\infty(D_\varepsilon^*)$ we obtain, since ψ is holomorphic,

$$\begin{aligned} D_1 := \Phi_1\bar\partial\Phi_0 f &= \Phi_1\bar\partial(\psi^{-1})^*(h^{-1/2}f)\\ &= \Phi_1(\psi^{-1})^*\bar\partial(h^{-1/2}f) = \psi^*\circ(\psi^{-1})^*\bar\partial(h^{-1/2}f)\left(\frac{\partial}{\partial\bar z}\right)\\ &= \frac{\partial}{\partial\bar z}(h^{-1/2}f)\,. \end{aligned} \tag{3.4}$$

To bring this to a "regular singular" form we use a further transformation. From (2.7) we have with $h_1\in C^\infty([0,\varepsilon]\times S^1)$

$$h(r,\varphi)^{1/2} = Nr^{N-1} + r^N h_1(r,\varphi)\,, \tag{3.5}$$

where $N := N(q)$. We introduce for ε small

$$\chi_1 : (0,\varepsilon)\times S^1 \ni (r,\varphi)\mapsto (x(r),\varphi)\in(0,\varepsilon^{N(q)})\times S^1\,,\qquad x(r) = r^{N(q)}\,. \tag{3.6}$$

We write $\delta := \varepsilon^{N(q)}$. Denoting by χ_2 the diffeomorphism defining polar coordinates,

$$\chi_2 : (0,\varepsilon)\times S^1 \to D_\varepsilon^*\,,$$

we obtain a diffeomorphism

$$\chi := \chi_2\circ\chi_1^{-1} : (0,\delta)\times S^1 \to D_\varepsilon^*\,.$$

Clearly,

$$\Phi_2 : C_0^\infty((0,\delta)\times S^1)\ni f\mapsto (\det D\chi^{-1})^{1/2} f\circ\chi^{-1}\in C_0^\infty(D_\varepsilon^*)$$

is unitary. To compute

$$D_2 := 2e^{-\sqrt{-1}\varphi}\Phi_2^{-1}D_1\Phi_2 \tag{3.7}$$

we need $\bar\partial$ on functions in polar coordinates (r, φ),

$$\frac{\partial}{\partial \bar z} = \frac{e^{\sqrt{-1}\varphi}}{2}\left(\partial_r + \frac{\sqrt{-1}}{r}\partial_\varphi\right), \tag{3.8}$$

and the relation

$$\det D\chi^{-1}(z) = N|z|^{N-2}. \tag{3.9}$$

Then we find for $f \in C_0^\infty((0, \delta) \times S^1) \cong C_0^\infty((0, \delta), C^\infty(S^1))$, computing in polar coordinates (r, φ) in D_ε^*,

$$\begin{aligned}
2e^{-\sqrt{-1}\varphi} D_1 \Phi_2 f(r, \varphi) &= \left(\partial_r + \frac{\sqrt{-1}}{r}\partial_\varphi\right)\left(\sqrt{N} r^{N/2-1} h(r, \varphi)^{-1/2} f(x(r), \varphi)\right)\\
&= \sqrt{N} r^{N/2-1}\Big[N r^{N-1} h(r, \varphi)^{-1/2}\frac{\partial f}{\partial x}(r^N, \varphi)\\
&\quad + r^{1-N/2}\frac{\partial}{\partial r}(r^{N/2-1} h(r, \varphi)^{-1/2}(f(r^N, \varphi)\\
&\quad + \frac{\sqrt{-1}}{r}\frac{\partial}{\partial \varphi}(h(r, \varphi)^{-1/2}) f(r^N, \varphi)\\
&\quad + \frac{\sqrt{-1}}{r} h(r, \varphi)^{-1/2}\frac{\partial f}{\partial \varphi}(r^N, \varphi)\Big]\\
&=: (\det D\chi^{-1})^{1/2}(r, \varphi)\Big[a \circ \chi^{-1}\frac{\partial f}{\partial x} \circ \chi^{-1}\\
&\quad + b \circ \chi^{-1}\sqrt{-1}\frac{\partial f}{\partial \varphi} \circ \chi^{-1} + c \circ \chi^{-1} f \circ \chi^{-1}\Big](r, \varphi)
\end{aligned}$$

or

$$D_2 f(x, \varphi) = \left(a\frac{\partial f}{\partial x} + b\sqrt{-1}\frac{\partial f}{\partial \varphi} + cf\right)(x, \varphi). \tag{3.10}$$

Next we have to compute the asymptotic behavior of the coefficients a, b, c as $x \to 0$. Using (3.5) we obtain

$$a(x, \varphi) = N x^{1-1/N} h(x^{1/N}, \varphi)^{-1/2} =: 1 + x^{1/N} a_1(x^{1/N}, \varphi), \tag{3.11a}$$

$$b(x, \varphi) = x^{-1/N} h(x^{1/N}, \varphi)^{-1/2} = N^{-1}x^{-1} + x^{1/N-1} b_1(x^{1/N}, \varphi), \tag{3.11b}$$

$$\begin{aligned}
c \circ \chi^{-1}(r, \varphi) &=: r^{1-N/2}\frac{\partial}{\partial r}(r^{N/2-1}N^{-1}r^{1-N} + r^{N/2-1+2-N} h_2(r, \varphi))\\
&\quad + \sqrt{-1} r^{-1+2-N}\frac{\partial h_2}{\partial \varphi}(r, \varphi)\\
&=: -\frac{1}{2} r^{-N} + r^{1-N} c_1(r, \varphi),
\end{aligned}$$

thus

$$c(x, \varphi) = -\frac{1}{2} x^{-1} + x^{1/N-1} c_1(x^{1/N}, \varphi) \tag{3.11c}$$

where $a_1, b_1, c_1 \in C^\infty([0, \varepsilon] \times S^1)$.

We are now in the position to apply the results of [B]. For each $p \in \Sigma$ we choose neighborhoods U_p, $\tilde U_{pi}$ as above and we put

$$U := \bigcup_{p\in\Sigma} U_p^*, \qquad M_1 := M\backslash U .$$

Putting together the unitary transformations (3.10) for all $\tilde U_{pi}^*$ we write $R := \sum_{p\in\Sigma} r(p)$ and denote by S_R^1 the disjoint union of R copies of S^1. Then the operator (3.7) acts in $H := L^2([0, \delta], L^2(S_R^1))$ with domain $C_0^\infty((0, \delta), C^\infty(S_R^1))$, and we can write it in the form

$$D_2 =: B_1(x)\partial_x + x^{-1}(\tilde S_0 + \tilde S_1(x)) . \tag{3.12}$$

Here

$$B_1(x) \text{ is multiplication by } a(x, \cdot) \text{ in each } L^2(S^1), \tag{3.13}$$

$$\tilde S_0 = \bigoplus_{\substack{p\in\Sigma \\ q\in\pi^{-1}(p)}} N(q)^{-1}\sqrt{-1}\partial_\varphi - \frac{1}{2}, \tag{3.14}$$

$$\tilde S_1(x) = x^{1/N} b_1(x^{1/N}, \varphi)\sqrt{-1}\partial_\varphi + x^{1/N} c_1(x^{1/N}, \varphi) \text{ on each } L^2(S^1) . \tag{3.15}$$

With $H_1 := H^1(S_R^1)$, the Sobolev space of order 1, $\tilde S_0$ is self-adjoint with domain H_1, and the assumption (3.1) in [B] is satisfied for U^* and the isometries constructed above. From (3.11b, c) we obtain (3.6b) in [B], with $\delta = 1/N$, which we need since $-\frac{1}{2} \in \operatorname{spec} \tilde S_0$. Using (3.11a) we verify loc. cit. (3.6b, c) with the same δ. The remaining assumptions are obviously satisfied.

Theorem 3.1. *All closed extensions of $\bar\partial$, with domain in $A^{0,0}(M)$ and range in $A^{0,1}(M)$, are Fredholm operators. With $\bar\partial_{\min}$ and $\bar\partial_{\max}$ denoting the minimal and maximal extension, respectively, the closed extensions correspond bijectively to the subspaces of the finite dimensional space*

$$W := \mathscr{D}(\bar\partial_{\max})/\mathscr{D}(\bar\partial_{\min}) . \tag{3.16}$$

If $\bar\partial_V$ is the closed extension corresponding to $V \subset W$ then

$$\operatorname{ind} \bar\partial_V = \operatorname{ind} \bar\partial_{\min} + \dim V . \tag{3.17}$$

Proof. In view of the preceding discussion this follows from Theorem 3.4 in [B]. Note that with $\pi : \mathscr{D}(\bar\partial_{\max}) \to W$ the canonical projection we have $\mathscr{D}(\bar\partial_V) = \pi^{-1}(V)$ and $\bar\partial_V = \bar\partial_{\max} \mid \mathscr{D}(\bar\partial_V)$. □

4. The Index Formula

We shall now follow the outline in [B, Sect. 4] to compute the index of $\bar\partial_{\min}$ and $\bar\partial_{\max}$.

Theorem 4.1

$$\operatorname{ind} \bar\partial_{\min} = \chi(\tilde M) .$$

Proof. We would like to apply [B, Theorem 4.2] but the assumption (4.1), i.e. $\tilde{S}_1(x) \equiv 0$, $B_1(x) \equiv I$ near $x = \delta$, is not necessarily satisfied. To achieve this we choose $\phi \in C_0^\infty(\mathbb{R})$ with $0 \leq \phi \leq 1$, $\phi = 1$ in a neighborhood of 1, and $\phi(x) = 0$ if $x \notin (\frac{1}{2}, \frac{3}{2})$. Then we deform $h^{1/2}$, simultaneously on all branches near all singular points, by putting locally near q (on $D_{2\varepsilon}$)

$$\begin{aligned} h_t(r, \varphi)^{1/2} :=& (1 - \phi(r/\varepsilon))h(r, \varphi)^{1/2} + \phi(r/\varepsilon)\,(th(r, \varphi)^{1/2} \\ &+ (1-t)N(q)r^{N(q)-1}), \qquad t \in [0, 1]\,. \end{aligned} \tag{4.1}$$

This defines a family of operators, D_t, on $\mathscr{D}(\bar{\partial}_{\min})$ which is continuous in view of (3.4). On the other hand, the arguments of Sect. 3 show that all operators D_t are Fredholm so

$$\operatorname{ind} \bar{\partial}_{\min} = \operatorname{ind} D_1 = \operatorname{ind} D_0\,. \tag{4.2}$$

Thus we obtain the situation required in loc. cit. where we have

$$S_0 = \bigoplus_{p \in \Sigma} \bigoplus_{q \in \pi^{-1}(p)} (N(q)^{-1}\sqrt{-1}\partial_\varphi - 1/2) \tag{4.3}$$

acting in the direct sum of $R = \sum_{p \in \Sigma} r(p)$ copies of $L^2(S^1)$. Clearly,

$$\operatorname{spec} S_0 = \bigcup_{\substack{p \in \Sigma \\ q \in \pi^{-1}(p)}} \{N(q)^{-1}k - 1/2 \mid k \in \mathbb{Z}\}\,. \tag{4.4}$$

Thus $\operatorname{spec} S_0$ is symmetric with respect to the origin and the η-functions η_{S_0} vanishes identically. Furthermore, it is readily seen that

$$\frac{1}{2} \dim \ker S_0 + \sum_{-1/2 < s < 0} \dim \ker (S_0 - s) = \sum_{\substack{p \in \Sigma \\ q \in \pi^{-1}(p)}} (N(q) - 1)/2\,. \tag{4.5}$$

To compute the last ingredient of the index formula we need to know the "index form" ω_{D_0}. Now changing h to h_0 in (4.1) amounts to a conformal change of the original metric. The new metric is also Kähler so we can compute ω_{D_0} by heat equation methods. The local version of the Atiyah-Singer index theorem in this case (cf. [Gi, Theorem 3.6.10]) gives (with $T'M$ the holomorphic tangent bundle)

$$\omega_{D_0} = \frac{1}{2} c_1(T'M)\,.$$

It is well known that (cf. e.g. [G+H])

$$c_1(T'M) = (2\pi)^{-1} K_0 \omega_0\,,$$

where K_0 is the Gauß curvature and ω_0 the volume form of M for the metric with conformal factor h_0. So we end up with

$$\omega_{D_0} = (4\pi)^{-1} K_0 \omega_0\,. \tag{4.6}$$

Now we derive from [B, Theorem 4.2] and (4.2), (4.5), (4.6),

$$\operatorname{ind} \bar{\partial}_{\min} = (4\pi)^{-1} \int_{M_1} K_0 \omega_0 - \frac{1}{2} \sum_{\substack{p \in \Sigma \\ q \in \pi^{-1}(p)}} (N(q) - 1) . \tag{4.7}$$

It remains to evaluate the integral on the right-hand side in (4.7) which we do using the Gauß-Bonnet Theorem for surfaces with boundary (cf. e.g. [dC, p. 274]). It follows that

$$(4\pi)^{-1} \int_{M_1} K_0 \omega_0 = \frac{1}{2} \chi_e(M_1) - \frac{1}{4\pi} \sum_{\substack{p \in \Sigma \\ q \in \pi^{-1}(p)}} \int_{c_q} \kappa_g , \tag{4.8}$$

where χ_e denotes the Euler characteristic and κ_g the geodesic curvature of c_q. Computing κ_g in polar coordinates on D_ε^* (cf. [dC, p. 252]) we find

$$\frac{1}{2\pi} \int_{c_q} \kappa_g = -N(q) ,$$

hence

$$(4\pi)^{-1} \int_{M_1} K_0 \omega_0 = \frac{1}{2} \chi_e(M_1) + \frac{1}{2} \sum_{p \in \Sigma} N(p) . \tag{4.9}$$

Now we observe that

$$\begin{aligned} \chi(\tilde{M}) = \frac{1}{2} \chi_e(\tilde{M}) &= \frac{1}{2} \left[\chi_e(M) + \sum_{p \in \Sigma} (r(p) - 1) \right] \\ &= \frac{1}{2} \left[\chi_e(M_1) + \sum_{p \in \Sigma} r(p) \right] \\ &= \frac{1}{2} \left[\chi_e(M_1) + \sum_{q \in \pi^{-1}(\Sigma)} (N(q) - (N(q) - 1)) \right] \\ &= \operatorname{ind} \bar{\partial}_{\min} , \end{aligned}$$

Where we have used (4.9) and (4.7) for the last equality. The proof if complete. □

Theorem 4.2.

$$\operatorname{ind} \bar{\partial}_{\max} = \chi(\tilde{M}) + \sum_{\substack{p \in \Sigma \\ q \in \pi^{-1}(p)}} (N(q) - 1) .$$

Proof. In view of Theorem 3.1, in order to calculate $\operatorname{ind} \bar{\partial}_{\max}$ it is enough to determine the dimension of W, the space of boundary conditions. To do so, we only have to observe that

$$(\bar{\partial}'_{\min})^* = \bar{\partial}_{\max}$$

and that our method applies to $\bar{\partial}'$ as well. In fact, since all our transformations are unitary it is easily checked that $-\bar{\partial}'$ satisfies the assumptions of [B, Theorem 4.2] with

$$\omega_{-\bar{\partial}'} = -\omega_{\bar{\partial}}\,,$$
$$S_0 = -\bigoplus_{p\in\Sigma}\ \bigoplus_{q\in\pi^{-1}(p)} (N(q)^{-1}\sqrt{-1}\partial_\varphi - 1/2)\,.$$

Thus we obtain from loc. cit. Theorem 4.2,

$$\begin{aligned}\operatorname{ind}(-\bar{\partial}_{\max}) &= \operatorname{ind}(-\bar{\partial}'_{\min})^* = -\operatorname{ind}(-\bar{\partial}'_{\min})\\ &= -\left[\int_{M_1} \omega_{-\bar{\partial}'} - \frac{1}{2}\dim\ker S_0 - \sum_{0<s<1/2} \dim\ker(S_0 - s)\right]\\ &= \int_{M_1} \omega_{\bar{\partial}} - \frac{1}{2}(\dim\ker S_0 + \sum_{-1/2<s<0} \dim\ker(S_0 - s)\\ &\quad + \sum_{|s|<1/2} \dim\ker(S_0 - s)\\ &= \operatorname{ind}\bar{\partial}_{\min} + \sum_{|s|<1/2} \dim\ker(S_0 - s)\,. \quad \square \end{aligned} \tag{4.10}$$

Remark. Formula (4.10) always holds under the assumptions of [B, Theorem 4.2], by the arguments given above.

Acknowledgement. The first author is indebted to the VW-Stiftung for support during the preparation of this work.

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Communicated by A. Connes

Received July 27, 1989