Let $\mathscr{A} - \mathscr{D}$ be given via (1.2) and (1.5) in Section II.1. On these spaces we define the operations * as the usual adjoint of an operator between Hilbert spaces. It is easy to see that the conditions (1.1)–(1.6) in Section II.1 are satisfied. Also the required axioms are fulfilled.

Use now Theorem II.4.1 in [GKW2] to conclude that diag (M_i^{-1}) is the right multiplicative diagonal of $I - g^*g$, where g is the unique triangular extension. Now the theorem follows directly from Theorem II.1.1. ▮

REFERENCES

[BarG] M. A. BARKAR AND I. C. GOHBERG, On factorizations of operators relative to a discrete chain of projectors in Banach space, *Mat. Issled.* **1** (1966), 32–54. [English translation, *Amer. Math. Soc. Transl.* (2) **90** (1970), 81–103]

[BEGL] A. BEN-ARTZI, R. L. ELLIS, I. GOHBERG, AND D. C. LAY, The maximum distance problem and band sequences, *Linear Algebra Appl.* **87** (1987), 93–112.

[BG] J. A. BALL AND I. GOHBERG, Shift invariant subspaces, factorization, and interpolation for matrices. I. The canonical case, *Linear Algebra Appl.* **74** (1986), 87–150.

[B] J. P. BURG, "Maximum Entropy Spectral Analysis," Doctoral Dissertation, Department of Geophysics, Stanford University, 1975.

[D] H. DYM, *J* contractive matrix functions, reproducing kernel Hilbert spaces and interpolation, *in* "CBMS Regional Conference Series in Mathematics," Providence, 1989.

[DG1] H. DYM AND I. GOHBERG, Extensions of matrix valued functions with rational polynomial inverse, *Integral Equations Operator Theory* **2** (1979), 503–528.

[DG2] H. DYM AND I. GOHBERG, On an extension problem, generalized Fourier analysis and an entropy formula, *Integral Equations Operator Theory* **3** (1980), 143–215.

[DG3] H. DYM AND I. GOHBERG, Extensions of band matrices with band inverses, *Linear Algebra Appl.* **36** (1981), 1–24.

[DG4] H. DYM AND I. GOHBERG, Extensions of kernels of Fredholm operators, *J. Analyse Math.* **42** (1982/83), 51–97.

[DG5] H. DYM AND I. GOHBERG, A maximum entropy principle for contractive interpolants, *J. Funct. Anal.* **65** (1986), 83–125.

[DG6] H. DYM AND I. GOHBERG, A new class of contractive interpolants and maximum entropy principles, *in* "Operator Theory: Advances Applications," pp. 117–150, Birkhäuser-Verlag, OT 29, Basel, 1988.

[GKW1] I. GOHBERG, M. A. KAASHOEK, AND H. J. WOERDEMAN, The band method for positive and contractive extension problems, *J. Operator Theory* **22** (1989), 109–155.

[GKW2] I. GOHBERG, M. A. KAASHOEK, AND H. J. WOERDEMAN, The band method for positive and contractive extension problems: An alternative version and new applications, *Integral Equations Operator Theory* **12** (1989), 343–382.

JOURNAL OF FUNCTIONAL ANALYSIS 95, 255–290 (1991)

The Expansion of the Resolvent near a Singular Stratum of Conical Type

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Communicated by R. B. Melrose

Received June 9, 1989

This paper extends the analysis of an isolated conical singularity in [BS2] to singular strata of arbitrary dimension. It gives an expansion of

$$\operatorname{tr}(\Delta + \lambda)^{-m}, \qquad \lambda \to +\infty,$$

in descending power of λ, for suitably large m. The coefficients are given by integrals of the usual densities associated with the Laplacian at nonsingular points, suitably regularized at the singular stratum, plus singular terms given as integrals over the singular stratum.

A concrete example of the situation treated is a "wedge" $W \subset M^n$ in a Riemannian manifold M. The boundary of W is smooth except along a singular stratum Σ parametrized by $s \in \mathbb{R}^{n-2}$. Near Σ, $\partial W = \Sigma_1 \cup \Sigma_2$, with Σ_1 and Σ_2 intersecting transversally in Σ. In a plane section perpendicular to Σ at a point s, Σ_1 and Σ_2 meet at an angle $\alpha(s) > 0$, measured inside W. Introduce coordinates (x_1, x_2, s) in a neighborhood of Σ, where

$s \in \mathbb{R}^{n-2}$ is a coordinate on Σ,
$\partial/\partial x_1 \perp \Sigma_1$, and $\Sigma_1 \sim \{x_1 = 0,\ x_2 \geqslant 0\}$,
$\Sigma_2 \sim \{x_1 = r \sin \alpha(s),\ x_2 = r \cos \alpha(s),\ r \geqslant 0\}$,
along Σ, $(\partial/\partial x_i, \partial/\partial x_j) = \delta_{ij}$.

* Partially supported by NSF Grant DMS 8703604.

0022-1236/91 $3.00

Then introduce polar coordinates in W,

$$x_1 = r \sin\theta, \qquad x_2 = r\cos\theta, \quad 0 \leqslant \theta \leqslant \alpha(s).$$

Since the vector field $\partial/\partial\theta$ is zero when $r=0$, the Laplacian takes the form

$$\begin{aligned}\Delta = &(-1+O(r))\partial_r^2 - (r^{-1}+O(1))\partial_r - (r^{-2}+O(r^{-1}))\partial_\theta^2\\ &+\Delta_\Sigma(r,\theta) + O(r^{-1})\partial_\theta + O(1)\partial_r\partial_\theta + O(r)\partial_r\partial_s\\ &+O(1)\partial_\theta\partial_s + O(1)\partial_s,\end{aligned}$$

acting in $L^2(v)$, where v is the appropriate volume element. Here $\Delta_\Sigma(r,\theta)$ is a Laplacian in the s variables, depending on parameters r and θ. We transform Δ to an operator in $L^2(dr\, d\theta\, v_s\, ds)$, where in (r,θ,s) coordinates, the volume element is $v = rv_s + O(r^2)$ with v_s the volume element on Σ. So the change of variable

$$\tilde{u} = (v_s/v)^{1/2} u$$

gives a new operator, which we write as leading terms at $r=0$ plus remainder:

$$\begin{aligned}\bar{\Delta} = &-\partial_r^2 - r^{-2}(\partial_\theta^2 + \tfrac{1}{4}) + \Delta_\Sigma(0)\\ &+O(r)\partial_s^2 + O(r)\partial_s\\ &+O(r^{-1})\partial_\theta^2 + O(r^{-1})\partial_\theta + O(r)\partial_r^2\\ &+O(1)\partial_r\partial_\theta + O(r)\partial_r\partial_s + O(1)\partial_\theta\partial_s\\ &+O(1)\partial_r + O(1)\partial_s + O(r^{-1}). \qquad (1)\end{aligned}$$

(Here $\Delta_\Sigma(0) = \Delta_\Sigma(0,\theta)$ is just the Laplacian on the singular stratum Σ, and is independent of θ.) The three leading terms correspond to the *normal operator* in Melrose's study of degenerate problems [MP]. We make a change of variable $\theta = \alpha(s)\varphi$ with $0 \leqslant \varphi \leqslant 1$, and write the normal operator as

$$-\partial_r^2 + r^{-2}A(s) + \Delta_\Sigma(0), \qquad (0.2)$$

where $A(s) = -\alpha^{-2}(s)\partial_\varphi^2 - 1/4$ on $0\leqslant\varphi\leqslant 1$, with suitable boundary conditions at $\varphi=0$ and $\varphi=1$. For the Dirichlet problem, they are $u(r,0)=u(r,1)=0$; for the Zaremba problem [Z], $u(r,0)=0$ and $u_\varphi(r,1)=0$. In the latter problem, the difference in the two boundary conditions (Dirichlet on Σ_1, Neumann on Σ_2) gives the effect of a conical singularity along Σ even when ∂W is smooth, and $\alpha(s)\equiv\pi$. The Fredholm properties of this problem have been studied by S. Simanca [Si].

This problem can be treated locally in r and s, but near $r=0$ it must be treated globally in φ; for the eigenvalues of $A(s)$ determine the singularities that occur as $r\to 0+$.

Melrose's program (which applies to many problems other than the conic ones treated here) is to first construct a uniform interior parametrix, inverting modulo functions which are C^∞ in the interior but singular on the boundary, and then to invert the normal operator to get a parametrix with the proper boundary behavior. We stay closer to the pre-ψdo approach to elliptic problems. Thinking of the given operator as a perturbation of the normal operator, we begin by inverting the latter, thus reversing the steps in Melrose's program. If the coefficients of Δ are suitably modified outside a small neighborhood of the singular stratum Σ, then the inverse of the normal operator provides the first term in a convergent Neumann series. This gives a parametrix near Σ, which is patched together with a classical ψdo parametrix in the interior to give a global parametrix.

In [BS2] we carried this out for the case where Σ is of dimension 0, and the operator has the form

$$\bar{\Delta} = -\partial_r^2 + r^{-2}A(r), \qquad (0.3)$$

with normal operator $-\partial_r^2 + r^{-2}A(0)$, $A(0)\geqslant -1/4$. If a denotes an eigenvalue of $A(0)$, and $\nu := (a+1/4)^{1/2}$, then the resolvent of the normal operator $-\partial_r^2 + r^{-2}A(0)$ is the direct sum of symmetric operators with kernels

$$(r\bar{r})^{1/2} I_\nu(r\sqrt{\lambda})K_\nu(\bar{r}\sqrt{\lambda}), \qquad r<\bar{r}, \qquad (0.4)$$

where I_ν and K_ν are Bessel functions. (When there are eigenvalues $a<3/4$ then the operator is not essentially self-adjoint, and a domain must be specified. The resolvent in (0.4) is for the Friedrichs extension from smooth functions vanishing near $r=0$, which in this case gives the domain with the highest possible order of vanishing at $r=0$.) Denoting by $G_0(\lambda)$ the direct sum of the operators with kernels (0.4), we obtained the resolvent of (0.3) near $r=0$ as a sum $G_0\sum_0^\infty R^j$. This was patched together with an interior resolvent to yield the expansion of $\operatorname{tr}(\Delta+\lambda)^{-m}$.

In the present paper we consider a formally self-adjoint elliptic operator Δ on a manifold W, of second order, with scalar principal symbol, with conic singularities along a compact submanifold $\Sigma\subset W$, $\partial\Sigma=\varnothing$. Near any point of Σ, Δ is to be represented in the form suggested by (0.1) and (0.2) above:

$$\Delta = -\partial_r^2 + \Delta_\Sigma + r^{-2}A(s) + R, \qquad (0.5)$$

where Δ_Σ is elliptic on Σ with scalar principal symbol; A is a smooth family of elliptic operators on sections of a bundle E over a compact manifold N, perhaps with boundary, with $A(s) \geqslant -1/4$; and the remainder R is a sum of smooth terms of the form

$$\begin{gathered} O(r)\partial_{s_i}\partial_{s_k}, \qquad O(r^{-1})[A(0)+1], \qquad O(r)\partial_r^2, \\ O(r)\partial_r\partial_{s_j}, \qquad O(1)\partial_r[A(0)+1]^{1/2}, \qquad O(1)\partial_s[A(0)+1]^{1/2}. \end{gathered} \tag{0.6}$$

Here ∂_{s_j} denotes a partial derivative with respect to local coordinates on Σ, defined in some $\Sigma' \subset \Sigma$, and $O(r^j)$ means r^j times a smooth family of bounded operators on $L^2(N, E)$. The Δ in (0.5) is an unbounded operator on a space which can be represented locally as

$$L^2(\Sigma') \otimes L^2((0, \varepsilon), dr) \otimes L^2(N, E).$$

The remainder terms in (0.6) are precisely those that tend to zero in an appropriate sense when Δ is rescaled in the (r, s) variables; see (5.14) and (6.9) below.

In local coordinates on Σ', denote the principal symbol of Δ_Σ by $|\sigma|_s^2$, where σ is the variable dual to s. Our first approximation to the resolvent of (0.5) is a ψdo on Σ' given in local coordinates by

$$\mathscr{G}_0 u(s) := \mathrm{Op}(G_0)u(s) := (2\pi)^{-k} \int_{\mathbb{R}^k} e^{i\langle s,\sigma\rangle} G_0(|\sigma|_s^2 + \lambda)\hat{u}(\sigma)\, d\sigma, \tag{0.7}$$

where $k = \dim \Sigma$, and

$$u \in C_0^\infty(\mathbb{R}^k, L^2(\mathbb{R}_+, L^2(N, E))),$$

$$\text{and } G_0(|\sigma|_\sigma^2 + \lambda) = (-\partial_r^2 + |\sigma|_s^2 + r^{-2}A(s) + \lambda)^{-1}. \tag{0.8}$$

(The inverse in (0.8) is taken as the resolvent of the Friedrichs extension of $-\partial_r^2 + r^{-2}A(s)$, for fixed s.) Thus we are led to consider operators of the form (0.7), (0.8), where $A(s)$ is now a family of unbounded operators on a Hilbert space H ($H = L^2(N, E)$ in our example above), with common domain, satisfying

$$A(s) \geqslant -1/4, \tag{0.9}$$

$$A(s)(A(0)+1)^{-1} \text{ is a smooth family of bounded operators} \tag{0.10}$$

and, to guarantee that G_0 is smooth in s,

$$A(s) + 1/4 \text{ has a smooth square root } \nu(s) \geqslant 0; \tag{0.11}$$

i.e., $\nu(s)(A(0)+1)^{-1/2}$ is a smooth bounded family. To produce trace class estimates, we assume that

$$(A(0)+1)^{-1} \text{ is in some Schatten class } C_p, \qquad p < \infty. \tag{0.12}$$

We use the square root $\nu(s)$ to write the kernel of G_0 in the form (0.4), where the order ν of the Bessel functions is now an operator family, in a sense made precise below. Thus we treat only the simplest realization of Δ. When Δ arises as a product D^*D of first order operators, an alternate construction allows the treatment of rather general realizations of D. This will be developed elsewhere.

After modifying the coefficients of Δ outside some neighborhood of $\{(r, s) = (0, 0)\}$, we show [Sect. 3] that

$$(\Delta + \lambda)\mathscr{G}_0 = I - \mathscr{R}$$

with $\mathscr{R}$ small so that, suitably interpreted,

$$(\Delta + \lambda)^{-1} = \mathscr{G}_0 \sum_0^\infty \mathscr{R}^j.$$

Then, by studying trace class properties of $\mathscr{G}_0^m$ for suitable m [Sect. 4], we represent

$$\operatorname{tr} \varphi(\Delta + \lambda)^{-m}, \qquad \varphi \in C_0^\infty(\mathbb{R}_+)$$

as an integral

$$\int_0^\infty \sigma(r, r\sqrt{\lambda})\, dr$$

which can be analyzed by the Singular Asymptotics Lemma of [BS1]. Theorem 5.2 demonstrates the existence of an expansion in powers of λ and $\log \lambda$, and Section 6 discusses the nature of the coefficients, which are regularized divergent integrals over the nonsingular part of W plus integrals of certain nonlocal invariants over Σ.

This work follows the main outline of [BS2]; but some of the arguments have been modified, of necessity, and so we do not suppose prior knowledge of that paper, except for several crucial self-contained lemmas which we quote here.

1. Splitting the Spectrum

The leading term in our parametrix is a ψdo with operator symbol

$$(-\partial_r^2 + r^{-2}A(s) + z^2)^{-1},$$

where $z^2 = |\sigma|_s^2 + \lambda$ and the inverse refers to the Friedrichs extension. We assume that $A(s) \geqslant -1/4$, and that $A(s) + 1/4$ has a smooth positive square root. Moreover, since $A(s)$ is, typically, a second order elliptic operator on a compact manifold N, it has an eigenvalue expansion; but the eigenvalues may coalesce and bifurcate as s varies. Nevertheless, locally near any particular s, we may split $A(s)$ into a "high eigenvalue" and a "low eigenvalue" part:

LEMMA 1.1. *Let $A_s > -1$ be a family of self-adjoint operators on a Hilbert space H, with $A_s(A_0+1)^{-1}$ smooth. (Thus the A_s have a common domain.) Let $C \subset \mathbb{C}$ be compact, with smooth boundary ∂C disjoint from* spec A_s, *for all s near 0. Then for s sufficiently small, there is a smooth family of unitary operators U_s such that for the family $\tilde{A}_s := U_s A_s U_s^*$, the spectral projection for* spec $\tilde{A}_s \cap C$ *is independent of s.*

Proof. $(A_s-\lambda)^{-1} = (A_0+1)^{-1}[(A_s-\lambda)(A_0+1)^{-1}]^{-1}$ is smooth in (λ, s) for $\lambda \notin \operatorname{spec} A_s$. So the orthogonal projections

$$P_s = \frac{i}{2\pi}\int_{\partial C} (A_s-\lambda)^{-1}\, d\lambda$$

form a smooth family. Set $P_s^\perp = I - P_s$ and $N_s = P_s(H)$, the range of P_s. The operator

$$Q_s := P_0 P_s + P_0^\perp P_s^\perp$$

is invertible for $s=0$, hence for s near 0; and

$$Q_s^* Q_s = P_s P_0 P_s + P_s^\perp P_0^\perp P_s^\perp$$

maps N_s to N_s and $N_s^\perp$ to $N_s^\perp$, so

$$U_s := Q_s (Q_s^* Q_s)^{-1/2}$$

maps N_s to N_0 and $N_s^\perp$ to $N_0^\perp$. Also

$$U_s U_s^* = Q_s (Q_s^* Q_s)^{-1} Q_s^* = I = U_s^* U_s;$$

so $U_s A_s U_s^*$ has the properties claimed in the lemma. ∎

In view of this "Splitting Lemma," locally, we can write $A(s) = U(s)^* \tilde{A}(s) U(s)$ where the unitary family $U(s)$ has locally bounded derivatives of all orders, and $\tilde{A}(s)$ has a constant spectral projection. Then $\tilde{A}(s) =: \tilde{A}_>(s) + \tilde{A}_<(s)$ with the "high eigenvalue part" $\tilde{A}_>(s)$ and the "low eigenvalue part" $\tilde{A}_<(s)$ each acting in a fixed Hilbert space. Thus $\tilde{A}_<(s)$ is represented by a matrix, and by assumption (0.11), $\tilde{A}_<(s) + 1/4$ has a smooth matrix square root $\nu_<(s) \geq 0$. We analyze the resolvent

$$(-\partial_r^2 + r^{-2}\tilde{A}_<(s) + z^2)^{-1}$$

of the Friedrichs extension of $-\partial_r^2 + r^{-2}\tilde{A}_<$, as in [BS2], by representing it as an integral operator with kernel

$$(r\bar{r})^{1/2} I_\nu(rz) K_\nu(\bar{r}z), \qquad r < \bar{r},$$

where ν is short for $\nu_<(s)$. The "matrix" Bessel functions I_ν and K_ν are defined and analyzed as follows.

2. ESTIMATES FOR MATRIX BESSEL KERNELS

Let ν be a self-adjoint C^∞ matrix function of parameter $s \in \mathbb{R}^k$. Define the matrix functions

$$\Gamma(\nu) = \int_0^\infty t^{\nu-1} e^{-t}\, dt, \qquad \nu > 0, \tag{2.1}$$

$$I_\nu(z) = \frac{(z/2)^\nu}{\Gamma(\nu+1/2)\Gamma(1/2)} \int_{-1}^{1} e^{-zt}(1-t^2)^{\nu-1/2}\, dt,$$

$$\nu > -1/2, \quad |\arg z| < \pi/2, \tag{2.2}$$

$$K_\nu(z) = (\pi/2z)^{1/2} \frac{e^{-z}}{\Gamma(\nu+1/2)} \int_0^{\infty e^{i\beta}} e^{-t} t^{\nu-1/2}(1+t/2z)^{\nu-1/2}\, dt,$$

$$\nu > -1/2, \quad \beta - \pi < \arg z < \beta + \pi, \quad |\beta| < \pi/2. \tag{2.3}$$

Extend these definitions by the standard recursions

$$I_\nu(z) = 2(\nu+1) z^{-1} I_{\nu+1}(z) + I_{\nu+2}(z), \tag{2.4a}$$

$$K_\nu(z) = -2(\nu+1) z^{-1} K_{\nu+1}(z) + K_{\nu+2}(z), \tag{2.4b}$$

where 1 and 2 in the subscripts denote the diagonal matrices I and $2I$. We also need the derivative formulas

$$\frac{dI_\nu}{dz} = I_{\nu+1}(z) + \frac{\nu}{z} I_\nu(z), \tag{2.5a}$$

$$\frac{dK_\nu}{dz} = -K_{\nu+1}(z) + \frac{\nu}{z} I_\nu(z). \tag{2.5b}$$

LEMMA 2.1. (a) *I_ν and K_ν solve $z^2 y''(z) + z y'(z) - (z^2+\nu^2) y(z) = 0$, for each s.*

(b) $K_\nu = K_{-\nu} = (\pi/2)(\sin \pi\nu)^{-1}(I_{-\nu} - I_\nu)$, *if* $\sin \pi\nu$ *is nonsingular.*

(c) *As* $z \to \infty$, $|\arg z| < \pi/2 - \varepsilon$,

$$K_\nu(z) \sim (\pi/2)^{1/2} e^{-z}(z^{-1/2} + O(z^{-3/2}))$$
$$I_\nu(z) \sim (2\pi)^{-1/2} e^{z}(z^{-1/2} + O(z^{-3/2})).$$

These expansions can be differentiated in s and z, and the resulting constants implied in $O(z^{-3/2})$ *are locally bounded in s.*

(d) *For* $|z| \leqslant 1/2$, $|\arg z| < \pi/2$, $a \leqslant \nu \leqslant b$,

$$\|\partial_s^\alpha \partial_z^k I_\nu(z)\| \leqslant C_{a,b,k,\nu} |z|^{a-k} |\log^{|\alpha|} z|.$$

The constant can be chosen uniformly in compact subsets of the parameter space $\mathbb{R}^k$.

(e) *For* $|z| \leqslant 1/2$, $|\arg z| < \pi/2$, $-a \leqslant \nu \leqslant a$ *with* $a > 0$,

$$\|\partial_s^\alpha \partial_z^k K_\nu(z)\| \leqslant C_{a,b,k,\nu} |z|^{-a-k} |\log^{|\alpha|} z|.$$

The constant can be chosen uniformly in compact subsets of the parameter space $\mathbb{R}^k$.

Proof. (a) For fixed s, $\nu(s)$ can be diagonalized, and then (2.1)–(2.3) decompose into a direct sum of standard integral representations of the Bessel functions solving the differential equation in (a); see [W, p. 172 (2), p. 168 (3), and p. 78 (8)].

(b) Again, diagonalize and use standard Bessel formulas.

(c) For K_ν with $\nu > -1/2$, we use the expansion of (2.3) given in [W], as follows. Take $|\arg z| < \pi/2 - \varepsilon$, $\beta = \arg z$. Then in (2.3), $\arg(t/z) = 0$, so for $u \geqslant 0$, $|1 + ut/2z| \geqslant 1$. By diagonalizing,

$$(1 + t/2z)^{\nu - 1/2} = \sum_{m=0}^{p-1} \binom{\nu - 1/2}{m} (t/2z)^m$$
$$+ p \binom{\nu - 1/2}{p} (t/2z)^p \int_0^1 (1-u)^{p-1} (1 + ut/2z)^{\nu - p - 1/2} \, du.$$

If $p > \nu - 1/2$ then $(1 + ut/2z)^{\nu - p - 1/2}$ is bounded, so

$$K_\nu(z) =: (\pi/2z)^{1/2} \frac{e^{-z}}{\Gamma(\nu + 1/2)} \int_0^{\infty e^{i\beta}} e^{-t} t^{\nu - 1/2}$$
$$\times \left[1 + \sum_{m=1}^{p-1} \binom{\nu - 1/2}{m} (t/2z)^m + b_p(\nu, t, z)\right] dt, \qquad (2.6)$$

where b_p is a suitably bounded function of all arguments. The expansion

(2.6) can be differentiated in z, and also in s, in view of Lemma 2.2 below on differentiating exponentials.

For general ν, we use Lemma 1.1 to split K_ν into a part with $\nu > -1/2$ and a part with $\nu < 1/2$; since $K_\nu = K_{-\nu}$, the previous argument covers both parts.

For I_ν, $\nu > -1/2$, we use a similar integral [W, pp. 74, 77]: for $|\beta| < \pi/2$ and $-\pi/2 < \arg z < \pi/2$

$$I_\nu(z) = e^{-\pi i\nu/2} J_\nu(z e^{\pi i/2})$$
$$= \frac{1}{2} e^{-\pi i\nu/2} [H_\nu^{(1)}(iz) + H_\nu^{(2)}(iz)]$$
$$= \frac{e^{-\pi i\nu/2}}{\Gamma(\nu + 1/2)} (2\pi i z)^{-1/2}$$
$$\times \left[e^{-z - \pi i\nu/2 - \pi i/4} \int_0^{\infty e^{i\beta}} e^{-u} u^{\nu - 1/2} (1 + u/2z)^{\nu - 1/2} \, du \right.$$
$$\left. + e^{z + \pi i\nu/2 + \pi i/4} \int_0^{\infty e^{i\beta}} e^{-u} u^{\nu - 1/2} (1 - u/2z)^{\nu - 1/2} \, du \right]. \qquad (2.7)$$

In the last integral, we take $|\beta - \arg z| > 1$. For $\nu \leqslant -1/2$, we use the recursion (2.4a).

(d) Assume first that $\nu > -1/2$ and apply (2.5a) to express the z-derivatives. Each term in the resulting formula is an integral of type (2.2), and these integrals can be differentiated with respect to s using Lemma 2.2 below. In the general case, apply (2.4a) to reduce to $\nu > -1/2$.

(e) We prove the estimate for $k = 0$, then extend it using (2.5b). From (2.3) with $\beta = 0$ we obtain

$$K_\nu(z) = \pi^{1/2} \Gamma(\nu + 1/2)^{-2} 2^{-\nu} z^{-\nu} e^{-z} \int_0^\infty e^{-t} t^{\nu - 1/2} (t + 2z)^{\nu - 1/2} \, dt.$$

Let us assume first that ν satisfies $\bar{a} \leqslant \nu(s) \leqslant a$ for some $\bar{a} > 0$. Then differentiate the integral using Lemma 2.2. Observing that $t \leqslant |t + 2z| \leqslant \sqrt{2(t^2 + 1)}$ the desired estimate follows easily, with a constant depending on $\bar{a}$.

Consider next the case when $-a \leqslant \nu(s) \leqslant a$ with $a < 1$. For $|z| \leqslant 1/2$ with $|\arg z| < \pi/2$ we then integrate (2.5b) along the ray from z to $z/2|z|$. Since $1 - a \leqslant \nu + 1 \leqslant a + 1$ we can differentiate the integral with respect to s and apply the estimate of the previous case to reach the conclusion of the lemma.

Finally, we consider $-a \leqslant \nu(s) \leqslant a$ with $a > 1/2$. Using Lemma 1.1 we can split K_ν into three parts, one with $-a \leqslant \nu \leqslant -\bar{a} < 0$, one with

$0<\bar{a}\leqslant \nu\leqslant a$, and one with $-1/2\leqslant \nu\leqslant 1/2$. The last part we have just considered; the other two parts are covered by our first case, since $K_{-\nu}=K_\nu$. ▮

The derivatives of exponentials above are estimated by:

LEMMA 2.2. *If* $a\leqslant \nu(s)\leqslant b$, *then for* $|\arg z|<\pi/2$

$$\|\partial_s^\alpha z^\nu\|\leqslant C_\alpha(|z|^a+|z|^b)\,|\log^{|\alpha|} z|$$

with C_α *depending only on the derivatives* $\partial_s^\beta \nu$ *for* $\beta\leqslant\alpha$.

Proof. Let $u(s,t)=e^{t\nu\log z}$, where $\nu=\nu(s)$. Then $u_t=(\nu\log z)u$ and $u(s,0)=I$. Denoting by u_s the gradient of u with respect to s, we find

$$\partial_t u_s=(\nu_s\log z)u+(\nu\log z)u_s, \qquad u_s(s,0)=0.$$

Hence

$$u_s(s,1)=\int_0^1 e^{(1-t)\nu\log z}\nu_s\log z e^{t\nu\log z}\,dt.$$

This gives the desired estimate for $\partial_s z^\nu=u_s(s,1)$; the higher derivatives are estimated by induction, differentiating the above integral. ▮

Now the operators with kernel defined by

$$\begin{aligned} k(x,y)&=(xy)^{1/2}I_\nu(xz)K_\nu(yz), \qquad x<y,\\ k(y,x)&=k(x,y), \end{aligned} \tag{2.8}$$

are estimated using Lemma 2.1 and "Schur's Test" (cf. [HS, p. 22]).

If $Kf(x)=\int k(x,y)f(y)\,dy$, and $\omega(x)>0$, then

$$\|K\|^2_{L^2}\leqslant\left[\sup_x\int\omega(x)\,\|k(x,y)\|\,\omega^{-1}(y)\,dy\right]$$
$$\times\left[\sup_y\int\omega(y)\,\|k(x,y)\|\,\omega^{-1}(x)\,dx\right].$$

LEMMA 2.3. *Let* l *be a positive continuous function on* $\mathbb{R}_+$ *with*

$$l(x)=\begin{cases}-\log x, & 0<x<1/2,\\ 1, & x\geqslant 1.\end{cases}$$

Suppose that k *is a matrix kernel with* $k(x,y)=0$, $y<x$, *and for* $x<y$

$$\|k(x,y)\|\leqslant C(xz)^\alpha\,(yz)^\beta\,(1+xz)^\gamma\,(1+yz)^\delta\,l^j(xz)\,l^m(yz)\,e^{zx-zy} \tag{2.9}$$

with

$$\alpha>-1/2, \qquad \alpha+\beta>-1, \qquad \alpha+\beta+\gamma+\delta\leqslant 0. \tag{2.10}$$

Then k *defines a bounded operator* K *on* L^2, *with*

$$\|K\|\leqslant C'(\alpha,\beta,\gamma,\delta)Cz^{-1} \qquad [C \text{ as in } (2.9)].$$

Proof. Use Schur's Test with weight $\omega(x)=(xz)^{-\alpha}\,l^{-j}(xz)$. Then changing variables xz to x, yz to y, dy to $z^{-1}\,dy$, gives

$$z\omega(x)\int\|k(x,y)\|\,\omega^{-1}(y)\,dy\leqslant C\int_x^\infty y^{\beta+\alpha}l^{m+j}(y)(1+x)^\gamma\,(1+y)^\delta e^{x-y}\,dy.$$

This integral is bounded as $x\to 0+$, and as $x\to+\infty$ it is $O(x^{\alpha+\beta+\gamma+\delta})=O(1)$, by (2.10), since $l(y)=1$ for $y\geqslant 1$. For the integral with respect to x,

$$z\omega(y)\int\|k(x,y)\|\,\omega^{-1}(x)\,dx$$

$$\leqslant\int_0^y x^{2\alpha}l^{2j}(x)\,y^{\beta-\alpha}l^{m-j}(y)(1+x)^\gamma\,(1+y)^\delta e^{x-y}\,dx$$

is like $Cy^{\beta+\alpha+1}l^{j+m}(y)\to 0$ as $y\to 0+$, since $\alpha+\beta>-1$. As $y\to+\infty$, the integral is dominated by

$$y^{\beta-\alpha+\delta}\left[e^{-y/2}\int_0^{y/2}x^{2\alpha}l^{2j}(x)(1+x)^\gamma\,dx+y^{2\alpha+\gamma}\int_{y/2}^y e^{x-y}\,dx\right]$$
$$=O(y^{\alpha+\beta+\gamma+\delta})=O(1). \text{ ▮}$$

Remark. A precise accounting shows that for $\alpha,\beta,\gamma,\delta$ in any bounded subset of (2.10),

$$\|K\|\leqslant Cz^{-1}(1+\beta+\alpha)^{(1+m+j)/2}\,(2\alpha+1)^{-j-1/2},$$

where C depends on the subset.

3. REMAINDER ESTIMATES

We now introduce a model operator, $\tilde{\Delta}$, acting in $L^2(\mathbb{R}_+\times\mathbb{R}^k, H)$ for some Hilbert space H and coinciding with Δ in the introduction near $(0,0)\in\mathbb{R}_+\times\mathbb{R}^k$. More specifically, we assume that with $\partial_0:=\partial/\partial r$, $\partial_i:=\partial/\partial s_i$, $1\leqslant i\leqslant k$,

$$\tilde{\mathcal{A}} = \sum_{i,j=0}^{k} A_{ij}(r,s)\partial_i\partial_j + \sum_{i=0}^{k} B_i(r,s)(A(0)+1)^{1/2}\partial_i$$
$$+ r^{-2}C(r,s)(A(0)+1), \tag{3.1}$$

where $A(s)$ is an operator family satifying (0.9)–(0.12), $A_{ij} = A_{ji}^*$, B_i, C are smooth functions in $\mathbb{R}_+ \times \mathbb{R}^k$ with values in $\mathscr{L}(H)$, uniformly bounded together with their derivatives, and $A_{ij}(0,s) = -a_{ij}(s)I$ such that $a_{00}(s) = -1$, $a_{0j}(s) = 0$ for $1 \leqslant j \leqslant k$, and

$$\Delta_\Sigma := -\sum_{i,j=1}^{k} \partial_i a_{ij}(s)\partial_j \tag{3.2}$$

is uniformly elliptic on $\mathbb{R}^k$. Assuming also $C(0,s) = A(s)(A(0)+1)^{-1}$, we may write

$$\tilde{\mathcal{A}} =: -\partial_r^2 + \Delta_\Sigma + r^{-2}A(s) + R, \tag{3.3}$$

where

$$R = \bar{A}_{00}(r,s)r\partial_0^2 + \sum_{j=1}^{k} \bar{A}_{0j}(r,s)r\partial_0\partial_j$$
$$+ \sum_{i,j=1}^{k} \bar{A}_{ij}(r,s)r\partial_i\partial_j + \sum_{i=0}^{k} \bar{B}_i(r,s)(A(0)+1)^{1/2}\partial_i$$
$$+ r^{-1}\bar{C}(r,s)(A(0)+1); \tag{3.4}$$

the operator functions $\bar{A}_{ij}$, $\bar{B}_i$, $\bar{C}$ have again values in $\mathscr{L}(H)$ and are smooth and uniformly bounded on $\mathbb{R}_+ \times \mathbb{R}^k$ together with their derivatives. Finally, we assume that $\tilde{\mathcal{A}}$ is symmetric and bounded below as an operator in $\mathscr{H} := L^2(\mathbb{R}_+ \times \mathbb{R}^k, H)$ with domain $C^\infty((0,\infty) \times \mathbb{R}^k, H_1)$, $H_1 := \mathscr{D}(A(0))$.

The "ansatz" for the resolvent is a ψdo with operator symbol,

$$\mathscr{G}_0 u(s) := (2\pi)^{-k} \int_{\mathbb{R}_k} e^{i\langle s,\sigma\rangle}(-\partial_r^2 + r^{-2}A(s) + z^2)^{-1}\hat{u}(\sigma)\,d\sigma \tag{3.5}$$

with $z^2 = \lambda + |\sigma|_s^2$ and $|\sigma|_s^2$ the principal symbol of Δ_Σ. Denote the symbol of $\mathscr{G}_0$ by

$$G_0 := (-\partial_r^2 + r^{-2}A(s) + z^2)^{-1}, \tag{3.6}$$

the resolvent of the Friedrichs extension. We estimate the norm of $\mathscr{G}_0$ on $L^2(\mathbb{R}_+ \times \mathbb{R}^k) \otimes H$ by the Calderón–Vaillancourt Theorem [CV], which states that

$$\|\mathscr{G}_0\| \leqslant C \sup_{|\alpha|,|\beta| \leqslant 3k} \|\partial_s^\alpha \partial_\sigma^\beta G_0\|_{L^2(\mathbb{R}_+,H)}. \tag{3.7}$$

According to Lemma 1.1 we have $A(s) = U(s)^*\tilde{A}(s)U(s)$ where U is a smooth family of unitary operators and $\tilde{A}(s) = \tilde{A}_<(s) + \tilde{A}_>(s)$, with both the low eigenvalue and the large eigenvalue part acting on a fixed Hilbert space. For the estimates (3.7) we can hence assume that $A(s) = \tilde{A}(s)$, since all derivatives of $U(s)$ are uniformly bounded in $\mathbb{R}^k$. Thus G_0 is the direct sum of $G_<$ with kernel

$$G_<(r,\bar{r}) = (r\bar{r})^{1/2} I_\nu(rz)K_\nu(\bar{r}z), \qquad r < \bar{r},$$
$$G_<(\bar{r},r) = G_<(r,\bar{r}), \tag{3.8}$$

and

$$G_> = (-\partial_r^2 + r^{-2}A_>(s) + z^2)^{-1}.$$

It follows from [BS2, Lemma 3.2 and Theorem 6.1] that the domain of the Friedrichs extension of $-\partial_r^2 + r^{-2}A_>(s)$ is independent of $s \in \mathbb{R}^k$. Since $z^2 = |\sigma|_s^2 + \lambda$, we have

$$\partial_s G_> = -G_>(r^{-2}\partial_s A_>)G_> - 2G_>^2\,\partial_s\,|\sigma|_s^2; \tag{3.9a}$$

the product $(r^{-2}\partial_s A_>)G_>$ is justified by [BS2, Lemma 3.2], since $\partial_s A_>(A(0)+1)^{-1}$ is bounded. Observing that

$$\partial_\sigma G_\gtrless = -G_\gtrless^2\,\partial_\sigma\,|\sigma|_s^2, \tag{3.9b}$$

that same lemma applies to higher derivatives, and gives

$$\|\partial_s^\alpha \partial_\sigma^\beta G_>\|_{L^2(\mathbb{R}_+,H)} \leqslant C_{\alpha\beta}\,|z|^{-2-|\beta|}. \tag{3.10}$$

For $G_<$ we use Lemmas 2.1 and 2.3. By the Splitting Lemma we may assume that ν is a direct sum, $\nu(s) = \bigoplus_i \nu_i(s)$, such that $0 \leqslant a_i \leqslant \nu_i(s) < b_i < a_i + 2$. From Lemma 2.1, in each such interval,

$$\|I_\nu(rz)\| \leqslant C(rz)^a\,(1+rz)^{-a-1/2}e^{rz},$$
$$\|K_\nu(\bar{r}z)\| \leqslant C(\bar{r}z)^{-b}\,(1+\bar{r}z)^{b-1/2}e^{-\bar{r}z},$$

so the kernel in (3.8) satisfies

$$\|G_<(r,\bar{r})\| \leqslant C(r\bar{r})^{1/2}\,(rz)^a\,(\bar{r}z)^{-b}\,(1+rz)^{-a-1/2}\,(1+\bar{r}z)^{b-1/2}e^{-z(\bar{r}-r)},$$
$$r < \bar{r}.$$

Replace $(r\bar{r})^{1/2}$ by $z^{-1}(rz\bar{r}z)^{1/2}$ and apply Lemma 2.3 to find $\|G_<\| \leqslant Cz^{-2}$. Similarly, using Lemma 2.1 and (3.9), we get

$$\|\partial_s^\alpha \partial_\sigma^\beta G_<\| \leqslant C_{\alpha\beta}z^{-2-|\beta|}. \tag{3.11}$$

Thus from (3.7) and (3.10), recalling $z^2 = |\sigma|_s^2 + \lambda$,

$$\|\mathscr{G}_0\| \leqslant C\lambda^{-1}. \tag{3.12}$$

Next, we estimate the remainder $\mathscr{R}$ in

$$(\tilde{\Delta} + \lambda)\mathscr{G}_0 = I - \mathscr{R}, \tag{3.13}$$

and make sense of the Neumann series $\mathscr{G}_0 \sum_0^\infty \mathscr{R}^j$. As in the simpler case of [BS2], $\mathscr{R}$ itself need not be bounded on L^2. But if we define

$$\omega(r) = r^{1/2}(1+r)^{-1/2}, \qquad \Omega = \text{multiplication by } \omega, \tag{3.14}$$

then we can prove that $\mathscr{G}_0 \Omega^{-1}$, $\Omega\mathscr{R}$, and $\Omega\mathscr{R}\Omega^{-1}$ are bounded; and the norm of $\Omega\mathscr{R}\Omega^{-1}$ can be made small so that

$$\mathscr{G}_0 + \mathscr{G}_0\, \Omega^{-1} \sum_{j=0}^{\infty} (\Omega\mathscr{R}\Omega^{-1})^j \Omega\mathscr{R} \tag{3.15}$$

converges, giving the resolvent of $\tilde{\Delta}$. For technical reasons, we also need to estimate the commutators of $\mathscr{R}$ with powers of $(1+r)$.

The terms in the remainder $\mathscr{R}$ arise from two sources: the difference between $\tilde{\Delta}$ and the normal operator $-\partial_r^2 + \Delta_\Sigma + r^{-2}A(s)$ [the terms in (3.4) above], and the remainder from the ψdo approximation to $(-\partial_r^2 + \Delta_\Sigma + r^{-2}A(s) + \lambda)^{-1}$. The latter terms are small when λ is large. The former are made small by further modifying the coefficients outside a small neighborhood of (0, 0), i.e., by multiplying $\mathscr{R}$ by a cut-off function in the r-variable. The most delicate is the term $O(r)\partial_r^2 \mathscr{G}_0$, a ψdo on $\mathbb{R}^k$ with operator symbol.

$$O(r)\partial_r^2 G_0. \tag{3.16}$$

We multiply this by a function $\varphi_0(\alpha r)$, where $\varphi_0(r) \equiv 1$ near 0, $0 \leqslant \varphi_0 \leqslant 1$, and $\varphi_0(r) \equiv 0$ for $r \geqslant 1$. Then $\varphi_0 \leqslant 1$ and $r\alpha\varphi_0(\alpha r) \leqslant 1$, so

$$\varphi_0(\alpha r) \leqslant 2(1+\alpha r)^{-1}. \tag{3.17}$$

Denote by R the operator "multiplication by r."

LEMMA 3.1. *When α and z are sufficiently large then the operator symbol*

$$(1+\alpha R)^{-1} R\Omega(1+R)^J \partial_s^\beta \partial_\sigma^\gamma \partial_r^2 G_0 (1+R)^{-J}\Omega^{-1}, \qquad J \in \mathbb{R}, \quad \beta, \gamma \in \mathbb{Z}_+^k, \tag{3.18}$$

has small norm in $L^2(\mathbb{R}_+, H)$.

Proof. Split $G_0 = G_< \oplus G_>$ by splitting $\tilde{A}$ as in Section 2. For $G_>$, [BS2, (3.5c)] gives

$$\begin{aligned}\|(1+\alpha R)^{-1} R\Omega(1+R)^J \partial_r^2 G_> (1+R)^{-J}\Omega^{-1}\|_{L^2(\mathbb{R}_+, H)} &\leqslant C \sup r(1+\alpha r)^{-1} \\ &\leqslant C/\alpha,\end{aligned} \tag{3.19}$$

which is small when α is large. The s-derivatives add factors

$$\Omega(1+R)^J R^{-2}(\partial A/\partial s_j) G_> (1+R)^{-J}\Omega^{-1}$$

and these are bounded, by [BS2, (3.5a)]. The σ-derivatives bring extra factors of $\Omega(1+R)^J \partial_\sigma |\sigma|_s^2\, G_> (1+R)^{-J}\Omega^{-1}$ in view of (3.6); by [BS2, (3.5d)] these factors are $O(z^{-1})$.

For $G_<$, $\partial_r^2[(r\bar{r})^{1/2} I_\nu(rz) K_\nu(\bar{r}z)]$ gives three terms, the worst of which is

$$r^{-3/2}\bar{r}^{1/2}(rz)^2 I_\nu'' K_\nu, \qquad r < \bar{r}. \tag{3.20}$$

(The other terms are similar for fixed z, but this one has the worst growth as $rz \to \infty$.) By splitting as after (3.10), we can assume $a \leqslant \nu \leqslant b$ with $0 \leqslant a \leqslant b < a+1$. Combining Lemma 2.1(c), (d), (e), we can dominate (3.20) by

$$r^{-3/2}(rz)^a (1+rz)^{3/2-a} \bar{r}^{1/2}(\bar{r}z)^{-b} (1+\bar{r}z)^{b-1/2} l(\bar{r}z) e^{-z(\bar{r}-r)}.$$

Taking into account the factor $r\varphi(\alpha r)$ and the conjugation with $\omega(r)$ and $(1+r)^J$ in (3.18), we have a kernel dominated, for $r < \bar{r}$, by

$$\begin{aligned}&\left(\frac{1+r}{1+\bar{r}}\right)^{J-1/2} (1+\alpha r)^{-1} (rz)^a (1+rz)^{3/2-a} (\bar{r}z)^{-b} \\ &\quad \times (1+\bar{r}z)^{b-1/2} l(\bar{r}z) e^{-z(\bar{r}-r)}.\end{aligned} \tag{3.21}$$

When $J \geqslant 1/2$, the leading term in (3.21) is $\leqslant 1$; thus assume $J \leqslant 1/2$. Then that leading term is $\leqslant ((1+rz)/(1+\bar{r}z))^{J-1/2}$, for $z \geqslant 1$. Moreover, if $z \geqslant \alpha$,

$$(1+\alpha r)^{-1} = \frac{z}{z+\alpha rz} \leqslant \frac{z}{\alpha + \alpha rz} = \frac{z/\alpha}{1+rz}.$$

Hence for $z \geqslant \alpha \geqslant 1$ and $r < \bar{r}$, our kernel is dominated by

$$(z/\alpha)(1+rz)^{J-a} (1+\bar{r}z)^{b-J} (rz)^a (\bar{r}z)^{-b} l(\bar{r}z) e^{-z(\bar{r}-e)}.$$

By Lemma 2.3, the norm of the corresponding operator is $O(1/\alpha)$, hence arbitrarily small for large α and $z \geqslant \alpha$. Similarly, for $r \geqslant \bar{r}$, the worst term in the kernel estimate is

$$\|r(1+\alpha r)^{-1}r^{-3/2}\bar{r}^{1/2}(rz)^2K_\nu''(rz)I_\nu(\bar{r}z)\|$$
$$\leqslant Cr(1+\alpha r)^{-1}r^{-3/2}\bar{r}^{1/2}(rz)^{-b}\,(1+rz)^{b+3/2}$$
$$\times\, l(rz)(\bar{r}z)^a\,(1+\bar{r}z)^{-a-1/2}e^{rz-\bar{r}z}$$
$$\leqslant C(z/\alpha)(rz)^{-(1/2)-b}\,(1+rz)^{b+1/2}$$
$$\times\, l(rz)(\bar{r}z)^{a+1/2}\,(1+\bar{r}z)^{-a-1/2}e^{-z(\bar{r}-r)}$$

when $z\geqslant\alpha$. Conjugate with $\omega(r)$ and $(1+r)^J$ as before, and find again that the norm is $O(1/\alpha)$.

The derivatives with respect to s and σ are similarly estimated, as in the proof of (3.12). ∎

A similar analysis applies to all the remainder terms listed in (3.4). In addition to Lemma 3.1 we need the following symbol estimates.

LEMMA 3.2. *When α and z are sufficiently large then the operator symbol*

$$(1+\alpha R)^{-1}R^{-1}(A(0)+1)(1+R)^J\Omega\partial_s^\beta\partial_\sigma^\gamma G_0\Omega^{-1}(1+R)^{-J},$$
$$J\in\mathbb{R},\quad \beta,\gamma\in\mathbb{Z}_+^k,$$

has small norm in $L^2(\mathbb{R}_+,H)$.

Proof. For $G_>$, the essential point is [BS2, (3.5a)] which gives

$$\|(A(s)+1)R^{-2+J}G_>R^{-J}\|\leqslant C.$$

Using $(1+R)^j\leqslant C(1+R^j)$ if $j\geqslant 0$ and $(1+\alpha R)^{-1}R^{-1}\leqslant\alpha^{-1}R^{-2}$ we obtain for $J\geqslant 1/2$

$$\|(1+\alpha R)^{-1}R^{-1}(A(s)+1)(1+R)^{J-1/2}R^{1/2}G_>R^{-1/2}(1+R)^{1/2-J}\|$$
$$\leqslant C\alpha^{-1}[\|(A(s)+1)R^{-2+1/2}G_>R^{-1/2}\|+\|(A(s)+1)R^{-2+J}G_>R^{-J}\|]$$
$$\leqslant C\alpha^{-1}. \tag{3.22}$$

If $J\leqslant 1/2$, we argue similarly, estimating $(1+R)^{1/2-J}$. By (0.10), $(A(0)+1)(A(s)+1)^{-1}$ is a smooth family of bounded operators; this completes the proof for $G_>$.

For $G_<$, denote by $P_<(s)$ the projection onto the low eigenspaces. Then

$$(A(0)+1)P_<(s)=(A(0)+1)(A(s)+1)^{-1}\,(A(s)+1)P_<(s)$$

is a smooth bounded family so we can ignore the factor $A(0)+1$ in this case. Then the proof of (3.22) for $G_<$ follows as the corresponding part of Lemma 3.1. The s and σ derivatives are also handled as before. ∎

LEMMA 3.3. *The operator symbol*

$$z^2(1+R)^J\Omega\partial_s^\beta\partial_\sigma^\gamma G_0\Omega^{-1}(1+R)^{-J},\qquad J\in\mathbb{R},\quad \beta\in\mathbb{Z}_+^k,$$

is uniformly bounded in norm on $L^2(\mathbb{R}_+,H)$.

Proof. For $G_>$, we now use [BS2, (3.5d)] to obtain

$$z^2\|(1+R)^J\Omega G_>\Omega^{-1}(1+R)^{-J}\|\leqslant C.$$

For $G_<$, we may assume that $J\leqslant 1/2$ and $a\leqslant\nu\leqslant b$, $0\leqslant a\leqslant b<a+1$, as in the proof of Lemma 3.1. For the kernel of $z^2(1+R)^J\Omega G_<\Omega^{-1}(1+R)^{-J}$ we obtain the bound

$$z\left(\frac{1+rz}{1+\bar{r}z}\right)^{J-1/2}(rz)^{a+1}(1+rz)^{-a-1/2}\,(\bar{r}z)^{-b}\,(1+\bar{r}z)^{b-1/2}e^{z(\bar{r}-r)}$$

if $z\geqslant 1$. Thus the assertion follows from Lemma 2.3. Again, s and σ derivatives are dealt with as before. ∎

These estimates imply the desired smallness of the remainder term.

THEOREM 3.1. *Given any numbers j and k, the operator (3.1) can be modified outside a small neighborhood of $(0,0)$ so that for the remainder*

$$\mathcal{R}=I-\tilde{\Delta}\mathcal{G}_0,$$

$\mathcal{R}$ conjugated with $\Omega(1+R)^j\,(1+|S|^2)^k$ is arbitrarily small, for λ sufficiently large.

Proof. We can ignore the powers of $(1+|s|^2)$; for

$$\int(1+|s|^2)e^{i\langle s-\bar{s},\sigma\rangle}H(s,\sigma)(1+|\bar{s}|^2)^{-1}\,d\sigma$$
$$=\int[(1+\Delta_\sigma)e^{i\langle s,\sigma\rangle}]e^{-i\langle\bar{s},\sigma\rangle}H(1+|\bar{s}|^2)^{-1}\,d\sigma.$$

Integrate by parts, and use $(1+|\bar{s}|^2)^{-1}$ to balance the powers of $\bar{s}$ coming fro derivatives of $e^{-i\langle\bar{s},\sigma\rangle}$. This reduces the estimate of the s-conjugations of

$$\mathrm{Op}(H)=\int e^{i\langle s-\bar{s},\sigma\rangle}H(s,\sigma)\,d\sigma$$

to estimates of $\mathrm{Op}(\partial_\sigma^\beta H)$, hence to estimates of H and its derivatives in s and σ.

Now, as noted before Lemma 3.1, we multiply each of the remainder terms in (3.4), right and left, by the function $\varphi_0(\alpha r)$ and introduce the new operator

$$\tilde{\tilde{\Delta}} := -\partial_r^2 + \Delta_\Sigma + r^{-2}A(s) + \varphi_\alpha R \varphi_\alpha,$$

where $\varphi_\alpha(r) := \varphi_0(\alpha r)$. Since $\tilde{\Delta}$ and $-\partial_r^2 + \Delta_\Sigma + r^{-2}A$ are symmetric, so is R and hence so is $\tilde{\tilde{\Delta}}$.

Since $\varphi_0 \equiv 1$ for small r, this does not change $\tilde{\Delta}$ in some neighborhood of $r=0$. The new remainder is

$$\varphi_\alpha^2 R + \varphi_\alpha[2r\varphi_\alpha' \bar{A}_{00}\partial_0 + r\varphi_\alpha' \Sigma_j \bar{A}_{0j}\partial_j + r^2\varphi_\alpha'' r^{-1}\bar{A}_{00} + r\varphi_\alpha' r^{-1}\bar{B}_0(A(0)+1)^{1/2}].$$

Here $r\varphi_\alpha'$ and $r^2\varphi_\alpha''$ have bounds independent of α, wile φ_α and φ_α^2 are $\leqslant 2(1+\alpha r)^{-1}$; so Lemmas 3.1 and 3.2 show that the terms arising from the difference between $\tilde{\tilde{\Delta}}$ and the normal operator $-\partial_r^2 + \Delta_\Sigma + r^{-2}A(s)$ can be made small. The other remainder terms arise from

$$u - (-\partial_r^2 + \Delta_\Sigma + r^{-2}A(s) + \lambda)(2\pi)^{-k} \int e^{i\langle s,\sigma\rangle} G_0 \hat{u}(\sigma)\, d\sigma.$$

These involve operator symbols $\partial_{s_i}\partial_{s_j}G_0$ and $\sigma_i\partial_{s_j}G_0$; their norms are small when λ is large, by Lemma 3.3, since $z^2 = |\sigma|_s^2 + \lambda$. ▮

4. Trace Class Properties

If $A(0)$ is a nonsingular elliptic operator of second order on a compact manifold N, $A(0)^m$ has a trace class resolvent if $2m > \dim N$; thus $(A(0)+1)^{-1}$ is of Schatten class C_p for $p > \frac{1}{2}\dim N$. This fact is incorporated in (0.12). We will show that $(\Delta+\lambda)^{-1}$ is of class C_q for $2q > 2p+k+1$, hence $(\Delta+\lambda)^{-m}$ is trace class if $2m > \dim M$, precisely the same relation as for a nonsingular problem (see Remark below).

The proof uses "test operators" adapted to the singularities of the problem.

Lemma 4.1. *Suppose that φ is C^∞ and*

$$\varphi(r) = \begin{cases} r^{-\alpha}, & 0 < r \leqslant 1, \\ >0, & 1 \leqslant r \leqslant 2, \\ r^{-\beta}, & r \geqslant 2. \end{cases}$$

If $\beta > 2$ and $3/2 \leqslant \alpha < 2$, then the operator

$$T := -\partial_r \varphi^{-1} \partial_r$$

with domain $\{u \in L^2(0,\infty) \mid Tu \in L^2\}$ has an inverse T^{-1} of trace class, with trace norm $\int_0^\infty r\varphi(r)\, dr$.

Proof. Set $\Phi(x) := \int_x^\infty \varphi(x)\, dx$. Then $\Phi \in L^1$, $\Phi \notin L^2$, and

$$Tu = f \Leftrightarrow u(x) = \int_x^\infty \varphi(t) \int_0^t f(y)\, dy\, dt - C_1\Phi(x) + C_2 \tag{4.1}$$

for constants C_1 and C_2. Define

$$Kf(x) = \int_x^\infty \varphi(t) \int_0^t f(y)\, dy\, dt =: \int_0^\infty k(x,y) f(y)\, dy,$$

where

$$k(x,y) = \begin{cases} \Phi(x), & y < x, \\ \Phi(y), & x < y. \end{cases}$$

Then $K = J^*J$, where J is the Hilbert–Schmidt operator

$$Jf(x) = \sqrt{\varphi(x)} \int_0^x f(y)\, dy.$$

Thus K is positive and trace class, and its trace norm is

$$\operatorname{tr} K = \|J\|_{HS}^2 = \int_0^\infty x\varphi(x)\, dx = \int_0^\infty \Phi(x)\, dx = \int_0^\infty k(x,x)\, dx.$$

Moreover, if $u \in L^2$ and $Tu = f \in L^2$ then $u = Kf$; for the constants C_1 and C_2 in (4.1) must vanish if $u \in L^2$. Thus $T^{-1} = K$ is trace class. ▮

On $\mathbb{R}^1$, we use a variant of the test operator from the proof of Lemma 3.5 in [BS2].

Lemma 4.2. *There is a self-adjoint operator on $L^2(\mathbb{R}^1)$,*

$$P = -a(s)\partial_s^2 + b(s)\partial_s + c(s)$$

with eigenvalues $\{j(j+1)\}_0^\infty$, *and as* $s \to \pm\infty$

$$a(s) \sim \pi s^2 |s|,$$
$$b(s) \sim -3\pi s |s|,$$
$$c(s) \sim -\pi |s|.$$

Proof. The Legendre operator

$$\tilde{P} = -\partial_x(\pi^2 - 4x^2)\partial_x$$

on $L^2(-\pi/2, \pi/2)$ has eigenvalues $\{4j(j+1)\}_0^\infty$. The map $Uf(x) = \sec x\, f(\tan x)$ is unitary from $L^2(\mathbb{R}^1)$ to $L^2(-\pi/2, \pi/2)$; it follows directly that

$$P = \tfrac{1}{4} U^{-1} \tilde{P} U$$

has the properties stated in the lemma. ▮

THEOREM 4.1. *If* $A(0) \in C_p$ *on* H, $p \geqslant 1$, *then for each* $j, l \in \mathbb{R}$, $\beta \in \mathbb{Z}_+^k$, *the operator*

$$\tilde{\mathscr{G}} := (1+R)^{j-3} (1+|S|^2)^{l-3/2} \operatorname{Op}(\partial_s^\beta G_0) \Omega^{-1} (1+R)^{-j} (1+|S|^2)^{-l}$$

is in the Schatten class $C_{p+(k+1)/2}$ *in* $L^2(\mathbb{R}_+ \times \mathbb{R}^k) \otimes H$.

Proof. Split $\tilde{\mathscr{G}} = \tilde{\mathscr{G}}_< \oplus \tilde{\mathscr{G}}_>$ by splitting $A(s)$ as in Section 1. On the first term, use the test operator T of Lemma 4.1 with $2 < \beta \leqslant 3$, $3/2 < \alpha < 2$. Writing $T(1+R)^{-3} =: a_2 \partial_r^2 + a_1 \partial_r + a_0$ we have a_i bounded at infinity and

$$a_2(r) = O(r^{3/2}),$$
$$a_1, a_0 = O(r^{1/2}), \qquad r \to 0.$$

We use again the Calderón–Vaillancourt Theorem to estimate the operator norms. As in the proof of Lemma 3.1, σ-derivatives are harmles. Hence by Lemmas 3.1, 3.2, and 3.3, $T\tilde{\mathscr{G}}_<$ is bounded. (To apply (3.18), note that $a_2(r)/r\omega(r)$ is bounded.) So $P_j \tilde{\mathscr{G}}_<$ is bounded, where P_j is the differential operator of Lemma 4.2 acting in the variable s_j. Hence

$$\left(T + \sum_{j=1}^k P_j\right) \tilde{\mathscr{G}}_<$$

is bounded. Moreover, $(T + \sum_1^k P_j)^{-1}$ is of class $C_{1+k/2}$, acting on $L^2(\mathbb{R}^k \times \mathbb{R}_+) \otimes H_<$, where $H_<$ is the finite-dimensional subspace

associated with $\tilde{\mathscr{G}}_<$. For, the eigenvalues of $T + \sum_1^k P_j$ are $\{\tau + \langle \gamma, \gamma+1 \rangle\}$ with τ in spec T and γ in $\mathbb{Z}_+^k$; and

$$\sum_{\tau,\gamma} (\tau + \langle \gamma, \gamma+1 \rangle)^{-q} \leqslant \sum_\tau \int_{\mathbb{R}^k} (\tau + |\sigma|^2)^{-q}\, d\sigma$$
$$= C \sum_\tau \tau^{-q+k/2} < \infty$$

if $-q + k/2 \leqslant -1$. Thus $\tilde{\mathscr{G}}_<$ is of class $C_{1+k/2}$.

For $\tilde{\mathscr{G}}_>$ we replace the test operator T by a "Legendre operator" $P_0 = (1/4)\tilde{P}$ on $L^2(\mathbb{R}_+)$, with $\tilde{P}$ as in the proof of Lemma 3.5 of [BS2], having eigenvalues $j(j+1)$. Then, as before, $P_0 \tilde{\mathscr{G}}_>$ is bounded, and so is $(1 + A(0) + \sum_1^k P_j)\tilde{\mathscr{G}}_>$; and $(1 + A(0) + \sum_0^k P_j)^{-1}$ is in $C_{p+(k+1)/2}$, since

$$\sum_{a \in \operatorname{spec} A(0)} \sum_{j=0}^\infty \sum_\gamma [j(j+1) + \langle \gamma, \gamma+1 \rangle + a + 1]^{-q}$$
$$\leqslant C \sum_a (a+1)^{-q+(k+1)/2} < \infty$$

if $q = p + (k+1)/2$, as claimed. ▮

Remark. If Δ acts on a manifold M with a singular stratum Σ of dimension k, then the conical cross-section N has dimension $n = \dim M - k - 1$. The second order elliptic operator $A(0)$ has resolvent $(A(0)+1)^{-1}$ in C_p for all $p > n/2$; Theorem 4.1 implies that $(\Delta + \lambda)^{-m}$ is trace class for $m = p + (k+1)/2$ if $m > (n+k+1)/2 = 1/2 \dim M$; this is the same restriction as for nonsingular operators.

Theorem 4.1 gives trace class estimates in $L^2(\mathbb{R}_+ \times \mathbb{R}^k) \otimes H$. We pass from these to pointwise estimates on the kernel by a k-dimensional version of the Trace Lemma in [BS2, Appendix].

LEMMA 4.3. *Let* H *be a Hilbert space, and* T *a trace class operator on* $L^2(\mathbb{R}^k, H)$ *such that all commutators* $[\partial_i, T]$, $[\partial_i, [\partial_j, T]]$, ... *of order* $\leqslant k$ *are also trace class. Then has a continuous kernel such that*

$$Tf(s) = \int_{\mathbb{R}^k} T(s, \bar{s}) f(\bar{s})\, d\bar{s}$$

with

$$\|T(s, \bar{s})\|_{\operatorname{tr} H} \leqslant \|[\partial_k, ..., [\partial_2, [\partial_1, T]] \cdots]\|_{\operatorname{tr} L^2(\mathbb{R}^k; H)}.$$

Further,

$$\int_{\mathbb{R}^k} \|T(s, s)\|_{\operatorname{tr} H}\, ds \leqslant \|T\|_{\operatorname{tr}}$$

and

$$\operatorname{tr} T = \int_{\mathbb{R}^k} \operatorname{tr}_H T(s, s)\, ds.$$

Proof. When $k=1$, this is the Trace Lemma in [BS2, p. 425]. The general case follows by induction. For, if all the commutators up to order k are trace class, then by the induction hypothesis $[\partial_1, T]$, acting on $L^2(\mathbb{R}^{k-1}, L^2(\mathbb{R}^1, H))$, has a continuous kernel with

$$\|[\partial_1, T](s_2, ..., s_k; \bar{s}_2, ..., \bar{s}_k)\|_{\operatorname{tr} L^2(\mathbb{R}^1; H)} \leqslant \|[\partial_k, ..., [\partial_1, T]\cdots]\|_{\operatorname{tr}}.$$

Moreover, T itself has a continuous kernel $T(s_2, ..., s_k; \bar{s}_2, ..., \bar{s}_k)$ with values in trace class operators on $L^2(\mathbb{R}^1, H)$. One checks that the commutator of this kernel with ∂_1 is precisely the kernel $[\partial_1, T](s_2, ..., s_k; \bar{s}_2, ..., \bar{s}_k)$ above. Hence, the present lemma follows with one more application of the case $k=1$. ▮

5. Asymptotic Expansions

Consider again the abstract second order elliptic operator (3.1),

$$\tilde{\Delta} = -\partial_r^2 + \Delta_\Sigma + r^{-2}A(s) + R, \tag{5.1}$$

where Δ_Σ is uniformly elliptic in $\mathbb{R}^k$, A satisfies the assumptions (0.9)–(0.12), and R is given in (3.4), with uniform norm estimates in $\mathbb{R}_+ \times \mathbb{R}^k$. Outside some small neighborhood of (0, 0) we have modified the coefficients of $\tilde{\Delta}$ in such a way that we have remainder estimates suitable for the following arguments. We can assume that $\tilde{\Delta}$ is symmetric and bounded from below in $\mathscr{H} := L^2(\mathbb{R}_+ \times \mathbb{R}^k, H)$ with domain $C_0^\infty((0, \infty) \times \mathbb{R}^k, H)$. Hence the Friedrichs extension exists; we will denote it, too, by $\tilde{\Delta}$, with domain $\mathscr{D}(\tilde{\Delta})$. In (3.5) we introduced the operator $\mathscr{G}_0(\lambda)$ in $L^2(\mathbb{R}_+ \times \mathbb{R}^k, H)$ as our first approximation to the resolvent $(\tilde{\Delta} + \lambda)^{-1}$. From Theorem 3.1 we have

$$(\Delta + \lambda)\mathscr{G}_0(\lambda) = I - \mathscr{R}(\lambda) \tag{5.2}$$

with

$$\|\Omega\mathscr{R}\Omega^{-1}\| < 1, \qquad \lambda \geqslant \lambda_0, \tag{5.3}$$

and $\mathscr{G}_0\Omega^{-1}$ bounded (see the proof of (5.8) below). Hence

$$\mathscr{G}(\lambda) := \mathscr{G}_0\Omega^{-1} \sum_{j \geqslant 0} (\Omega\mathscr{R}\Omega^{-1})^j \Omega \tag{5.4}$$

is well defined; we are going to show that in fact

$$\mathscr{G}(\lambda) = (\tilde{\Delta} + \lambda)^{-1}. \tag{5.5}$$

We have from (3.1)

$$\tilde{\Delta} = \sum_{i,j=0}^{k} A_{ij}\partial_i\partial_j + \sum_{i=0}^{k} B_i(A(0)+1)^{1/2}\partial_i + r^{-2}C(A(0)+1 \tag{5.6}$$

with coefficients $A_{ij} = A_{ji}^*$, B_i, C which are smooth functions in $\mathbb{R}_+ \times \mathbb{R}^k$ with values in $\mathscr{L}(H)$, $A_{ij}(0, s) = -a_{ij}(s)I$, and uniformly bounded in $\mathbb{R}_+ \times \mathbb{R}^k$ together with their derivatives. Consider the space

$$\mathscr{D}_1(\tilde{\Delta}) := \{u \in \mathscr{H} \mid \partial_i u \in \mathscr{H} \text{ if } 1 \leqslant i \leqslant k,\ \tilde{\Delta}u \in \mathscr{H}, \text{ and } \|u(r)\|_{L^2(\mathbb{R}^k, H)} = O(r^{1/2}) \text{ as } r \to 0\}. \tag{5.7}$$

Here $\tilde{\Delta}$ is applied in the sense of distributions. Note that $\mathscr{D}_1(\tilde{\Delta}) \subset H^1((0, \infty), L^2(\mathbb{R}^k, H)) \subset C((0, \infty), L^2(\mathbb{R}^k, H))$, by elliptic regularity (cf., e.g., [BS2, Theorem 2.1]), which makes the last condition in (5.7) meaningful.

Lemma 5.1. $\mathscr{D}_1(\tilde{\Delta}) \subset \mathscr{D}(\tilde{\Delta})$.

Proof. We extend the argument in [BS2, Theorem 6.1] to this case. With φ, χ, ψ_n as defined there we put

$$\tilde{\psi}_n(r, s) := \varphi(|s|/n)\varphi(r/n)\psi_n(r),$$
$$\tilde{\psi}_{nm} := \tilde{\psi}_n - \tilde{\psi}_m, \qquad n, m \geqslant 2.$$

Then $\tilde{\psi}_{nm}$ is uniformly bounded and converges to 0 pointwise. Also, $\tilde{\psi}_n u \in \mathscr{D}(\tilde{\Delta})$ for $u \in \mathscr{D}_1(\tilde{\Delta})$, by interior regularity and mollification. It is, therefore, enough to show that

$$\lim_{n,m\to\infty} \operatorname{Re}[(\tilde{\Delta}\tilde{\psi}_{nm}u, \tilde{\psi}_{nm}u) - (\tilde{\psi}_{nm}\tilde{\Delta}u, \tilde{\psi}_{nm}u)] = 0, \qquad u \in \mathscr{D}_1(\tilde{\Delta}).$$

To see this we use (5.6). For the terms coming from $A_{00}\partial_0^2$ we write $\psi' = \partial\psi/\partial r = \partial_0\psi$, and obtain

$$\begin{aligned}(A_{00}(\tilde{\psi}''_{nm}u + 2\tilde{\psi}'_{nm}u'), \tilde{\psi}_{nm}u) &= 2(\tilde{\psi}'_{nm}u', A_{00}\tilde{\psi}_{nm}u) - (\tilde{\psi}'_{nm}u, A_{00}(\psi'_{nm}u + \tilde{\psi}_{nm}u') + A'_{00}\tilde{\psi}_{nm}u)\\ &\quad - (\tilde{\psi}'_{nm}u', A_{00}\tilde{\psi}_{nm}u),\end{aligned}$$

hence

$$\operatorname{Re}(A_{00}(\tilde{\psi}''_{nm}u + 2\tilde{\psi}'_{nm}u'), \tilde{\psi}_{nm}u) = -\operatorname{Re}(\tilde{\psi}'_{nm}u, A_{00}\tilde{\psi}'_{nm}u + A'_{00}\tilde{\psi}_{nm}u).$$

So this term is bounded by $C(\|\tilde{\psi}'_{nm}u\|^2+\|\tilde{\psi}_{nm}u\|^2)$. All other term have similar bounds. Hence it is enough to show that $\|\tilde{\psi}'_{nm}u\|\to 0$, $n, m\to\infty$. But for $m\geqslant n$ (5.7) gives

$$\|\tilde{\psi}'_{nm}u\|^2_{\mathscr{H}}\leqslant C_u\left[\int_0^2 r(\psi'_n(r)^2+\psi'_m(r)^2)\,dr+\frac{1}{n^2}\right]$$

and the assertion follows as in [BS2, Theorem 6.1].

LEMMA 5.2. *$\mathscr{G}(\lambda)$ maps $\mathscr{H}$ into $\mathscr{D}_1(\tilde{A})$, for large λ.*

Proof. By construction, for $f\in\mathscr{G}$, $\tilde{A}\mathscr{G}f\in\mathscr{H}$, as a distribution. In view of (5.4) and (5.7) it is hence enough to prove that

$$\partial_i\mathscr{G}_0\Omega^{-1} \text{ is bounded for } 1\leqslant i\leqslant k \tag{5.8}$$

and

$$\|\mathscr{G}_0\Omega^{-1}f(r)\|_{L^2(\mathbb{R}^k,H)}\leqslant C_f r^{1/2}. \tag{5.9}$$

For the proof of (5.8) we write $\mathscr{G}_0=\mathscr{G}_{0<}+\mathscr{G}_{0>}$, splitting $\tilde{A}$ as in Section 2. Estimating again by the Calderón–Vaillancourt Theorem we obtain the boundedness of $\partial_i\mathscr{G}_{0<}\Omega^{-1}$ as in Lemmas 3.1, 3.2, and 3.3, whereas $\partial_i\mathscr{G}_{0>}\Omega^{-1}$ is estimated using (3.9a), (3.9b), [BS2, Lemma 3.2], and the inequality $\omega(x)^{-1}\leqslant C(1+1/x^2)$.

For (5.9), we split once more. Let $u=\mathscr{G}_{0>}\Omega^{-1}f$, and $h(r):=r^{-1/2}u(r)$. The argument for (3.10) gives similar bounds for $R^{-1/2}G_>\Omega^{-1}$ and $\partial_r R^{-1/2}G_>\Omega^{-1}$ (with less decay as $z\to\infty$). Hence h and $h':=\partial_r h$ are in $L^2(\mathbb{R}_+\times\mathbb{R}^k, H)$. By standard regularity, $h(1)\in L^2(\mathbb{R}^k,H)$, so $\|h(r)\|=\|h(1)+\int_1^r h'\|\leqslant C$, proving (5.9) for $\mathscr{G}_{0>}$.

LEMMA 5.3. $\|\mathscr{G}_{0<}\Omega^{-1}f(r)\|_{L^2(\mathbb{R}^k,H)}O(r^{1/2})$.

Proof. We can assume that v is scalar, since all estimates are made with a fixed s.

First we prove that if $a<b$, $z>0$, then

$$\partial_z^j I_v(az)K_v(bz)\leqslant C_j z^{-j} I_v(az/2)K_v(bz/2). \tag{5.10}$$

For by [MOS, p. 98],

$$I_v(az)K_v(bz)=\tfrac{1}{2}\int_0^\infty t^{-1}e^{-z^2t}e^{-(a^2+b^2)/4t}I_v(ab/2t)\,dt.$$

Since $\partial_z^j e^{-z^2t}=z^{-j}P_j(z^2t)e^{-z^2t}$ with a polynomial P_j, we find

$$\partial_z^j I_v(az)K_v(bz)=(2z^j)^{-1}\int_0^\infty t^{-1}P_j(z^2t)e^{-z^2t}e^{-(a^2+b^2)/4t}I_v(ab/2t)\,dt.$$

Since $P_j(z^2t)e^{-z^2t}\leqslant C_je^{-z^2t/4}$ and $I_v\geqslant 0$ on $\mathbb{R}_+$, (5.10) follows.

Next we show, again for $a\leqslant b$, that for $0<\delta\leqslant 1/2$

$$\left|\int_{\mathbb{R}^k} I_v(az)K_v(bz)e^{i\langle s-\bar{s},\sigma\rangle}\,d\sigma\right|$$
$$\leqslant C_\delta\min(|s-\bar{s}|^{\delta-k}(b-a)^{-2\delta}, |s-\bar{s}|^{-k-1})e^{-(b-a)/4}l(b). \tag{5.11a}$$

To start, we have for $a<b$, and for any multiindex α,

$$(s-\bar{s})^\alpha\int_{\mathbb{R}^k} I_v(az)K_v(bz)e^{i\langle s-\bar{s},\sigma\rangle}\,d\sigma$$
$$=\int_{\mathbb{R}^k} e^{i\langle s-\bar{s},\sigma\rangle}(i\partial_\sigma)^\alpha I_v(az)K_v(bz)\,d\sigma.$$

Since $z^2=\lambda+|\sigma|_s^2$, $|\partial_\sigma^\beta z|\leqslant C|z|^{1-|\beta|}$. Then by (5.10) and Lemma 2.1

$$\begin{aligned}
&|(i\partial_\sigma)^\alpha I_v(az)K_v(bz)|\\
&\leqslant C|z|^{-|\alpha|}I_v(az/2)K_v(bz/2)\\
&\leqslant C|z|^{-|\alpha|}e^{-(b-a)z/2}l(bz)(az)^v(bz)^{-v}(1+az)^{-v-1/2}(1+bz)^{v-1/2}\\
&\leqslant C|z|^{-|\alpha|-\varepsilon}|b-a|^{-\varepsilon}l(b)e^{-(b-a)/4}
\end{aligned}$$

for any $\varepsilon\geqslant 0$, since $e^{-(b-a)z/4}\leqslant C|b-a|^{-\varepsilon}z^{-e}$ and $z\geqslant 1$. With $|\alpha|=k+1$ and $\varepsilon=0$ this gives

$$\left|\int_{\mathbb{R}^k} I_v(az)K_v(bz)e^{i\langle s-\bar{s},\sigma\rangle}\,d\sigma\right|\leqslant C|s-\bar{s}|^{-k-1}l(b)e^{-(b-a)/4}. \tag{5.11b}$$

With $|\alpha|=k$ and $\varepsilon=\delta$ it gives

$$(b-a)^\delta|s-\bar{s}|^k\left|\int I_v(az)K_v(bz)e^{i\langle s-\bar{s},\sigma\rangle}\,d\sigma\right|\leqslant Cl(b)e^{-(b-a)/4}.$$

With $|\alpha|=k-1$ and $\varepsilon=1+\delta$ it gives

$$(b-a)^{1+\delta}|s-\bar{s}|^{k-1}\left|\int_{\mathbb{R}^k} I_v(az)K_v(bz)e^{i\langle s-\bar{s},\sigma\rangle}\,d\sigma\right|\leqslant Cl(b)e^{-(b-a)/4}.$$

Add the previous two inequalities, use $x+1 \geqslant x^\delta$ for $x \geqslant 0$, $0 \leqslant \delta < 1$, with $x = (b-a)/|s-\bar{s}|$, and find

$$(b-a)^{2\delta} |s-\bar{s}|^{k-\delta} \int_{\mathbb{R}^k} I_\nu(az) K_\nu(bz) e^{i\langle s-\bar{s}, \sigma\rangle} \, d\sigma \leqslant Cl(b) e^{-(b-a)/4}.$$

This, with (5.11b), proves (5.11a).

Now, to prove Lemma 5.3, we have

$$\begin{aligned} r^{-1/2} \mathscr{G}_{0<} \omega^{-1} f(r,s) &= (2\pi)^{-k} \int_{\mathbb{R}^k} \int_{\mathbb{R}_+} r^{-1/2} G_{0<}(r, \bar{r}, s; z) \bar{r}^{-1/2} \\ &\quad \times (1+\bar{r})^{1/2} \hat{f}(\bar{r}, \sigma) e^{i\langle s, \sigma\rangle} \, d\bar{r} \, d\sigma \\ &=: \int_{\mathbb{R}^k} \int_{\mathbb{R}_+} r^{-1/2} \hat{G}_{0<}(r, \bar{r}; s, \bar{s}; z) \bar{r}^{-1/2} (1+\bar{r})^{1/2} f(\bar{r}, \bar{s}) \, d\bar{r} \, d\bar{s}, \end{aligned}$$

where

$$\hat{G}_{0<}(r, \bar{r}; s, \bar{s}; z) = (2\pi)^{-k} \int_{\mathbb{R}^k} e^{i\langle s-\bar{s}, \sigma\rangle} r^{1/2} I_\nu(rz) K_\nu(\bar{r}z) \bar{r}^{1/2} \, d\sigma, \qquad r < \bar{r}.$$

By (5.11),

$$\begin{aligned} & r^{-1/2} \hat{G}_{0<}(r, \bar{r}; s, \bar{s}; z) \bar{r}^{-1/2} (1+\bar{r})^{1/2} \\ &\quad \leqslant C e^{-|r-\bar{r}|/4} \min(|s-\bar{s}|^{\delta-k} |r-\bar{r}|^{-2\delta}, |s-\bar{s}|^{-k-1}) \\ &\quad\quad \times l(\max(r, \bar{r}))(1+\bar{r})^{1/2}. \end{aligned}$$

By Schur's test, for $r \leqslant 1$,

$$\begin{aligned} & \| r^{-1/2} \mathscr{G}_{0<} \omega^{-1} f(r) \|_{L^2(\mathbb{R}^k, H)} \\ &\quad \leqslant \int_{\mathbb{R}_+} \left\| \int_{\mathbb{R}^k} r^{-1/2} \hat{G}_{0<} \bar{r}^{-1/2} (1+\bar{r})^{1/2} f(\bar{r}, \bar{s}) \, d\bar{s} \right\|_{L^2(\mathbb{R}^k, H)} d\bar{r} \\ &\quad \leqslant C \left[\int_0^r e^{-|r-\bar{r}|/4} |r-\bar{r}|^{-2\delta} l(r)(1+\bar{r})^{1/2} \|f(\bar{r})\|_{L^2(\mathbb{R}^k, H)} \, d\bar{r} \right. \\ &\quad\quad \left. + \int_r^\infty e^{-|r-\bar{r}|/4} |r-\bar{r}|^{-2\delta} l(\bar{r})(1+\bar{r})^{1/2} \|f(\bar{r})\|_{L^2(\mathbb{R}^k, H)} \, d\bar{r} \right] \\ &\quad \leqslant C \left[l(r) r^{1/2-2\delta} + \left(\int_0^\infty e^{-\bar{r}/2} \bar{r}^{-4\delta} l(\bar{r})^2 (1+\bar{r}) \, d\bar{r} \right)^{1/2} \right] \|f\|_{\mathscr{H}} \\ &\quad \leqslant C. \end{aligned}$$

The lemma is proved. ∎

THEOREM 5.1. *$\mathscr{G}$ is the resolvent of the Friedrichs extension, and $\mathscr{D}_1(\tilde{\Delta}) = \mathscr{D}(\tilde{\Delta})$.*

Proof. It follows from Lemmas 5.1 and 5.2 that we have the operator equality

$$(\tilde{\Delta} + \lambda) \mathscr{G}(\lambda) = I.$$

Since the Friedrichs extension is self-adjoint, $\mathscr{G}$ must be the resolvent. Thus, $\mathscr{D}(\tilde{\Delta}) = \operatorname{im} \mathscr{G}(1) \subset \mathscr{D}_1(\tilde{\Delta})$ so by Lemma 5.1, $\mathscr{D}_1(\tilde{\Delta}) = \mathscr{D}(\tilde{\Delta})$. ∎

Finally, we need the scaling properties of $\mathscr{G}$ and its kernel. We are going to use "scaling to a base point" in $\mathbb{R}_+ \times \mathbb{R}^k$; for fixed $s_0 \in \mathbb{R}^k$ we consider the maps

$$(r, s) \mapsto (tr, s_0 + t(s - s_0)), \qquad t > 0. \tag{5.12}$$

It will be clear that the estimates we are going to obtain are uniform in s_0, so, for simplicity of notation, we take $s_0 = 0$. With (5.15) we associate the unitary transformation

$$U_t f(r, s) := t^{(k+1)/2} f(tr, ts) \tag{5.13}$$

and the scaled operators

$$\begin{aligned} \tilde{\Delta}_t &:= t^2 U_t \tilde{\Delta} U_t^* \\ &= \sum_{i,j=0}^k A_{ij}(tr, ts) \partial_i \partial_j + t \sum_{i=0}^k B_i(tr, ts)(A(0)+1)^{1/2} \partial_i \\ &\quad + r^{-2} C(tr, ts)(A(0)+1) \\ &=: -\partial_r^2 + \Delta_{\Sigma,t} + r^{-2} A(ts) + tR_t, \\ \mathscr{G}_t &:= (\tilde{\Delta}_t + \lambda)^{-1}. \end{aligned} \tag{5.14}$$

By construction, $\tilde{\Delta}_t$ is self-adjoint and equals the Friedrichs extension of the differential expression in (5.14). By Lemma 5.2,

$$\begin{aligned} \mathscr{D}(\tilde{\Delta}_t) = U_t \mathscr{D}(\tilde{\Delta}) = \{ u \in \mathscr{H} \mid \partial_i u \in \mathscr{H} \text{ for } 1 \leqslant i \leqslant k,\ \tilde{\Delta}_t u \in \mathscr{H}, \text{ and} \\ \|u(r)\|_{L^2(\mathbb{R}^k, H)} = O(r^{1/2}) \text{ as } r \to 0 \}. \end{aligned}$$

Hence $\mathscr{G}_t$ is constructed in the same way as $\mathscr{G} = \mathscr{G}_1$. We only have to replace G_0 by

$$G_{0,t} := (-\partial_r^2 + r^{-2} A(ts) + \lambda + |\sigma|_{s,t}^2)^{-1}, \tag{5.15}$$

where $|\sigma|^2_{s,t}$ is the principal symbol of

$$\Delta_{\Sigma,t} := \sum_{i,j=1}^{k} a_{ij}(ts)\partial_i\partial_j \tag{5.16}$$

(such that $G_{0,t}$ is the resolvent of the Friedrichs extension of the scaled normal operator), and R by tR_t, where R_t is given by (5.14). It is clear from this description that all estimates we have obtained so far are valid uniformly for $t \in [0, 1]$. We turn to the trace class properties of $\mathscr{G}_t$.

LEMMA 5.4. *Let $\varphi \in C_0^\infty(\mathbb{R}_+ \times \mathbb{R}^k)$. Then for $m \geqslant p + (k+1)/2$, $\varphi\mathscr{G}_t^m(\lambda)$ is in the trace class $C_1(\mathscr{H})$ with uniform trace norm estimates for $t \in [0, 1]$, $\lambda \geqslant 1$. Moreover, $\mathscr{G}_t^m$ has a continuous kernel*

$$\mathscr{G}_t^m(s, \bar{s}; \lambda) \in C_1(L^2(\mathbb{R}_+, H)) \tag{5.17}$$

which has in turn a kernel

$$\mathscr{G}_t^m(r, \bar{r}; s, \bar{s}; \lambda) \in C_1(H) \tag{5.18}$$

such that

$$\operatorname{tr} \varphi\mathscr{G}_t^m(\lambda) = \int_{\mathbb{R}_+ \times \mathbb{R}^k} \varphi(r, s) \operatorname{tr}_H \mathscr{G}_t^m(r, r; s, s; \lambda)\, dr\, ds. \tag{5.19}$$

Proof. We want to use Theorem 4.1 and the Trace Lemma 4.3. We introduce the following notation: let $A \in \mathscr{L}(\mathscr{H})$ and let $I = (i_1, ..., i_l) \in \mathbb{Z}^l_+$ be a multiindex of length $l \geqslant 0$. We define the multiple commutators of A with the operators ∂_i, $1 \leqslant i \leqslant k$, by

$$\mathscr{C}_0[A] := A, \tag{5.20}$$

$$\mathscr{C}^I[A] := \mathscr{C}^{(i_1, ..., i_l)}[A] := [\partial_{i_1}, \mathscr{C}^{(i_2, ..., i_l)}[A]]. \tag{5.21}$$

We want to prove that for $J \in \mathbb{R}$, $t \in [0, 1]$, $q \geqslant p + (k+1)/2$

$$\|(1+R)^{J-3}(1+|S|^2)^{(J-3)/2}\mathscr{C}^I[\mathscr{G}_t](1+R)^{-J}(1+|S|^2)^{-J/2}\|_{C_q(\mathscr{H})} \leqslant C_{J,I}. \tag{5.22}$$

This will prove the lemma. For, commutators act as derivations on $\mathscr{L}(\mathscr{H})$, so we may write with $\psi(r, s) := (1+r)(1+|s|^2)^{1/2}$ and certain numbers $c(I_1, ..., I_m)$

$$\varphi\mathscr{C}^I[\mathscr{G}_t^m] = \varphi\psi^{3m} \sum_{I_1 + \cdots + I_m = I} (c(I_1, ..., I_m)(\psi^{-3m}\mathscr{C}^{I_1}[\mathscr{G}_t]\psi^{3(m-1)}) \times (\psi^{-3(m-1)}\mathscr{C}^{I_2}[\mathscr{G}_t]\psi^{3(m-2)}) \cdots (\psi^{-3}\mathscr{C}^{I_m}[\mathscr{G}_t]));$$

now $\varphi\psi^{3m}$ is bounded, and by (5.22) and Theorem 4.1, $\varphi\mathscr{C}^I[\mathscr{G}_t^m]$ is trace class with uniform trace norm estimate in $t \leqslant 1$. Then (5.17)–(5.19) follow from Lemma 4.3 and the Trace Lemma in the appendix of [BS2].

For the proof of (5.22) we use the following estimates: for $J \in \mathbb{R}$, $t \in [0, 1]$ and $q \geqslant p + (k+1)/2$, and with ψ the operator "multiplication by ψ"

$$\|\Psi^{J-3}\mathscr{C}^I[\mathscr{G}_{0,t}]\Omega^{-1}\Psi^{-J}\|_{C_q(\mathscr{H})} \leqslant C_{J,I}; \tag{5.23}$$

and, for the remainder $\mathscr{R}_t = I - (\tilde{\Delta}_t + \lambda)\mathscr{G}_{0,t}$,

$$\|\Psi^J\Omega\mathscr{C}^I(\mathscr{R}_t)\Omega^{-1}\Psi^{-J}\|_{\mathscr{H}} \leqslant D_{J,I}; \tag{5.24}$$

and, given $J_0 \geqslant 0$ we can assume that the model operator has been so constructed that

$$D_{J,\varnothing} \leqslant 1/2 \quad \text{if} \quad |J| \leqslant J_0 \quad \text{and} \quad \lambda \geqslant \lambda_0 \tag{5.25}$$

For the proof of (5.23) we use the commutator relation

$$[\partial_i, \operatorname{Op}(G)] = \operatorname{Op}\left(\frac{\partial G}{\partial s_i}\right) \tag{5.26}$$

and we obtain

$$\Psi^{J-3}\mathscr{C}^I[\mathscr{G}_{0,t}]\Omega^{-1}\Psi^{-J} = \Psi^{J-3}\operatorname{Op}(\partial_s^I G_{0,t})\Omega^{-1}\Psi^{-J}.$$

So (5.23) follows from Theorem 4.1. For the proof of (5.24) we recall that by (5.14), $\mathscr{R}_t$ is a sum of ψdo's, and the commutators only add s-derivatives of the symbols, by (5.26). Thus the remainder estimates in Theorem 3.1 do not change, and the result is uniform in $t \leqslant 1$.

To prove (5.25) we cosider first the terms in $\mathscr{R}_t$ coming from $tR_t\mathscr{G}_{0,t}$. The modifying function φ_0 introduced in Lemma 3.1 is replaced by $t\varphi_0(\alpha tr)$ whereas the remainder estimate is otherwise uniform in $t \leqslant 1$. Then the inequality (3.17) is replaced with

$$t\varphi_0(\alpha tr) \leqslant 2t(1 + \alpha tr)^{-1} \leqslant 2(1 + \alpha r)^{-1}$$

so the proof of Lemma 3.1 carries through to show that all these terms can be made uniformly small if $|J| \leqslant J_0$, α is sufficiently large, and λ is large. The other terms in $\mathscr{R}_t$ come from

$$I - (-\partial_r^2 + \Delta_{\Sigma,t} + r^{-2}A(ts) + \lambda)\mathscr{G}_{0,t}(\lambda)$$

and are $O(\lambda^{-1})$ uniformly in $t \leqslant 1$, just as in the proof of Theorem 3.1. This completes the proof of (5.25) and hence the proof of the lemma. ∎

Let Δ be the operator described in the introduction. The asymptotic expansion of $(\Delta+\lambda)^{-m}$ near the singular stratum will be determined by the corresponding expansion of $\operatorname{tr} \varphi\mathscr{G}^m(\lambda)$, $\varphi \in C_0^\infty(\mathbb{R}_+ \times \mathbb{R}^k)$. To derive it we will use the Singular Asymptotics Lemma (SAL) from [BS1] as cited in [BS2]. Thus we write

$$\tilde{\sigma}(r, s; \zeta) := \operatorname{tr}_H \mathscr{G}^m(r, r; s, s; \zeta^2/r^2), \tag{5.27a}$$

$$\sigma(r, \zeta) := \int_{\mathbb{R}^k} \varphi\tilde{\sigma}(r, s; \zeta)\, ds, \tag{5.27b}$$

and we get from Theorem 5.1

$$\operatorname{tr} \varphi\mathscr{G}^m(\lambda) = \int_0^\infty \int_{\mathbb{R}^k} \varphi(r, s)\tilde{\sigma}(r, s; r\sqrt{\lambda})\, ds\, dr$$

$$= \int_0^\infty \sigma(r, r\sqrt{\lambda})\, dr. \tag{5.28}$$

We verify the integrability condition (1.5b) of the SAL in [BS2] by means of the scaling. If follows from (5.14) that

$$\mathscr{G}^m(r, \bar{r}; s, \bar{s}; \lambda) = t^{2m-k-1}\mathscr{G}_t^m(r/t, \bar{r}/t; s/t, \bar{s}/t; t^2\lambda) \tag{5.29}$$

and consequently

$$\tilde{\sigma}(t, 0; \zeta) = t^{2m-k-1} \operatorname{tr}_H \mathscr{G}_t^m(1, 1, 0, 0; \zeta^2). \tag{5.30}$$

Set $\tilde{\sigma}^{(j)}(r, s; \zeta) = \partial_r^j \tilde{\sigma}$ and $\sigma^{(j)}(r, \zeta) = \partial_r^j \sigma(r, \zeta)$. The cited integrability condition follows from:

LEMMA 5.5. *Given $J_0 > 0$ we can choose λ_0 and α_0 such that for $\lambda \geqslant \lambda_0$, $\alpha \geqslant \alpha_0$ in Lemmas* 3.1–3.8, $0 \leqslant j \leqslant J_0$, $0 \leqslant t \leqslant 1$, *$\tilde{\sigma}^{(j)}$ exists, and*

$$\int_0^1 u^j\, |\sigma^{(j)}(ut, u\lambda_0^{1/2})|\, du \leqslant C_{J_0}. \tag{5.31}$$

Proof. From (5.27a) and (5.29) we have

$$I_j(u, t) := u^j \tilde{\sigma}^{(j)}(ut; 0; u\lambda_0^{1/2}) = \partial_t^j(t^{2m-k-1} \operatorname{tr}_H \mathscr{G}_t^m(u, u; 0, 0; \lambda_0)).$$

Now choose $\varphi \in C_0^\infty(\mathbb{R}_+ \times \mathbb{R}^k)$ with $\varphi(r, s) = 1$ if $r \leqslant 1$, $|s| \leqslant 1$. Then apply the Trace Lemma in [BS2] to get

$$\int_0^1 |I_j(u, t)|\, du \leqslant \|\partial_t^j(t^{2m-k-1}\varphi\mathscr{G}_t^m(0, 0; \lambda_0))\|_{\operatorname{tr}(\mathbb{R}_+, H)},$$

where the subscript denotes the trace norm for operators on $L^2(\mathbb{R}_+, H)$.

Use Lemma 4.3 next to derive

$$\int_0^1 |I_j(u, t)|\, du \leqslant \mathscr{C}^{(1, \ldots, k)}[\partial_t^j(t^{2m-k-1}\varphi\mathscr{G}_t^m(\lambda_0))]\|_{\operatorname{tr}(\mathbb{R}_+ \times \mathbb{R}^k, H)}. \tag{5.32}$$

We have to show that the right hand side of (5.32) is uniformly bounded if $j \leqslant J_0$, $0 \leqslant t \leqslant 1$. For $j = 0$ this has been proved in Lemma 5.4. Following the pattern of that proof, we generalized (5.23), (5.24) to

$$\|\Psi^{J-3-j}\partial_t^j \mathscr{C}^I[\mathscr{G}_{0,t}]\Omega^{-1}\Psi^{-J}\|_{\mathscr{C}_q(\mathscr{H})} \leqslant C_{j,J,I}, \tag{5.33}$$

$$\|\Psi^{J-j}\Omega\partial_t^j C^I[\mathscr{R}_t]\Omega^{-1}\Psi^{-J}\|_{\mathscr{H}} \leqslant D_{j,J,I}, \tag{5.34}$$

for any multiindex I, $J \in \mathbb{R}$, $j \in \mathbb{Z}_+$, $t \in [0, 1]$, and $q \geqslant p + (k+1)/2$. To prove (5.33) we have to show first that $\operatorname{Op}(\partial_s^I G_{0,t})$ is a smooth function of $t \in [0, 1]$. But using (5.15) we see that

$$\partial_t G_{0,t} = \sum_{i=1}^k s_i \partial_i G_{0,t}. \tag{5.35}$$

This proves differentiability and shows at the same time that the argument used to prove (5.23) still works except for a factor $(1+|s|^2)^{j/2}$ which is balanced by the additinal factor ψ^{-j} in (5.33). We can deal analogously with (5.34) if we observe that the terms in $\mathscr{R}_t$ have the form $\operatorname{Op}(A_t G_t)$ where G_t is an appropriate operator symbol of order $\leqslant 0$ satisfying (5.35), and $A_t = \tilde{A}(tr, ts)$ for some smooth function $\tilde{A}$ with values in $\mathscr{L}(H)$. Therefore, the proof of (5.34) must take into account powers of r and $|s|$ which were not present in (5.24). These are again taken care of by ψ^{-j}.

The above estimates use the scaling (5.12) with base point $s_0 = 0$. They can be carried out equally well with any base point s_0 and are uniform in s_0, so

$$\int_0^1 |u^j \tilde{\sigma}^{(j)}(ut, s; u\lambda_0^{1/2})|\, du \leqslant C_{J_0}.$$

Since the φ in (5.27b) is 0 for $|s| \geqslant 1$, (5.31) follows by integration of this last inequality in s. ∎

We can now prove the main result of this paper.

THEOREM 5.2. *Let Δ be the Friedrichs extension of a second order symmetric Laplace type operator as described in the introduction. If $m > \dim M/2$ then $(\Delta+\lambda)^{-m}$ is trace class and there is an asymptotic expansion*

$$\operatorname{tr}(\Delta+\lambda)^{-m} \sim_{\lambda \to \infty} \lambda^{\dim M/2 - m} \sum_{j \geqslant 0} a_j \lambda^{-j/2} + \sum_{j \geqslant 2m-k} b_j \lambda^{-j/2} \log \lambda.$$

Proof. We first obtain this expansion to arbitrarily high order for the model operator $\tilde{\Delta}$ with resolvent $\mathscr{G}$. We apply the SAL of [BS1] to the right hand side of (5.28). By Lemma 5.5, the assumption (1.5b) in [BS2] is satisfied. It only remains to verify (1.5a) in [BS2], that we have an asymptotic expansion

$$\sigma(r,\zeta) \sim_{\zeta\to\infty} \sum_{\alpha\to-\infty} \sigma_\alpha(r)\zeta^\alpha \tag{5.36}$$

with smooth coefficients, $\sigma_\alpha \in C^\infty(\mathbb{R}_+)$. To see this we recall from the identifies (5.28) and (5.30) that

$$\begin{aligned}\sigma(r,\zeta) &= \int_{\mathbb{R}^k} \varphi(r,s)\,\tilde{\sigma}(r,s;\zeta)\,ds,\\ \tilde{\sigma}(r,0;\zeta) &= r^{2m-k-1}\operatorname{tr}_H \mathscr{G}_r^m(1,1;0,0;\zeta^2).\end{aligned} \tag{5.37}$$

It is therefore sufficient to prove an asymptotic expansion of the type (5.36) for the right hand side of (5.37), which is uniform in the base point of the scaling. We can also restrict attention to an arbitrairly small neighborhood of $r=0$ by choosing the r-support of φ sufficiently small. By (5.14) and the construction of the model operator $\tilde{\Delta}$ in Section 3, $\mathscr{G}_r$ is the resolvent of a second order differential operator, $\tilde{\Delta}_r$, acting on $C^\infty(\tilde{E})$, where $\tilde{E}$ is the pullback of E to $(0,\infty)\times\Sigma\times N$. Moreover, for small r, $\tilde{\Delta}_r$ is elliptic in view of (5.14), so we obtain the desired expansion for $\operatorname{tr}_{L^2(N,E)} \mathscr{G}_r(1,1;0,0;\zeta^2)$ by standard elliptic theory. Obviously, this expansion is uniform in the base point. Thus we obtain an asymptotic expansion of $\varphi\mathscr{G}^m(\lambda)$ of the desired form: the term of highest order in the interior expansion is $c\lambda^{\dim M/2-m}$, and the contributions from [BS2, (1.6a) and (1.6c)] are of smaller order in view of the factor r^{2m-k-1}.

It remains to see that $\varphi(\Delta+\lambda)^{-m}$ has the same asymptotic expansion as $\varphi\mathscr{G}^m(\lambda)$. Let $\mathscr{G}_i^m(\lambda)$ be an interior parametrix for $(\Delta+\lambda)^m$, and choose $\psi, \tilde{\psi}\in C_0^\infty((0,\infty)\times\mathbb{R}^k)$ with $\tilde{\psi}=1$ on a neighborhood of $\operatorname{supp}\psi$, while $\operatorname{supp}\psi$ and $\operatorname{supp}\tilde{\psi}$ are near $r=0$, so that Δ and $\tilde{\Delta}$ agree in these supports. We have, with slight abuse of notation,

$$\psi\mathscr{G}_i^m\tilde{\psi}(\tilde{\Delta}+\lambda)^m = \psi\mathscr{G}_i^m\tilde{\psi}(\Delta+\lambda)^m =: \psi+\mathscr{S}, \tag{5.38}$$

where

$$\|\partial^\alpha\mathscr{S}\|_{\mathrm{tr}} \leqslant C_{N,\alpha}\lambda^{-N}$$

for all N and α. Multiply (5.38) on the right by $(\Delta+\lambda)^{-m}\chi$, where $\chi\in C_0^\infty(\mathbb{R}\times\mathbb{R}^k)$ with $\operatorname{supp}\chi\cap\operatorname{supp}\tilde{\psi}=\varnothing$, to find

$$\psi(\Delta+\lambda)^{-m}\chi = -\mathscr{S}(\Delta+\lambda)^{-m}\chi.$$

Hence

$$\|\psi\partial^\alpha(\Delta+\lambda)^{-m}\chi\|_{\mathrm{tr}} \leqslant C_{N,\alpha}\lambda^{-N}. \tag{5.39}$$

Now take φ as above and choose $\tilde{\varphi}\in C_0^\infty(\mathbb{R}_+\times\mathbb{R}^k)$ with $\tilde{\varphi}=1$ near $\operatorname{supp}\varphi$. Then, since derivatives of $\tilde{\varphi}$ vanish near the support of φ, we obtain from (5.39) that

$$\|(\tilde{\Delta}+\lambda)^m\tilde{\varphi}(\Delta+\lambda)^{-m}\varphi-\varphi\|_{\mathrm{tr}} \leqslant C_N\lambda^{-N}.$$

Hence

$$|\operatorname{tr}\varphi((\Delta+\lambda)^{-m}-\mathscr{G}^m(\lambda))| \leqslant C_N\lambda^{-N}.$$

The theorem is proved. ∎

6. The Nature of the Singular Terms

From (5.28) we get the contribution to $\operatorname{tr}(\Delta+\lambda)^{-m}$ from a neighborhood of a given point s_0 on Σ by using a cutoff function $\varphi(r,s)$ supported near $(0,s_0)$. To simplify notation we replace $\tilde{\sigma}(r,s;\zeta)$ by $\sigma(r,s;\zeta)$ and have

$$\operatorname{tr}\varphi(\Delta+z^2)^{-m} = \int_0^\infty \int_{\mathbb{R}^k} \varphi(r,s)\sigma(r,s;rz)\,ds\,dr \tag{6.1}$$

with

$$\sigma(r,s;\zeta) = \operatorname{tr}_H \mathscr{G}^m(r,r;s,s;\zeta^2/r^2), \tag{6.2}$$

and $H=L^2(N,E)$. We analyze σ by scaling to a base point s_0 as in (5.12); the result in (5.30) has $s_0=0$, and in general, with (5.12),

$$\sigma(t,s_0;z) = t^{2m-k-1}\operatorname{tr}_H \mathscr{G}_t^m(1,1;s_0,s_0;z^2). \tag{6.3}$$

According to [BS2, (1.6)], taking $\partial_r^j\varphi(0,s)=0$ for $j=1,2,\ldots$,

$$\operatorname{tr}\varphi(\Delta+z^2)^{-m} \sim \sum_{l\geqslant 0} z^{-l-1}\int_0^\infty \frac{\zeta^l}{l!}\int_{\mathbb{R}^k}\varphi(0,s)\sigma^{(l)}(0,s;\zeta)\,ds\,d\zeta \tag{6.4a}$$

$$+\sum_\alpha \int_0^\infty\int_{\mathbb{R}^k}\varphi(r,s)\sigma_\alpha(r,s)(rz)^\alpha\,ds\,dr \tag{6.4b}$$

$$+\sum_{\alpha=-1}^{-\infty}\int_{\mathbb{R}^k}\varphi(0,s)\sigma_\alpha^{(-\alpha-1)}(0,s)\,ds\,\frac{z^\alpha\log z}{(-\alpha-1)!}, \tag{6.4c}$$

where $\sigma^{(l)}(r, s; \zeta) = \partial_r^l \sigma(r, s; \zeta)$, and

$$\sigma(r, s; \zeta) \sim \Sigma_\alpha \sigma_\alpha(r, s) \zeta^\alpha, \qquad \zeta \to \infty, \tag{6.5}$$

is a given expansion obtained from (5.37), arising from the standard ψdo calculus away from the singular stratum, as in [S1] or [S2]. Thus the integrands in (6.4b) are the usual ones; they may have singularities on the stratum Σ, and (6.4b) represents a regularization of improper integrals by analytic continuation.

The functions σ_α in (6.5) are computed recursively from the symbol of Δ, so the coefficients in (6.4c) are recursively defined from that symbol. They are integrals of smooth densities defined on the singular stratum Σ.

The integrals in (6.4a) are defined by analytic continuation, and so the order of integration can be reversed; thus the coefficient of z^{-l-1} there is likewise the integgral of a smooth density defined on Σ, which we now discuss in more detail.

The coefficients in (6.4a) are recursively defined from the Taylor expansion in r of a suitable *operator symbol* of Δ along the singular stratum Σ; the operators in question act on $L^2(N, E)$, so are "global on the cross-section" N of the cone with vertex at s. To verify this, consider the relation

$$\mathscr{G}_t = \mathscr{G}_{0,t} + \mathscr{G}_t \mathscr{R}_t \tag{6.6}$$

with $\mathscr{G}_{0,t} = \mathrm{Op}(G_{0,t})$,

$$G_{0,t} = (-\partial_r^2 + |\sigma|^2_{s(t)} + r^{-2} A(s(t)) + \zeta^2)^{-1} \tag{6.7}$$

and

$$s(t) = s_0 + t(s - s_0) \tag{6.8}$$

and

$$\mathscr{R}_t = t\left[r \sum_{i,j=0}^{k} A_{ij}(tr, s(t)) \partial_i \partial_j + \sum_{j=0}^{k} B_j(tr, s(t)) \Lambda \partial_j + r^{-1} B(tr, s(t)) \Lambda^2 \right] \mathscr{G}_{0,t}$$
$$+ t\,\mathrm{Op}\left(-2\sqrt{-1} \sum_{i,j=1}^{k} g_{ij}(s(t)) \sigma_i \partial_i G_0 - t \sum_{i,j=1}^{k} g_{ij}(s(t)) \partial_i \partial_j G_0 \right), \tag{6.9}$$

where $\partial_0 = \partial/\partial r$, $\Lambda = (A(0)+1)^{1/2}$, and A_{ij}, B_j, B are smooth families of bounded operators on $L^2(N, E)$.

Since $\mathscr{R}_0 = 0$, the derivatives of $\mathscr{G}_t$ at $t=0$ are defined recursively by (6.6), and this allows the calculation of the terms

$$\sigma^{(j)}(0, s; z) = \partial_t^j (t^{2m-k-1} \operatorname{tr}_H \mathscr{G}_t^m(1, 1; s, s; z^2))|_{t=0}$$

in (6.4a).

We illustrate the nature of these singular terms more concretely by describing the first two. In view of the factor t^{2m-k-1} in (6.3) and the derivative in (6.4a), the first nonvanishing term in (6.4a) is

$$z^{k-2m} \int_{\mathbb{R}^k} \varphi(0, s) \int_0^\infty \operatorname{tr}_H \mathscr{G}_0^m(1, 1; s, s; \zeta^2) \zeta^2) \zeta^{2m-k-1}\, d\zeta\, ds$$
$$= z^{k-2m} \int_{\mathbb{R}^k} \varphi(0, s) c_{k-2m}(s)\, ds,$$

where

$$c_{k-2m}(s) = (2\pi)^{-k} \int_0^\infty \zeta^{2m-k-1} \sum_\nu \int_{\mathbb{R}^k} \frac{1}{(m-1)!} \left(\frac{-1}{2\zeta} \frac{\partial}{\partial \zeta} \right)^{m-1}$$
$$\times I_{\nu(s)} K_{\nu(s)}(\sqrt{\zeta^2 + |\sigma|_s^2})\, d\sigma\, d\zeta$$

can be evaluated using the Mellin transform of $I_\nu K_\nu$, cf. [BS2, (2.11)].

For the next term, with z^{k-2m-1}, we need $\operatorname{tr} \partial_t \mathscr{G}_t^m|_{t=0}$, computed from (6.6). We take the base point $s_0 = 0$ for simplicity, and find from (5.15)

$$\partial_t G_{0,t}|_{t=0} = \sum_i s_i (\partial_{s_i} G_0)|_{s=0} =: \langle s, G_1 \rangle.$$

Set $G_{00} = (-\partial_r^2 + r^{-2} A(0) + |\sigma|_0^2 + \zeta^2)^{-1}$. Then

$$\partial_t \mathscr{G}_t^m|_{t=0} = \mathrm{Op}(\langle s, G_1 \rangle) \mathrm{Op}(G_{00}^{m-1})$$
$$+ \mathrm{Op}(G_{00}) \mathrm{Op}(\langle s, G_1 \rangle) \mathrm{Op}(G_{00}^{m-2}) + \cdots. \tag{6.10}$$

The factor of s can be moved to the left of each product using

$$\mathrm{Op}(G s_j) = s_j\, \mathrm{Op}(G) + \mathrm{Op}(-i \partial_{\sigma_j} G).$$

For the other terms of $\partial_t \mathscr{G}_t^m$, we need the derivative at $t=0$ of (6.9). This is

$$\left[r \sum_{i,j=0}^{k} A_{ij}(0, 0) \partial_i \partial_j + \sum_{j=0}^{k} B_j(0, 0) \Lambda \partial_j + r^{-1} B(0, 0) \Lambda^2 \right] \mathrm{Op}(G_{00})$$
$$- 2\sqrt{-1} \sum_{i,j=1}^{k} g_{ij}(0, 0)\, \mathrm{Op}(\sigma_i \partial_j G_0),$$

which we denote by $\mathscr{R}_1$. Thus $\partial_t \mathscr{G}_t^m|_{t=0}$ is a sum of products of ψdo's on $\mathbb{R}^k$ with symbols which are operators on $L^2(\mathbb{R}_+) \otimes L^2(N, E)$, with polynomial dependence on s as seen in (6.10). For such symbols the functional calculus has no residual terms, so we obtain (in principle) an explicit formula for

$$\operatorname{tr}_H \partial_t \mathscr{G}_t^m(1, 1; 0, 0; \zeta^2)|_{t=0}$$

and hence an explicit smooth function $c_{k-2m-1}(s)$ giving the coefficient of z^{k-2m-1} in (6.4a) in the form

$$\int_\Sigma \varphi(0, s)\, c_{k-2m-1}(s)\, ds.$$

Of course, to work out these coefficients in a given case is challenging. We intend to return to this question in a further publication.

ACKNOWLEDGMENTS

Both authors are indebted to Richard Melrose for numerous conversations, and for making available the notes of his course on manifolds with corners. They also gratefully acknowledge the hospitality and support of the Max Planck Institute for Mathematics in Bonn, May 1987. The first author has been partially supported by the Deutsche Forschungsgemeinschaft and the Stiftung Volkswagenwerk. The second author is indebted to the National Science Foundation Grant DMS 8703604, and to the Brandeis Science Library for refuge from the infamous heat of the summer of 1988.

REFERENCES

[BS1] J. BRÜNING AND R. SEELEY, Regular singular asymptotics, *Adv. in Math.* **58** (1985), 133–148.

[BS2] J. BRÜNING AND R. SEELEY, The resolvent expansion for second order regular singular operators, *J. Funct. Anal.* **73** (1987), 369–429.

[CV] A. CALDERÓN AND R. VAILLANCOURT, On the boundedness of pseudodifferential operators, *J. Math. Soc. Japan* **23** (1971), 374–378.

[HS] P. HALMOS AND V. SUNDER, "Bounded Integral Operators in L^2 Spaces," Springer, Berlin, 1978.

[S1] R. SEELEY, Complex powers of an elliptic operator, *in* "Proceedings Symposia in Pure Mathematics," Vol. 10, pp. 288–307, Amer. Math. Soc., Providence, RI.

[S2] R. SEELEY, The resolvent of an elliptic boundary problem, *Amer. J. Math.* **91** (1969), 889–920.

[MOS] W. MAGNUS, F. OBERHETTINGER, AND R. P. SONI, "Formulas and Theorems for the Special Functions of Mathematical Physics," 3rd ed., Springer-Verlag, Berlin, NY, 1966.

[MP] R. MELROSE AND P. PIAZZA, Analytic *K*-theory on manifolds with corners. *Adv. in Math.*, to appear.

[Si] S. SIMANCA, Mixed elliptic boundary value problems, *Comm. Partial Differential Equations* **12** (1987), 123–200.

[W] G. WATSON, "A Treatise on the Theory of Bessel Functions," 2nd ed., University Press, Cambridge, 1948.

[Z] S. ZAREMBA, Sur un problème mixte relatif à l'équation de Laplace, *Bull. Intern. Acad. Sci. Cracovie* (1010), 314–344.

JOURNAL OF FUNCTIONAL ANALYSIS **95**, 291–319 (1991)

Hopf–von Neumann Algebra Bicrossproducts, Kac Algebra Bicrossproducts, and the Classical Yang–Baxter Equations

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Communicated by A. Connes

Received October 2, 1988; revised May 22, 1989

Two groups G_1, G_2 are said to be a *matched pair* if each acts on the space of the other and these actions, (α, β) say, obey a certain compatibility condition. For every matched pair of locally compact groups $(G_1, G_2, \alpha, \beta)$ we construct an associated coinvolutive Hopf–von Neumann algebra $\mathscr{M}(G_1)^\beta \bowtie_\alpha L^\infty(G_2)$ by simultaneous cross product and cross coproduct. For non-trivial α, β these *bicrossproduct* Hopf–von Neumann algebras are non-commutative and non-cocommutative. If the modules for the actions α, β are also matched then these bicrossproducts are Kac algebras. In this case we show that the dual Kac algebra is of the same form with the roles of G_1, G_2 and of α, β interchanged. Examples exist with G_1 a simply connected Lie group and choices of G_2 determined by suitable solutions of the Classical Yang–Baxter Equations on the complexification of the Lie algebra of G_1.

1. INTRODUCTION AND PRELIMINARIES

The idea of semidirect products of operator algebras by group actions has proven a very fruitful one for generating new examples. A natural formulation for such group actions is in the language of coinvolutive Hopf–von Neumann algebras and Kac algebras, e.g., [Str]. One would like to obtain still more examples of coinvolutive Hopf–von Neumann and Kac algebras, and in this paper this will be achieved by combining a semidirect product algebra construction with a dual semidirect coproduct coalgebra construction. It turns out that the constraint for these to fit together to form a coinvolutive Hopf–von Neumann algebra, called the *bisemidirect product* or *bicrossproduct*, is a generalization of the *Classical Yang–Baxter Equations* (CYBE). Thus many examples of the necessary data are known.

* Work supported in part by the Herchel Smith Scholarship of Emmanuel College, Cambridge, England. Current address: University of Cambridge, UK.

0022-1236/91 $3.00