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ON L^2-INDEX THEOREMS FOR COMPLETE MANIFOLDS OF RANK-ONE TYPE

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To the memory of Vojislav Avacumović

1. Introduction. Index problems on noncompact manifolds have attracted much interest recently. In this paper we consider a complete Riemannian manifold M, of dimension m, and a generalized Dirac operator, D, on M. (These operators are described in detail in [LM].) This means that D is an elliptic first-order differential operator acting on the smooth sections of a hermitian vector bundle E over M. Now E has a Clifford module structure and a metric connection, ∇, compatible with Clifford multiplication (denoted by $\cdot$), and locally, with any orthonormal frame (F_i) for TM, we have

$$D\sigma = \sum_{i\geq 1} F_i \cdot \nabla_{F_i} \sigma, \qquad \sigma \in C^\infty(E). \tag{1.1}$$

It is easily seen that D with domain $C_0^\infty(E)$ is symmetric in $L^2(E)$ and that D has a unique closed extension, also to be denoted by D. But if in addition E has a ∇-parallel splitting, $E = E^+ \oplus E^-$, such that $F \cdot E^\pm \subset E^\mp$ for $F \in TM$, then $D^+ := D|C_0^\infty(E^+)$ is no longer symmetric. Thus, we can ask whether the unique closed extension, D^+, is a Fredholm operator, or more generally, whether the spaces $\ker D^\pm \cap L^2(E^\pm)$ are finite-dimensional. If so,

$$L^2\text{-ind } D^+ := \dim \ker D^+ - \dim \ker D^- \tag{1.2}$$

is well defined, and we can try to derive an L^2-index formula. The first example of such a formula seems to occur in [APS], for the geometric operators on manifolds with cylindrical ends. Since then, there has been considerable and ongoing interest in such formulas; see, e.g., [ADS], [BaMo], [GL], [M1], [M2], [M3], [M4], [St1], [St2], among others.

Nevertheless, there is still no systematic approach to such theorems. In [B] we made a modest first step towards an abstract setting the applications of which, however, seemed to be limited to manifolds which are asymptotically warped products. The reason for this limitation was that we required a specific limit for the Dirac operator at infinity. In the present paper we relax this requirement considerably. This is possible by generalizing analogously the notion of *regular singular operator* introduced in [BS2] and [B]. The main result is formulated as Theorem 4.4. Thus, we can treat considerably more general manifolds like multiply warped products, which we treat as main example here in Section 5. But our theory also

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applies to manifolds with finite volume and pinched negative curvature; this will be treated elsewhere. In general, the sort of geometric control implying the crucial assumptions (2.9) through (2.13) corresponds roughly to the behavior of locally symmetric spaces of rank one with finite volume. For this reason we call complete manifolds, which satisfy (2.1) and the above mentioned conditions for *all* generalized Dirac operators, manifolds of *rank-one type*. We will further clarify the geometric content in a future publication.

The index formula in Theorem 4.4 is calculable (in principle) only in the Fredholm case, but even then we feel that it is not in its final form. In the non-Fredholm case there occurs always a quantity h_1 (see (2.34)) which is virtually impossible to compute. In the cylindrical case, it has been shown by Müller [M3] that h_1 is related to zero-energy resonances of a certain natural scattering problem. It is thus remarkable that we get a computable L^2-signature formula in some cases where $h_1 > 0$. (See Theorem 5.7 which generalizes Corollary (4.11) in [APS].)

The plan of this work is as follows: In Section 2 we relate the computation of L^2-indices to the computation of Fredholm indices for certain associated weighted operators; in particular, we provide a convenient framework for proving the finiteness of L^2-indices.

In Section 3 we transform the weighted operator to a generalized regular singular operator, and we study its Fredholm properties. Both Section 2 and 3 follow the outline given in [B], with some crucial modifications.

Section 4 is devoted to computation of the Fredholm index. This is somewhat subtle since we have to allow deformations that change the domain.

In Section 5 we study the L^2-signature. We give a very general separation of variables for manifolds with productlike ends (Theorem 5.3) and discuss various special cases, notably multiply warped products.

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2. The class of operators. We consider a more general class of operators than the one studied in [B]. To describe it, let M be a Riemannian manifold and U be an open subset such that

$$M_1 := M\backslash U \text{ is a compact manifold with boundary.} \tag{2.1}$$

Consider next two hermitian vector bundles E_1, E_2 over M and a first-order elliptic differential operator D: $C^\infty(E_1) \to C^\infty(E_2)$. It is enough to think of D as a generalized Dirac operator in the sense of Gromov and Lawson [GL]. We assume a "nice" representation of D on U in the following sense.

There are a Hilbert space H, with a dense subspace H_1, and isometries

$$\Phi_i\colon L^2(E_i|U) \to L^2((0, \infty), H). \tag{2.2}$$

They induce isomorphisms

$$H_0^1(E_i|\bar{U}) \to H_0^1([0, \infty), H) \cap L^2((0, \infty), H_1), \qquad i = 1, 2.$$

Moreover, there is a smooth function (in the sense of strong differentiability) $\mathbb{R}_+ \ni y \mapsto S(y) \in \mathscr{L}(H_1, H)$ such that $S(y)$ is self-adjoint in H with domain H_1, and for $u \in C_0^\infty([0, \infty), H_1)$, $y \geq 0$,

$$\Phi_2 D \Phi_1^{-1} u(y) = -\partial_y u(y) + S(y)u(y). \tag{2.3}$$

Though we can develop most of our theory in an abstract setting, the operators we are going to consider will be of rather special type. Consequently, the family $S(y)$ will be special. For the present applications we may, therefore, assume that

$S(y)$, $y \geq 0$, is a smooth family of self-adjoint elliptic differential operators of first order, acting on $C^\infty(F)$ for some hermitian vector bundle F over the compact Riemannian manifold $N := \partial M_1$. (2.3′)

Examples of this situation are asymptotically warped products as described in [B]; more general examples will be given in Section 5 below. We have to impose further conditions on D in order to obtain a finite L^2-index. By this we mean the quantity

$$L^2\text{-ind } D := \dim \ker D \cap L^2(E_1) - \dim \ker D' \cap L^2(E_2) \tag{2.4}$$

where $D'\colon C^\infty(E_2) \to C^\infty(E_1)$ denotes the formal adjoint. If $D_{\min}$ and $D_{\max}$ denote the minimal and maximal closed extension of D, respectively, where $D_{\min}$ is the closure and $D_{\max} = (D'_{\min})^*$, then we can write

$$L^2\text{-ind } D = \dim \ker D_{\max} - \dim \ker D'_{\max}.$$

Thus, if L^2-ind D is defined, it need not be the index of some closed extension. If $D_{\max} = D_{\min}$, however, this is the case. It should be noted that in this case the finiteness of L^2-ind D does not imply that D is Fredholm. In fact, in this work we are mostly interested in the non-Fredholm case.

From (2.2) we also obtain isometries

$$L^2(E_j) \to L^2(E_j|M_1) \oplus L^2((0, \infty), H) =: \mathscr{H}_j, \tag{2.5}$$

$j = 1, 2$. This allows us to identify D with a closed operator in $\mathscr{H}_1$ with values in $\mathscr{H}_2$. We write $u = (u_i, u_b)$ for elements in $\mathscr{H}_j$, with u_i the "inner" and u_b the "boundary" component. We define a multiplication on $\mathscr{H}_j$ by elements of $C^{\sim}(\mathbb{R}_+) := \{\varphi \in C^\infty(\mathbb{R}_+) | \varphi$ is constant near 0 and $\infty\}$ by

$$\varphi u = (\varphi(0)u_i, \varphi u_b) \in \mathscr{H}_j. \tag{2.6}$$

Then we require that

$$(\mathbf{I} \oplus \Phi_j)^{-1} \varphi u = \overline{\varphi}(\mathbf{I} \oplus \Phi_j)^{-1} u \tag{2.7}$$

for some $\overline{\varphi} \in C^\infty(M)$, with $\overline{\varphi} \in C_0^\infty(M)$ if $\varphi \in C_0^\infty(\mathbb{R}_+)$.

For the domain $\mathscr{D}(D)$ of D we have

$$\begin{aligned} \mathscr{D}(D) = \{(u_i, u_b) \in \mathscr{H}_1 | u_i \in H^1(E_1 | M_1); u_b \in H^1_{\mathrm{loc}}([0, \infty), H) \cap L^2_{\mathrm{loc}}((0, \infty), H_1), \\ (-\partial_y + S(y))u_b(y) \in L^2((0, \infty), H); u_i | \partial M_1 = \Phi_1^{-1} u_b | \partial M_1\}. \end{aligned} \tag{2.8}$$

To obtain a finite L^2-index for D, we have to impose further conditions on the operator function S in (2.2). First of all, we postulate a "spectral concentration property" for a renormalization of S as follows.

There is a smooth family $Q(y)$ of spectral projections of $S(y)$ such that

$$f(y)|Q(y)S(y)| \leq C_1 Q(y), \qquad C_1 < 1/2, \tag{2.9}$$

and

$$f(y)|(\mathbf{I} - Q(y))S(y)| \geq C_2(\mathbf{I} - Q(y)), \qquad C_2 > 1/2. \tag{2.10}$$

Here, f is a smooth positive function satisfying

$$f'(y) = a + o(1) \qquad \text{as } y \to \infty, \qquad 0 \leq a < 1. \tag{2.11}$$

With $\bar{S}(y) := f(y)S(y)$ the operator equation in (2.3) can be written as

$$Du_b(y) = -\partial_y u_b(y) + f(y)^{-1}\bar{S}(y)u_b(y). \tag{2.12}$$

This form resembles the warped-product case dealt with in [B], but we have the important difference that $\bar{S}(y)$ will, in general, not converge as $y \to \infty$. To extend the methods of [B] we finally have to bound the variation of $\bar{S}$ in the following sense: the function

$$\alpha(y) := \|\bar{S}'(y)(|\bar{S}(y)| + \mathbf{I})^{-1}\|_H \tag{2.13a}$$

has to satisfy the estimate

$$\int_0^\infty f(y)\alpha^2(y)\, dy =: C_3 < \infty. \tag{2.13b}$$

The conditions (2.9) through (2.13) seem fairly sharp in this generality. For more special families $S(y)$ they can be relaxed, e.g., if we assume commutativity. We will return to this question at a later occasion.

Following the pattern of [B], we now compare L^2-ind D with the index of an associated regular singular operator. We define

$$F(y) := \int_0^y \frac{du}{f(u)} \tag{2.14}$$

and with $\psi \in C^\infty(\mathbb{R}_+)$, $\psi(0) = 0$, $\psi(\infty) = 1$,

$$g(y) := [((1 - \psi)(y)f(0) + \psi f(y))e^{\psi F(y)}]^{1/2}. \tag{2.15}$$

LEMMA 2.1. *For $1 < b < 1/a$ (where a is defined in (2.11)) and y sufficiently large, we have*

$$F(y) \geq \log y^b. \tag{2.16}$$

In particular, $\lim_{y \to \infty} F(y) = \infty$. *The function*

$$s(y) := \int_y^\infty \frac{du}{g(u)^2} \tag{2.17}$$

maps $(0, \infty)$ *diffeomorphically onto* $(0, s_0)$, $s_0 := s(0)$, *and for large* y

$$s(y) = e^{-F(y)}. \tag{2.18}$$

Moreover, the map

$$\Psi: L^2((0, s_0), H) \ni v \mapsto g^{-1} v \circ s \in L^2(\mathbb{R}_+, H)$$

is unitary, and for $u \in C_0^\infty((0, s_0), H_1)$ *we have*

$$\Psi^{-1} g D g \Psi u(x) = [\partial_x + (g^2/f) \circ s^{-1}(x) \bar{S} \circ s^{-1}(x)] u(x). \tag{2.19}$$

Proof. See the proof of Lemma 2.1 in [B]. □

Later on, we will frequently use the additional hypothesis

$$Q(y) \equiv Q(0) =: Q, \qquad y \geq 0. \tag{2.20}$$

Now we introduce the weighted operator D_g by

$$D_g u := g D g u = (f(0) D u_i, g(y)[-\partial_y + S(y)] g u_b(y)), \; u \in \mathscr{D}(D), \; u_b \in C_0^\infty([0, \infty), H_1). \tag{2.21}$$

With $I := (0, s_0)$ we obtain from Lemma 2.1 the following result, exhibiting D_g as a generalized regular-singular operator.

LEMMA 2.2. *D_g with domain given in* (2.21) *is unitarily equivalent to the following operator $\tilde{D}$: Put*

$$\tilde{\mathscr{H}}_i := L^2(E_i|M_1) \oplus L^2(I, H), \qquad i = 1, 2, \tag{2.22a}$$

and

$$\begin{aligned} \mathscr{D}_0(\tilde{D}) &:= \{u := (u_i, u_b) \in \tilde{\mathscr{H}}_1 | \mathbf{I} \oplus \Psi(u) \in \mathscr{D}(D), u_b \in C_0^\infty((0, s_0], H_1)\}, \\ \tilde{D}u &:= (f(0)Du_i, (\partial_x + x^{-1}\tilde{S}(x))u_b(x)), \qquad u \in \mathscr{D}_0(\tilde{D}) \end{aligned} \tag{2.22b}$$

where

$$\tilde{S}(x) := x(g^2/f) \circ s^{-1}(x)\bar{S}(s^{-1}(x)), \qquad x \in I. \tag{2.22c}$$

In particular,

$$\tilde{S} \textit{ satisfies the analogue of } (2.3'). \tag{2.22d}$$

Making $\tilde{Q}(x) := Q(s^{-1}(x))$ with Q in (2.9), *we have for sufficiently small x*

$$(\mathbf{I} - \tilde{Q}(x))|\tilde{S}(x)| \geq C_2(\mathbf{I} - \tilde{Q}(x)), \tag{2.23a}$$

$$\tilde{Q}(x)|\tilde{S}(x)| \leq C_1\tilde{Q}(x), \tag{2.23b}$$

with $C_1 < 1/2 < C_2$ (and corresponding to (2.20) *we will sometimes use the assumption*

$$\tilde{Q}(x) = \tilde{Q}(0) \qquad \textit{for } x \in I). \tag{2.24}$$

If (2.13b) *holds, then the function*

$$\tilde{\alpha}(x) := \|\tilde{S}'(x)(|\tilde{S}(x)| + \mathbf{I})^{-1}\|_H \tag{2.25a}$$

satisfies

$$\int_0^{s_0} x\tilde{\alpha}(x)^2\, dx =: C_4^2 < \infty \tag{2.25b}$$

and, with C_5 depending only on C_1 and C_2, we have

$$\|\tilde{Q}'(x)\|_H \leq C_5\tilde{\alpha}(x), \qquad x \in (0, s_0). \tag{2.25c}$$

We remark that by an appropriate choice of ψ in (2.15) we can assume that (2.23) holds in any given neighborhood $(0, s_1]$ of 0, $s_1 < s_0$. The main result of the next section, Theorem 3.6, will show that all closed extensions of D_g are Fredholm operators. From this fact we easily derive the following theorem.

THEOREM 2.3. *Under the assumptions* (2.2), (2.3′), (2.7), (2.9) *through* (2.11), *and* (2.13), *the L^2-index of D is finite.*

Proof. It follows as in [B, (2.20)] that $g^{-1} \in L^\infty(\mathbb{R}_+)$. Hence, $u \in \ker D_{\max}$ implies $g^{-1}u \in \ker D_{g,\max}$, and since $\ker D_{g,\max}$ is finite-dimensional, the injectivity of the map $u \mapsto g^{-1}u$ implies that $\ker D$ is finite-dimensional, too. A similar argument applies to $\ker D'_{\max}$. □

To derive an index formula we need in addition the assumptions (2.20) or (2.24) which will be assumed for the remainder of this section. We start with defining a suitable closed extension of D_g. We obtain as in [B, Lemma 2.2] the following lemma.

LEMMA 2.4. $g^{-1}\mathscr{H}_1 \cap \mathscr{D}(D_{g,\max}) \supset \mathscr{D}(D_{g,\min})$.

Proof. It will follow from Lemma 3.4 below that $u = (u_i, u_b) \in \mathscr{D}(D_{g,\max})$ is in $\mathscr{D}(D_{g,\min})$ if and only if

$$\|gu_b(y)\|_H^2 \le Cs(y). \tag{2.26}$$

This together with (2.16), (2.11), and (2.18) implies the assertion. □

Now we introduce an extension $D_{g,V}$ of D_g by

$$D_{g,V} := D_{g,\max} | g^{-1}\mathscr{H}_1 \cap \mathscr{D}(D_{g,\max}) \tag{2.27a}$$

where

$$V := g^{-1}\mathscr{H}_1 \cap \mathscr{D}(D_{g,\max}) / \mathscr{D}(D_{g,\min}). \tag{2.27b}$$

This extension is closed since $D_{g,\min}$ and $D_{g,\max}$ are Fredholm and V is finite-dimensional. The boundary conditions for $D_{g,V}$ depend on the induced ordinary differential operators (with Q from (2.20)) □

$$T_Q := -\partial_y + QS(y) =: -\partial_y + S_Q(y),$$

$$T'_Q := \partial_y + S_Q(y),$$

acting in $L^2(\mathbb{R}_+, QH)$. We introduce the spaces

$$K^{(\prime)} := \{e \in QH \mid T_Q^{(\prime)} u(y) = 0 \text{ has an } L^2\text{-solution } u \text{ with } u(0) = e\},$$

and we observe that $K \perp K'$. (See [B, Sec. 4].) Then we can decompose orthogonally

$$QH =: K \oplus K' \oplus L \tag{2.28}$$

with some, possibly trivial, subspace L. We need the following fact. (See [B, Lemma 4.1].)

LEMMA 2.5. *Let $v \in C^1(\mathbb{R}_+, QH_1)$ be a solution of the equation*

$$T_Q v = g^{-1} w$$

with $w \in L^2(\mathbb{R}_+, QH)$ and denote by $W_Q(y, y_1)$ the solution operator of T_Q with $W_Q(y_1, y_1) = \mathbf{I}$, y, $y_1 \geq 0$. Then there are $\gamma v \in K$ and $v_1 \in C^1(\mathbb{R}_+, QH)$ with the estimate

$$\|v_1(y)\|_H^2 = O(e^{-F(y)}) \qquad \text{as } y \to \infty \tag{2.29}$$

such that

$$v(y) = W_Q(y, 0)\gamma v + v_1(y). \tag{2.30}$$

γv and v_1 depend linearly on v and are uniquely determined by (2.29) *and* (2.30).

Proof. We have for $y \geq 0$

$$v(y) = W_Q(y, 0)v(0) + \int_0^y W_Q(y, y')g^{-1}w(y')\, dy'. \tag{2.31}$$

Since $W_Q(y, y')$ solves the equation $T_Q W = 0$ with respect to y, we have for $e \in QH$, by (2.9)

$$\begin{aligned}\|W_Q(y, y')e\|^2 &= \|e\|^2 + 2\int_{y'}^y \langle S_Q W_Q(t, y_1)e, W_Q(t, y_1)e\rangle\, dt \\ &\leq \|e\|^2 + 2C_1 \int_{y'}^y f(t)^{-1}\|W_Q(t, y_1)e\|^2\, dt;\end{aligned}$$

hence, from Gronwall's lemma

$$\|W_Q(y, y')\| \leq e^{C_1|F(y) - F(y')|}. \tag{2.32}$$

(2.32), together with $g(y')^{-1} = f(y')^{-1/2}e^{-F(y')/2}$ for y' large, shows that the integral in (2.31) converges at ∞. Since $W_Q(y, y') = W_Q(y, 0)W_Q(y', 0)^{-1}$, we can write

$$\begin{aligned} v(y) &=: W_Q(y, 0)\gamma v - \int_y^\infty W_Q(y, y')g^{-1}w(y')\,dy' \\ &=: W_Q(y, 0)\gamma v + v_1(y). \end{aligned} \tag{2.33}$$

Finally, we estimate with (2.32), for large y,

$$\begin{aligned} \|v_1(y)\|^2 &\le \|w\|^2_{L^2(\mathbb{R}_+, H)} e^{-F(y)} \int_y^\infty f(y')^{-1} e^{-(1-2C_1)(F(y')-F(y))}\,dy' \\ &= e^{-F(y)} \|w\|^2_{L^2(\mathbb{R}_+, H)} \int_0^\infty e^{-(1-2C_1)z}\,dz \\ &= e^{-F(y)} \|w\|^2_{L^2(\mathbb{R}_+, H)} (1 - 2C_1)^{-1}. \end{aligned}$$

Assume now a second representation

$$v(y) = W_Q(y, 0)\overline{\gamma v} + \bar{v}_1(y).$$

Then, with W_Q' the solution operator for T_Q', we find

$$\begin{aligned} \gamma v - \overline{\gamma v} &= W_Q'(y, 0)^* W_Q(y, 0)(\gamma v - \overline{\gamma v}) \\ &= W_Q'(y, 0)^*(\bar{v}_1 - v_1)(y). \end{aligned}$$

Using (2.29) and (2.32) for W_Q', we obtain

$$\|\gamma v - \overline{\gamma v}\| \le C e^{(C_1 - 1/2)F(y)}, \qquad y \ge 0;$$

hence, $\gamma v = \overline{\gamma v}$, $v_1 = \bar{v}_1$. □

Now we can prove the following relation between L^2-ind D and ind $D_{g,\min}$. Put

$$h_0 := \dim V, \qquad h_1 := \dim \ker D^*_{g,V} - \dim \ker D'_{\max}. \tag{2.34}$$

THEOREM 2.6. *Under the assumptions of Theorem* 2.3 *we have*

$$L^2\text{-ind } D = \operatorname{ind} D_{g,\min} + h_0 + h_1. \tag{2.35}$$

If we assume also (2.20), *then*

$$h_0 = \dim K, \qquad h_1 \le \dim L. \tag{2.36}$$

If the metric on M or on E_1, E_2, is changed on a compact set or if the weight function g is changed on a compact set such that it remains positive and equal to $f(0)^{1/2}$ near 0, then h_0 does not change; h_1 is invariant under compact perturbations of g.

Proof. We know from Theorem 2.3 that dim ker $D_{\max}$ and dim ker $D'_{\max}$ are finite. Since $g^{-1} \in L^\infty(\mathbb{R}_+)$, it follows from the construction of $D_{g,V}$ that $u \mapsto g^{-1}u$ defines a bijective map ker $D_{\max} \to$ ker $D_{g,V}$. Combining this with Lemma 2.2 and Theorem 3.2 below, we obtain

$$L^2\text{-ind } D = \dim\ker D_{g,V} - \dim\ker D^*_{g,V} + h_1$$

$$= \operatorname{ind} D_{g,V} + h_1 = \operatorname{ind} D_{g,\min} + h_0 + h_1 .$$

To prove $h_0 = \dim K$ we define a linear map $\tilde{\gamma}: \mathscr{D}(D_{g,V}) \to QH$ with image K and kernel $\mathscr{D}(D_{g,\min})$. Pick $\varphi \in C^\infty(\mathbb{R})$ with $\varphi(y) = 0$ if $y \leq 1$, $\varphi(y) = 1$ if $y \geq 2$. For $u = (u_i, u_b) \in \mathscr{D}(D_{g,\max})$ we have $(1 - \varphi)u \in \mathscr{D}(D_{g,\min})$ by (2.26), and $\varphi(\mathbf{I} - Q)u \in \mathscr{D}(D_{g,\min})$ by Lemma 3.4. We obtain for $v := \varphi g Q u$

$$T_Q v = g^{-1}\varphi Q D_g u - \varphi' g Q u =: g^{-1} w$$

with $w \in L^2(\mathbb{R}_+, H)$. Using Lemma 2.5, we define $\tilde{\gamma}$ by

$$\tilde{\gamma}(u) := \gamma v = \gamma(\varphi g Q u).$$

The decomposition (2.30) gives

$$v(y) = W_Q(y, 0)\gamma v + v_1(y) =: v_0(y) + v_1(y),$$

and the estimate (2.29) shows that $v_1 \in L^2(\mathbb{R}_+, H)$. Hence, $u \in \mathscr{D}(D_{g,V})$ if and only if $v \in L^2(\mathbb{R}_+, H)$ if and only if $v_0 \in L^2(\mathbb{R}_+, H)$ if and only if $\gamma v \in K$. Thus, $\tilde{\gamma}$ maps into K, and this map is easily seen to be surjective. If $u \in \ker \tilde{\gamma}$, then it follows from (2.29) and (2.26) that

$$\varphi Q u = g^{-1} v_1 \in \mathscr{D}(D_{g,\min});$$

hence, $u \in \mathscr{D}(D_{g,\min})$. Conversely, $u \in \mathscr{D}(D_{g,\min})$ implies, by (2.26), (2.29), and (2.32), as in the uniqueness proof of Lemma 2.5,

$$\|\gamma v\|_H \leq C e^{(C_1 - 1/2)F(y)}, \qquad y \geq 0.$$

It follows readily that $\gamma v = 0$.

To prove the second estimate in (2.36), we introduce the quantities V', γ', $\tilde{\gamma}'$, W'_Q for D', analogous to V, γ, $\tilde{\gamma}$, W_Q. Then the argument in [B, Lemma 2.3] shows that

$$\ker D'_{g,V'} = g^{-1} \ker D'_{\max} \subset \ker D^*_{g,V} .$$

Also, $\tilde{\gamma}'$ maps $\mathscr{D}(D'_{g,\max})$ to QH with $\tilde{\gamma}'^{-1}(K') = \mathscr{D}(D'_{g,V'})$. Hence, $\tilde{\gamma}'$ induces an injective map $\ker D^*_{g,V}/\ker D'_{g,V'} \to (K' \oplus L)/K' \simeq L$ if we show that

$$\tilde{\gamma}'u \perp K \qquad \text{for } u \in \ker D^*_{g,V}.$$

To see this, let $e \in K$ and put $e' := \tilde{\gamma}'u$ for some $u \in \ker D^*_{g,V}$. With φ as above and $\varphi_\alpha(y) := \varphi(y - \alpha)$, $\alpha \geq -1$, it is clear that we can replace φ by φ_α in the previous considerations. Then, for $\alpha \geq 0$ we have $\bar{u} := g^{-1}\varphi_\alpha W_Q e \in \mathscr{D}(D_{g,V})$, and it follows from Lemma 2.5 that

$$\begin{aligned} 0 &= (\bar{u}, D^*_{g,V}u) = (D_{g,V}\bar{u}, u) \\ &= (-\varphi'_\alpha W_Q e, \varphi_{-1} g u) =: -(\varphi'_\alpha W_Q e, \varphi_{-1} W'_Q e' + v_1) \\ &= -\langle e, e' \rangle - (\varphi'_\alpha W_Q e, v_1). \end{aligned}$$

Letting $\alpha \to \infty$, we obtain $\langle e, e' \rangle = 0$.

It remains to study the invariance of h_0, h_1 under compact perturbations. Observe first that the map $T\colon u \mapsto gu$ induces a bijection

$$T\colon \mathscr{D}(D_{g,v}) \to \mathscr{D}(gD_{\max}).$$

It follows from the closed graph theorem that T is continuous, and hence is an isomorphism by the open mapping theorem. But then we conclude that

$$T(\mathscr{D}(D_{g,\min})) = \mathscr{D}(gD_{\min})$$

since T preserves the support. Hence,

$$h_0 = \dim \mathscr{D}(gD_{\max})/\mathscr{D}(gD_{\min})$$

where the spaces occuring on the right-hand side do not change if g or the metric is altered on a compact set.

The invariance of h_1 follows from

$$\begin{aligned} \ker D^*_{g,V} &= \{u \in L^2(E_2) \mid (u, gDgv) = 0 \text{ for all } v \in \mathscr{D}(D_{g,V})\} \\ &= \{\tilde{u} \in gL^2(E_2) \mid (\tilde{u}, D\tilde{v}) = 0 \text{ for all } \tilde{v} \in \mathscr{D}(gD_{\min})\}. \end{aligned}$$

Again, the spaces involved do not change if g experiences a compact perturbation. □

In general, h_1 is very difficult to compute. It vanishes, however, if the operator D is Fredholm.

LEMMA 2.7. *Assume that D satisfies the assumptions of Theorem* 2.3 *and* (2.20) *and that D has a unique closed extension which is Fredholm. Then $h_1 = 0$.*

Proof. Denote the unique closed extension also by D. It is well known that the Fredholm property implies that

$$0 \notin \operatorname{spec}_e D^*D \oplus DD^* \tag{2.37}$$

where spec_e denotes the essential spectrum. It follows from the decomposition principle and (2.20) that with A and A' the Friedrichs extension of $T_Q' T_Q$ and $T_Q T_Q'$ in $L^2(\mathbb{R}_+, QH)$, respectively, we also have

$$0 \notin \operatorname{spec}_e A \cup \operatorname{spec}_e A'. \tag{2.38}$$

Now denote by J the space of solutions of the equation

$$Au(y) = T_Q' T_Q u(y) = 0, \qquad y \geq 0,$$

which we may identify with $\mathbb{C}^{2n}$ via $u \mapsto (u(0), v(0))$, where $v := T_Q u$ and $n := \dim Q$. A well-known result in the spectral theory of ordinary differential operators (see, e.g., [W, Theorem 11.4]) asserts that (2.38) implies

$$\dim J_1 \geq n \tag{2.39}$$

where

$$J_1 := J \cap L^2(\mathbb{R}_+, QH).$$

Now consider the spaces

$$J_2 := \{u \in J \mid v(0) \in K \oplus L\},$$

$$J_3 := \{u \in J_2 \mid u(0) \in K,\ v(0) = 0\}.$$

Then we claim that

$$J_1 \cap J_2 = J_3. \tag{2.40}$$

Accepting (2.40) for the moment, the lemma follows from (2.36) and

$$\begin{aligned} 2n \geq \dim(J_1 + J_2) &= \dim J_1 + \dim J_2 - \dim J_1 \cap J_2 \\ &= \dim J_1 + n + \dim K + \dim L - \dim K \\ &\geq 2n + \dim L. \end{aligned}$$

For the proof of (2.40) we pick $u \in J$ and obtain with $v(y) = T_Q u(y) = W_Q'(y, 0)v(0)$

$$u(y) = W_Q(y, 0)u(0) + \int_0^y W_Q(y, y')W_Q'(y', 0)v(0)\, dy'.$$

Hence, with $W_Q'(y, \bar{y})^* = W_Q(y, \bar{y})^{-1}$, $W_Q(y, \bar{y}) = W_Q'(\bar{y}, y)^*$,

$$\begin{aligned}\langle u(y), v(y)\rangle &= \langle W_Q(y, 0)u(0), W_Q'(y, 0)v(0)\rangle \\ &\quad + \int_0^y \langle W_Q(y, y')W_Q'(y', 0)v(0), W_Q'(y, 0)v(0)\rangle\, dy' \\ &= \langle u(0), v(0)\rangle + \int_0^y \langle W_Q(0, y')W_Q'(y', 0)v(0), v(0)\rangle\, dy' \\ &= \langle u(0), v(0)\rangle + \int_0^y \|v(y')\|^2\, dy'.\end{aligned}$$

This implies for $T > 0$ that

$$\begin{aligned}&\int_0^T (T - y)/2 \int_0^y \|v(y')\|^2\, dy'\, dy \\ &\qquad = \int_0^T (T - y)/2(\langle u(y), v(y)\rangle - \langle u(0), v(0)\rangle)\, dy \\ &\qquad \le \left(\int_0^T \|u(y)\|^2\, dy\right)^{1/2} \left(\int_0^T (T - y)^2/4\, \|v(y)\|^2\, dy\right)^{1/2} \\ &\qquad\quad + (T^2/4)\, \|u(0)\|\, \|v(0)\|,\end{aligned}$$

and since

$$\int_0^T (T - y)/2 \int_0^y \|v(y')\|^2\, dy'\, dy = \int_0^T (T - y)^2/4\, \|v(y)\|^2\, dy,$$

we arrive at the inequality

$$\begin{aligned}&\frac{T^2}{4}\left[\frac{3}{16}\int_0^{T/2} \|v(y)\|^2\, dy - \|u(0)\|\, \|v(0)\|\right] \\ &\qquad \le \int_0^T \|u(y)\|^2\, dy. \qquad (2.41)\end{aligned}$$

Clearly, $J_3 \subset J_1 \cap J_2$. Conversely, let $u \in J_1 \cap J_2$. Since $T_Q' v = 0$ and $v(0) \in K \oplus L$, $v(0) \neq 0$ implies $v \notin L^2(\mathbb{R}_+, QH)$. But then $u \notin L^2(\mathbb{R}_+, QH)$ by (2.41); so we conclude $v(0) = 0$. Then $v = T_Q u \equiv 0$, and $u \in L^2(\mathbb{R}_+, QH)$ implies $u(0) \in K$, i.e., $u \in J_3$. □

3. The boundary parametrix. We will now study the weighted operator D_g introduced in (2.21). By Lemma 2.2 this operator is, near the boundary, unitarily equivalent to

$$T := \partial_x + x^{-1}\tilde{S}(x) \tag{3.1}$$

acting in $C_0^\infty(I, H_1)$. Here, $\tilde{S}(x)$ is explicitly given by (2.22c). In what follows, we will treat the abstract situation described in Lemma 2.2; i.e., we consider an operator $\tilde{D}$ satisfying the assumptions (2.22a, b, d) and (2.23a, b) for *all* $x \in I$. Moreover, we will at first assume that a stronger version of (2.24) holds, namely,

we can split $\mathbf{I} - \tilde{Q} = \tilde{Q}_> + \tilde{Q}_<$, independent of x, such that

$$\tilde{S}(x)\tilde{Q}_> \geq C_2\tilde{Q}_>,\, \tilde{S}(x)\tilde{Q}_< \leq -C_2\tilde{Q}_<. \tag{3.2}$$

We will also assume that $I = (0, s_0]$ is such that $s_0 \leq C_2$.

The operator $\tilde{D}$ is *not* regular singular in the sense of [BS2] or [B] since we do not have convergence of $\tilde{S}(x)$ as $x \to 0$. However, we do have the sort of control given by (2.25). Under these assumptions we are going to construct a boundary parametrix for $\tilde{D}$. Since we have no limiting operator as in [BS2] or [B], we cannot use eigenspace decomposition. Instead, we use the theory of abstract evolution equations as exposed, e.g., in [Kr]. In view of (3.2) we can treat the cases

$$\tilde{S}_>(x) := \tilde{Q}_>\tilde{S}(x), \qquad \tilde{S}_0(x) := \tilde{Q}\tilde{S}(x), \qquad \tilde{S}_<(x) := \tilde{Q}_<\tilde{S}(x)$$

separately. Starting with $\tilde{S}_0(x)$, we denote by $W_0(x, y)$ the solution operator, i.e., the unique matrix-valued function solving

$$(\partial_x + x^{-1}\tilde{S}_0(x))W_0(x, y) = 0, \tag{3.3a}$$

$$W_0(y, y) = \mathbf{I}, \qquad x, y \in I. \tag{3.3b}$$

Then we define the parametrix by

$$P_0 f(x) := \int_0^x W_0(x, y)f(y)\,dy, \qquad x \in I, \qquad f \in L^2(I, \tilde{Q}H). \tag{3.4}$$

To imitate this procedure for $\tilde{S}_>$, we recall the following lemma.

LEMMA 3.1. *For $\delta \in I$ consider the triangle*

$$\Delta_\delta := \{(x, y) \in I \times I \mid \delta \leq y \leq x \leq s_0\}.$$

On Δ_δ we can define an operator-valued function $W_>(x, y) \in \mathscr{L}(\tilde{Q}_> H)$ with the following properties.

(a) *For $e \in H_1$, $(x, y) \in \Delta_\delta$, the unique solution of the initial-value problem*

$$(\partial_x + x^{-1}\tilde{S}_>(x))u(x) = 0, \qquad x > y, \tag{3.5a}$$

$$u(y) = e, \tag{3.5b}$$

is given by

$$u(x) = W_>(x, y)e.$$

(b) *$W_>$ is uniformly bounded and strongly continuous in Δ_δ.*

(c) *For $(x, y) \in \Delta_\delta$ and $y \le z \le x$, we have*

$$W_>(x, y) = W_>(x, z)W_>(z, y)$$

and

$$W_>(y, y) = I.$$

(d) *$W_>(x, y)$ maps H_1 to itself, and*

$$U(x, y) := \tilde{S}_>(x)W_>(x, y)\tilde{S}_>(y)^{-1}$$

is bounded and strongly continuous in Δ_δ.

(e) *On H_1, $W_>$ is strongly continuously differentiable in Δ_δ, and*

$$\partial_x W_>(x, y) = \tilde{S}_>(x)W_>(x, y), \qquad \partial_y W_>(x, y) = -W_>(x, y)\tilde{S}_>(y).$$

Proof. The proof follows from [Kr, p. 195] if we show that the Cauchy problem for (3.1) is uniformly correct. But we have the estimate for $\lambda \ge 0$

$$\|(-x^{-1}\tilde{S}_>(x) - \lambda)\|^{-1} \le (x^{-1}C_2 + \lambda)^{-1} \le (1 + \lambda)^{-1};$$

so the assertion follows from [Kr, Theorem 3.11]. □

Lemma 3.1 does not give a uniform norm estimate for $W_>$ in I. To derive it we construct an approximation to $W_>$ in the following way. Introduce the contour

$$c(t) := d + |t| + it, \qquad t \in \mathbb{R}, \qquad 1/2 < d < C_2$$

and define for $0 < y < x$

$$\tilde{W}_>(x, y) := (2\pi i)^{-1} \int_c e^{\zeta \log(y/x)}(\tilde{S}_>(x) - \zeta)^{-1}\, d\zeta. \tag{3.6}$$

With $f \in C_0^\infty(I, H_1)$ we define for $\delta \in I$

$$\tilde{u}(x) := \int_\delta^x \tilde{W}_>(x, y) f(y)\, dy, \tag{3.7}$$

and we want to use Duhamel's principle to compare $W_>$ and $\tilde{W}_>$. Applying T to (3.7) formally, we obtain

$$(\partial_x + x^{-1}\tilde{S}_>(x))\tilde{u}(x) = f(x) + \int_\delta^x \tilde{R}(x, y) f(y)\, dy \tag{3.8}$$

where

$$\tilde{R}(x, y) := -(2\pi i)^{-1} \int_c e^{\zeta \log(y/x)} (\tilde{S}_>(x) - \zeta)^{-1} \tilde{S}'_>(x)(\tilde{S}_>(x) - \zeta)^{-1}\, d\zeta. \tag{3.9}$$

If we know that u is differentiable in $[\delta, s_0)$ with values in H_1 and that the right-hand side is continuous in $[\delta, s_0)$, then it follows from [Kr, Theorem 3.1] that

$$\begin{aligned}\int_\delta^x \tilde{W}_>(x, y) f(y)\, dy = &\int_\delta^x W_>(x, y) f(y)\, dy \\ &+ \int_\delta^x W_>(x, y) \int_\delta^y \tilde{R}(y, z) f(z)\, dz\, dy.\end{aligned} \tag{3.10}$$

Arguing again formally, we define for $0 < y < x \le s_0$

$$\tilde{R}^0(x, y) := \mathbf{I}, \qquad \tilde{R}^1(x, y) := \tilde{R}(x, y),$$

$$\begin{aligned}\tilde{R}^{j+1}(x, y) &:= \int_y^x \tilde{R}(x, z)\tilde{R}^j(z, y)\, dz \\ &=: \tilde{R} * \tilde{R}^j(x, y),\end{aligned}$$

and we expect a representation of $W_>$ by the Neumann series

$$W_>(x, y) = \sum_{j \ge 0} (-1)^j \tilde{W}_> * \tilde{R}^j(x, y). \tag{3.11}$$

If this has been established, estimates for $\tilde{W}_>$ and $\tilde{R}$ will imply estimates for $W_>$.

LEMMA 3.2. *If C_4 in (2.25) is sufficiently small, then we have in $0 < y < x \le s_0$ the estimate*

$$\| W_>(x, y) \|_H \le C(y/x)^{1/2} \left(\frac{1 + \log x/y}{\log x/y} \right)^{1/3}.$$

Proof. We first want to justify (3.10). To do so we observe that $\tilde{W}_>(x, y)$ has a continuous derivative with respect to $x \in (y, s_0)$, given by

$$\begin{aligned}\partial_x \tilde{W}_>(x, y) &= -(2\pi i)^{-1} \int_c e^{\zeta \log(y/x)}(\tilde{S}_>(x) - \zeta)^{-1}[\zeta/x + \tilde{S}'_>(x)(\tilde{S}_>(x) - \zeta)^{-1}]\, d\zeta \\ &= -x^{-1}\tilde{S}_>(x)\tilde{W}_>(x, y) + \tilde{R}(x, y). \qquad (3.12)\end{aligned}$$

We also have the norm estimates

$$\begin{aligned}\|\tilde{W}_>(x, y)\| &\leq C_\varepsilon (y/x)^d (\log x/y)^{-\varepsilon} \int_c |\zeta|^{-1} |\zeta - d|^{-\varepsilon} d|\zeta| \\ &\leq C_\varepsilon (y/x)^d (\log x/y)^{-\varepsilon}, \qquad (3.13)\end{aligned}$$

$$\|\tilde{R}(x, y)\| \leq C_\varepsilon (y/x)^d (\log x/y)^{-\varepsilon} \tilde{\alpha}(x), \qquad (3.14)$$

uniformly in $\delta \leq y < x \leq s_0$, and for $0 < \varepsilon < 1/2$. Then we conclude from standard estimates that, in $[\delta, s_0)$, $\tilde{u}$ in (3.7) is differentiable with derivative

$$\begin{aligned}\tilde{u}'(x) &= f(x) - x^{-1}\tilde{S}_>(x)\tilde{u}(x) + \int_\delta^x \tilde{R}(x, y) f(y)\, dy, \\ &=: -x^{-1}\tilde{S}_>(x)\tilde{u}(x) + g(x). \qquad (3.15)\end{aligned}$$

Moreover, since $f \in C_0^\infty(I, H_1)$, it is readily seen that g is continuous in $[\delta, s_0]$ and that $\tilde{u}(x) \in H_1$ for all x. Hence, it follows from [Kr, p. 195] that

$$\tilde{u}(x) = \int_\delta^x W_>(x, y) g(y)\, dy$$

or

$$\begin{aligned}\int_\delta^x W_>(x, y) f(y)\, dy &= \int_\delta^x \tilde{W}_>(x, y) f(y)\, dy - \int_\delta^x W_>(x, y) \int_\delta^y \tilde{R}(y, z) f(z)\, dz\, dy \\ &= \int_\delta^x \tilde{W}_>(x, y) f(y)\, dy - \int_\delta^x W_> * \tilde{R}(x, y) f(y)\, dy,\end{aligned}$$

which is (3.10). Since $C_0^\infty(I, H_1)$ is dense in $L^2(I, H)$, we conclude that

$$W_>(x, y) = \tilde{W}_>(x, y) - W_> * \tilde{R}(x, y), \qquad \delta \leq y < x \leq s_0.$$

From this it follows by induction that for all $N \in \mathbb{N}$

$$W_>(x, y) = \sum_{j=0}^{N} (-1)^j \tilde{W}_> * \tilde{R}^j(x, y) + (-1)^{N+1} W_> * \tilde{R}^{N+1}(x, y). \tag{3.16a}$$

Next, we fix $A_1 > 1$, and we claim that, for $j \geq 1$ and $1 \leq x/y \leq A_1$,

$$\|\tilde{R}^j(x, y)\|_H \leq C_{1/3}^j C_6^{j-1} (y/x)^d (\log x/y)^{-1/3} \tilde{\alpha}(x) \tag{3.16b}$$

with $C_{1/3}$ from (3.14) and (see (2.25))

$$C_6 := C_4 B(1/3, 1/3)^{1/2} \sup_{1 \leq z \leq A_1} (\log z)^{1/6}$$

where B denotes Euler's beta function.

In fact, if $j = 1$, this follows from (3.14) with $\varepsilon = 1/3$ since $0 < y < x$. Assume next that (3.16b) is proved for some $j \geq 1$. Then from (3.14), (3.16b), and (2.25), we get

$$\begin{aligned}
\|\tilde{R}^{j+1}(x, y)\|_H &= \left\| \int_y^x \tilde{R}(x, z) \tilde{R}^j(z, y)\, dz \right\|_H \\
&\leq C_{1/3}^{j+1} C_6^{j-1} (y/x)^d \tilde{\alpha}(x) \int_y^x (\log x/z)^{-1/3} (\log z/y)^{-1/3} \tilde{\alpha}(z)\, dz \\
&\leq C_{1/3}^{j+1} C_6^{j-1} (y/x)^d \tilde{\alpha}(x) C_4 \left(\int_y^x (\log x/z)^{-2/3} (\log z/y)^{-2/3} \frac{dz}{z} \right)^{1/2} \\
&= C_{1/3}^{j+1} C_4 C_6^{j-1} (y/x)^d \tilde{\alpha}(x) \left(\int_0^{\log x/y} (\log x/y - t)^{-2/3} t^{-2/3}\, dt \right)^{1/2} \\
&= C_{1/3}^{j+1} C_4 C_6^{j-1} (y/x)^d \tilde{\alpha}(x) (\log x/y)^{1/6 - 1/3} B(1/3, 1/3)^{1/2} \\
&\leq C_{1/3}^{j+1} C_6^j (y/x)^d (\log x/y)^{-1/3} \tilde{\alpha}(x).
\end{aligned}$$

If now $C_4 = C_4(A_1)$ is so small that $C_{1/3} C_6 < 1$, then the series (3.16a) converges uniformly in $\delta \leq y < x \leq s_0$, in view of Lemma 3.1, (b), and the above estimates. Hence, we derive the bound

$$\|W_>(x, y)\|_H \leq A_2 (y/x)^d \left(\frac{1 + \log x/y}{\log x/y} \right)^{1/3}, \qquad 1 \leq x/y \leq A_1, \tag{3.17}$$

with A_2 independent of δ and A_1.

Next, we let $A_1 \geq 2$, and we choose $C_4 = C_4(A_1)$ so small that (3.17) holds for $1 \leq x/y \leq A_1^2$. For arbitrary $x \geq y > 0$ we now put $z_j := A_1^j y$, and we determine

$N \in \mathbb{Z}_+$ by the requirement $z_N \leq x < z_{N+1}$, i.e.,

$$N = \left[\frac{\log x/y}{\log A_1}\right];$$

we assume $N \geq 2$. Then we estimate, using Lemma 3.1, (c), and (3.17),

$$\begin{aligned}
\| W_>(x, y)\| &\leq \| W_>(x, z_{N-1})\| \prod_{j=0}^{N-1} \| W_>(z_{j+1}, z_j)\| \\
&\leq A_2^N (y/x)^d \left(\frac{1 + \log x/z_{N-1}}{\log x/z_{N-1}}\right)^{1/3} \left(\frac{1 + \log A_1}{\log A_1}\right)^{(N-1)/3} \\
&\leq (A_2 A_3)^N (y/x)^d
\end{aligned}$$

where we have set

$$A_3 := \sup_{z \geq 2} \left(\frac{1 + \log z}{\log z}\right)^{1/3}.$$

Now we have, assuming $A_2 A_3 \geq 1$ as we may,

$$(A_2 A_3)^N \leq e^{(\log A_2 A_3/\log A_1)\log x/y} =: (x/y)^{\tilde{d}} = (y/x)^{-\tilde{d}}.$$

Since $\lim_{A_1 \to \infty} \tilde{d} = 0$, we can now choose $A_1 \geq 2$ definitely in such a way that $d - \tilde{d} \geq 1/2$, and we choose C_4 such that (3.17) holds with this choice of A_1. It follows that $W_>(x, y)$ satisfies the asserted estimate for all $x \geq y > 0$. □

Remark. It is clear from the proof that the exponents in the estimate are not the best possible.

In view of the last result, we will from now on *assume* that C_4 is small enough.

We now define the boundary parametrix for $\tilde{S}_>$ by

$$P_> f(x) := \int_0^x W_>(x, y) f(y)\, dy, \qquad x \in I, \qquad f \in L^2(I, H). \tag{3.18}$$

This is well defined in view of Lemma 3.2.

To obtain the parametrix for $\tilde{S}_<$, we introduce the solution operator $W^>$ associated to the equation

$$(\partial_x - x^{-1} \tilde{S}_<(x)) u(x) = 0,$$

and we put

$$W_<(x, y) := W^>(y, x)^*, \qquad 0 < x < y \leq s_0. \tag{3.19}$$

Then we define

$$P_< f(x) := \int_{s_0}^{x} W_<(x, y) f(y)\, dy. \tag{3.20}$$

The full boundary parametrix is then given by

$$Pf(x) := (P_< + P_0 + P_>) f(x). \tag{3.21}$$

We now collect the main properties of P.

THEOREM 3.1. (1) *For $f \in C_0^\infty(I, H_1)$ we have $Pf \in C^1(I, H) \cap C(I, H_1)$ and*

$$(\partial_x + x^{-1}\tilde{S}(x))Pf(x) = f(x), \qquad x \in I.$$

(2) *For $f \in L^2(I, H)$ we have $Pf \in C(I, H)$ and*

$$\|Pf(x)\|_H \le x^{1/2} \|f\|_{L^2(I,H)}, \qquad x \in I.$$

(3) *With $\varphi \in C_0^\infty(I)$ we have*

$$\|\varphi Pf\|^2_{L^2(I,H_1)} + \|\varphi Pf\|^2_{H^1(I,H)} \le C_\varphi \|f\|^2_{L^2(I,H)}.$$

(4) *With $\varphi \in C_0^\infty(-s_0, s_0)$, φP is compact in $L^2(I, H)$.*
(5) *For $f \in C_0^\infty(I, H_1)$ we have $P^*f \in C^1(I, H) \cap C(I, H_1)$ and*

$$-(\partial_x - x^{-1}\tilde{S}(x))P^*f(x) = f(x), \qquad x \in I.$$

Proof. (1) This is Theorem 3.2 in [Kr, p. 196] (note that the proof there needs a slight correction since $U(t, \tau)$ is defined only for $t \ge \tau$) for $P_> f$ and $P_0 f$. The proof for $P_< f$ is analogous.

(2) It follows from Lemma 3.2 that for $0 < y < x < s_0$

$$\|W_<(y, x)\|_H + \|W_>(x, y)\|_H \le C(y/x)^{1/2}(\log x/y)^{-1/3}$$

which implies the desired estimate, by Cauchy-Schwarz, for $P_> f$ and $P_< f$.

For $W_0(x, y)$ we obtain as in the proof of Lemma 2.5

$$\|W_0(x, y)\|_H \le C(x/y)^{C_1} \tag{3.22}$$

which implies the estimate for $P_0 f$.

(3) Again, we prove the estimate separately for $P_>$, $P_<$, and P_0. Let $u := \varphi P_> f$ with $f \in C_0^\infty(I, H_1)$ and $\varphi \in C_0^\infty(I)$. Then

$$(\partial_x + x^{-1}\tilde{S}_>(x))u(x) = \varphi' P_> f(x) + \varphi f(x) =: Tu(x),$$

and with $C_\varphi > 0$

$$(Tu, u) = \int_0^{s_0} x^{-1}\langle \tilde{S}_> u, u\rangle(x)\, dx \geq C_\varphi \|\tilde{S}_>^{1/2} u\|_{L^2}^2 .$$

Furthermore,

$$\|Tu\|_{L^2}^2 = \|u'\|_{L^2}^2 + \|X^{-1}\tilde{S}_> u\|_{L^2}^2 + 2\,\mathrm{Re}(u', X^{-1}\tilde{S}_> u)$$

and

$$|2\,\mathrm{Re}(u', X^{-1}\tilde{S}_> u)| = 2\left|\int_0^{s_0} (x^{-2}\langle \tilde{S}_> u, u\rangle(x) - x^{-1}\langle \tilde{S}'_> u, u\rangle(x))\, dx\right|$$
$$\leq C_\varphi \|\tilde{S}_>^{1/2} u\|_{L^2}^2 .$$

Since $P_>$ is continuous in $L^2(I, H)$, the assertion follows in this case.

The proof for $P_<$ is similar, and the proof for P_0 is just the regularity of ordinary differential equations.

(4) Let B be a bounded set in $L^2(I, H)$; we have to show that PB has a finite ε-net for any $\varepsilon > 0$. Fix $\varepsilon > 0$ and choose $\psi \in C_0^\infty(I)$ such that

$$\|(1 - \psi)\varphi P f\| < \varepsilon \qquad \text{if } f \in B;$$

this is possible by (2). Therefore, we may assume that $\varphi \in C_0^\infty(I)$. Fix $x_0 \in I$ and let $(e_s, s)_{s \in \mathbb{R}}$ be the spectral resolution of $\tilde{S}(x_0)$. Using (3), we have for $f \in B$

$$C \geq \|\varphi P f\|_{L^2(I, H_1)}^2 \geq C\|(|\tilde{S}(x_0)| + 1)\varphi P f\|_{L^2(I,H)}^2$$
$$\geq C \sum_{|s| > N} (s^2 + 1)\|\langle \varphi P f, e_s\rangle\|_{L^2(I)}^2 + \|P_N \varphi P f\|_{L^2(I,H)}^2$$

where P_N denotes the orthogonal projection in H onto the space spanned by $\{e_s \mid |s| \leq N\}$. It follows that

$$\|(I - P_N)_\varphi P f\|_{L^2(I,H)}^2 \leq C(N^2 + 1)^{-1},$$

and it is enough to prove the assertion for the set $P_N \varphi P B$. But (3) implies that $\varphi P B \subset H^1(I, P_N H) \subset C_0(I, P_N H)$ and that $\varphi P B$ is equicontinuous. In fact,

$$\|\varphi P f(x) - \varphi P f(y)\|_H \leq \int_y^x \left\| \frac{d}{dz}\varphi P f(z)\right\|_H dz$$
$$\leq |x - y|^{1/2} \|\varphi P f\|_{H^1(I,H)} \leq C|x - y|^{1/2}.$$

Thus, the proof is completed with the Arzela-Ascoli theorem, applied to the set $P_N \varphi P B$.

(5) A straightforward computation shows that

$$-P_{>}^{*} f(x) = \int_{s_0}^{x} W_{>}(y, x)^{*} f(y)\, dy,$$

$$-P_0^{*} f(x) = \int_{s_0}^{x} W_0(y, x)^{*} f(y)\, dy,$$

$$-P_{<}^{*} f(x) = \int_{0}^{x} W^{>}(x, y) f(y)\, dy.$$

Thus, the proof is the same as in (1) except for $-P_0^*$. But since this concerns a matrix equation, the result is also clear. Note, however, that $P_0^* f$ will, in general, not satisfy the estimate (2). Instead, using (3.22), we see that for $x \in I, f \in L^2(I, H)$ the estimate

$$\|P^* f(x)\|_H \leq C x^{-C_1} \|f\|_{L^2(I,H)} \tag{3.23}$$

holds. □

We show next that P is, near the boundary, a left inverse for $\tilde{D}_{\max}$ on functions vanishing sufficiently fast.

LEMMA 3.3. *Let $u \in \mathscr{D}(\tilde{D}_{\max})$ satisfy*

$$\|u_b(x)\|_H \leq C_u x^{1/2}, \qquad x \in I.$$

Then for $\varphi \in C_0^\infty(-s_0, s_0)$ we have

$$\varphi u = P \tilde{D}_{\max} \varphi u. \tag{3.24}$$

Proof. Let $v \in C_0^\infty(I, H_1)$ and $\tilde{\psi} \in C_0^\infty(\mathbb{R})$ with $\tilde{\psi}(x) = 1$ if $|x| \leq 1$ and $\tilde{\psi}(x) = 0$ if $|x| \geq 2$. Put $\psi_n(x) := 1 - \psi(nx)$ and compute

$$\begin{aligned}(P\tilde{D}_{\max}\varphi u, v) &= \lim_{n\to\infty} (\tilde{D}_{\max}\varphi u, \psi_n P^* v)\\ &= \lim_{n\to\infty} (\varphi u, \tilde{D}'_{\min}\psi_n P^* v)\\ &= \lim_{n\to\infty} [(\varphi u, \psi_n v) - (\varphi u, \psi_n' P^* v)];\end{aligned}$$

here, we have used Theorem 3.1, (5). To estimate the second term we use the assumption on u and (3.23):

$$|(\varphi u, \psi_n' P^* v)| \leq Cn \int_0^{2/n} \|\varphi u(x)\|_H \|P^* v(x)\|_H \, dx$$

$$\leq Cn \int_0^{2/n} x^{1/2 - C_1} \, dx \leq C n^{C_1 - 1/2}.$$

Since $C_1 < 1/2$, the assertion follows. □

Lemma 3.3 leads us to define

$$\mathscr{D}_1 := \{u \in \mathscr{D}(\tilde{D}_{\max}) \mid \|u_b(x)\|_H = O(x^{1/2}) \text{ as } x \to 0\}, \tag{3.25a}$$

$$\tilde{D}_1 := \tilde{D}_{\max} | \mathscr{D}_1 . \tag{3.25b}$$

LEMMA 3.4.

$$\tilde{D}_1 = \tilde{D}_{\min} .$$

Moreover, for $\varphi \in C_0^\infty(-s_0, s_0)$ *and* $u \in \mathscr{D}(\tilde{D}_{\max})$, *we have*

$$(\mathbf{I} - \tilde{Q})\varphi u \in \mathscr{D}_1 . \tag{3.26}$$

Proof. The argument in [BS1, Theorem 6.1] extends to this case to show that

$$\mathscr{D}_1 \subset \mathscr{D}(\tilde{D}_{\min}).$$

But Lemma 3.3 implies that $\tilde{D}_1$ is closed; so $\tilde{D}_1 = \tilde{D}_{\min}$.

For the proof of (3.26) we observe that $(\mathbf{I} - \tilde{Q})\varphi u \in \mathscr{D}(\tilde{D}_{\max})$ since $\tilde{Q}$ commutes with $\tilde{D}_{\max}$. Thus, we find, as in the proof of Lemma 3.3,

$$\varphi(\mathbf{I} - \tilde{Q})u = P\tilde{D}_{\max}\varphi(\mathbf{I} - \tilde{Q})u$$

and hence the assertion by Theorem 3.1, (2), if we can prove that (in the notation above)

$$\lim_{n \to \infty} (\varphi u, \psi_n'(P_>^* + P_<^*)v) = 0.$$

Using (the analogue of) Theorem 3.1, (2), we estimate

$$|(\varphi u, \psi_n'(P_>^* + P_<^*)v)|^2 \leq C_v \int_0^{2/n} \|u(x)\|_H^2 \, dx \, n^2 \int_0^{2/n} x \, dx$$

$$\leq C_v \int_0^{2/n} \|u(x)\|_H^2 \, dx$$

which completes the proof. □

So far, we have worked with the condition (3.2) which says that the projections $\tilde{Q}_>$, $\tilde{Q}_<$, and $\tilde{Q}$ are all independent of x. In general, however, we only have the spectral concentration property expressed in (2.9) and (2.10). Fortunately, this is still sufficient to prove that all closed extensions of $\tilde{D}$ are Fredholm operators. To see this, we view $\tilde{D}$ as perturbation of an operator $\tilde{D}^0$ satisfying (3.2).

LEMMA 3.5. *Assume that $\tilde{D}$ satisfies the assumptions* (2.22a, b, d) *and* (2.23a, b). *Then there is a smooth family $U(x)$ of unitary operators in H such that*

$$\tilde{S}^0(x) := U(x)^*\tilde{S}(x)U(x) \quad \textit{satisfies } (3.2) \tag{3.27}$$

and

$$\|U'(x)\|_H \le C\tilde{\alpha}(x). \tag{3.28}$$

Proof. From (2.25c) we have

$$\|\tilde{Q}'(x)\|_H \le C_5\tilde{\alpha}(x).$$

Next, we observe the relations

$$2\tilde{Q}_>(x) = (\mathbf{I} - \tilde{Q}(x))(\tilde{S}(x)|\tilde{S}(x)|^{-1} + \mathbf{I}),$$

$$(I - \tilde{Q}(x))|\tilde{S}(x)|^{-1} = (2\pi i)^{-1}\left[\int_c - \int_{-c}\right]\zeta^{-1}(\tilde{S}(x) - \zeta)^{-1}\,d\zeta,$$

which follow from the spectral theorem. They imply that $\tilde{Q}_>(x)$ is differentiable with an estimate

$$\|\tilde{Q}'_>(x)\|_H \le C\tilde{\alpha}(x);$$

so the analogous result also holds for $\tilde{Q}_<(x)$.

Now we construct unitary transformation functions $U_0(x)$, $U_>(x)$, $U_<(x)$ as in [Ka, pp. 100], satisfying

$$U_0(x)^*\tilde{Q}(x)U_0(x) = \tilde{Q}(s_0),$$

$$U_{\gtrless}(x)^*\tilde{Q}_{\gtrless}(x)U_{\gtrless}(x) = \tilde{Q}_{\gtrless}(s_0),$$

$$\|U_0'(x)\|_H + \|U_>'(x)\|_H + \|U_<'(x)\|_H \le C\tilde{\alpha}(x).$$

Then

$$U(x) := U_0(x) + U_>(x) + U_<(x)$$

is unitary and has the desired properties. □

Now we use the unitary transformation

$$\Psi\colon L^2(I, H) \ni f \mapsto Uf \in L^2(I, H).$$

Then Lemma 3.5 shows that we may assume that

$$\tilde{D}u_b(x) = [\partial_x + x^{-1}\tilde{S}^0(x) + B(x)]u_b(x) \tag{3.29a}$$

where $\tilde{S}^0(x)$ satisfies (3.2) and $B(x) = U'(x)U(x)^*$ is bounded in H with

$$\|B(x)\|_H \leq C\tilde{\alpha}(x). \tag{3.29b}$$

We can now state and prove the main result of this section.

THEOREM 3.2. *Assume that $\tilde{D}$ satisfies the assumptions* (2.22a, b, c, d) *and* (2.23a, b) *and that the constant C_4 in* (2.25b) *is sufficiently small. Moreover, assume that for every $\varphi \in C_0^\infty(-s_0, s_0)$ with $\varphi = 1$ near 0 there is a compact operator $P_\varphi \in \mathscr{L}(\tilde{\mathscr{H}}_2, \mathscr{D}(\tilde{D}_{\min}))$ and compact operators $K_\varphi^i \in \mathscr{L}(\tilde{\mathscr{H}}_i)$ such that*

$$\tilde{D}_{\min}P_\varphi = 1 - \varphi + K_\varphi^2, \qquad P_\varphi\tilde{D}_{\min} = 1 - \varphi + K_\varphi^1. \tag{3.30}$$

Assume also that for each $\psi \in C_0^\infty(-s_0, s_0)$ with $\psi = 1$ near 0 *and $\varphi = 1$ near* supp ψ *we can construct $K_\varphi^i = K_{\varphi,\psi}^i$ such that $\psi K_\varphi^i = 0$, $\psi K_\varphi^{i*} = 0$.*

(1) *With $\tilde{D}_1$ as in* (3.25), *we have $\tilde{D}_1 = \tilde{D}_{\min}$.*
(2) *All closed extensions of $\tilde{D}$ are Fredholm operators, corresponding bijectively to the subspaces of the finite-dimensional space*

$$V_0 := \mathscr{D}(\tilde{D}_{\max})/\mathscr{D}(\tilde{D}_{\min}).$$

Denoting by $\tilde{D}_V$ the operator corresponding to $V \subset V_0$, we have

$$\operatorname{ind} \tilde{D}_V = \operatorname{ind} D_{\min} + \dim V.$$

Proof. (1) We may assume that $\tilde{D}$ fulfills (3.29). We conclude as above, using the arguments of [BS1, Theorem 6.1], that $\mathscr{D}_1 \subset \mathscr{D}(\tilde{D}_{\min})$. Thus, we will have the desired equality if we show that $\tilde{D}_1$ is a closed operator.

For $u \in \mathscr{D}_1$ we have by (3.29b), (3.25a), and (2.25b),

$$\int_0^{s_0} \|B(x)u(x)\|_H^2\, dx \leq C_u \int_0^{s_0} x\tilde{\alpha}(x)^2\, dx \leq C_4^2 C_u. \tag{3.31}$$

Thus, we find with $\varphi \in C_0^\infty(-s_0, s_0)$

$$[\partial_x + x^{-1}\tilde{S}^0(x)]\varphi u(x) = \tilde{D}\varphi u(x) - B\varphi u(x) =: v(x)$$

with $v \in L^2(I, H)$. If we denote by P^0 the boundary parametrix constructed with $\tilde{S}^0$, we conclude from Lemma 3.3 that

$$\varphi u = P^0 \tilde{D} \varphi u - P^0 B \varphi u,$$

and by iteration, setting $f := \tilde{D}\varphi u$,

$$\varphi u = \sum_{j=0}^{N} (-1)^j P^0 (BP^0)^j f + (-1)^{N+1} P^0 (BP^0)^N B \varphi u.$$

Using Theorem 3.1, (2), we find as in the proof of (3.31) that

$$\|BP^0\| < 1 \tag{3.32}$$

if C_4 is sufficiently small. Thus, we obtain

$$\varphi u = \sum_{j=0}^{\infty} (-1)^j P^0 (BP^0)^j f =: P^0 V \tilde{D} \varphi u \tag{3.33}$$

for some bounded operator V in $L^2(I, H)$. From this we see immediately that $\tilde{D}_1$ is closed.

(2) If we prove that $\tilde{D}_{\min}$ and $\tilde{D}'_{\min} = \tilde{D}^*_{\max}$ are both Fredholm operators, then all closed extensions of $\tilde{D}$ are Fredholm, and the assertions concerning V_0 and ind $\tilde{D}_V$ are easy consequences. But our assumptions on $\tilde{D}$ imply the same assumptions for $\tilde{D}'$. Therefore, it is enough to prove that $\tilde{D}_{\min}$ is Fredholm. This will be achieved if we construct an operator $P_{\min} \in \mathscr{L}(\tilde{\mathscr{H}}_2, \mathscr{D}(\tilde{D}_{\min}))$ with the property

$$\tilde{D}_{\min} P_{\min} = \mathbf{I} + K \tag{3.34}$$

for some compact operator $K \in \mathscr{L}(\tilde{\mathscr{H}}_2)$. To construct $P_{\min}$ we only need a boundary parametrix, in view of (3.30) (which expresses the ellipticity of D away from the singularities).

Choose $\varphi, \psi \in C_0^\infty(-s_0, s_0)$ with $0 \le \varphi, \psi \le 1$, $\varphi = 1$ near 0, and $\psi = 1$ near supp φ. We define

$$\tilde{P}_{\min} := \psi P^0 \varphi + P_\varphi,$$

and we compute with Theorem 3.1, (1), (3.29), and (3.30)

$$\begin{aligned}\tilde{D}_{\max} \tilde{P}_{\min} &= \mathbf{I} + \psi' P^0 \varphi + \psi B P^0 \varphi + K_\varphi^2 \\ &=: \mathbf{I} + \tilde{K} + \psi B P^0 \varphi \\ &=: \mathbf{I} + \tilde{K} + R.\end{aligned}$$

Here, K is compact by Theorem 3.1, (4), and $\|R\| < 1$ by (3.32). Moreover, $\tilde{P}_{\min}$ maps into $\mathscr{D}(\tilde{D}_{\min})$ by the first part of the proof. Thus, we obtain (3.34) with

$$P_{\min} := \tilde{P}_{\min}(\mathbf{I} + R)^{-1}, \qquad K := \tilde{K}(\mathbf{I} + R)^{-1}. \qquad \square$$

Remarks. (1) Later on, we will apply this result to the weighted operator gDg, introduced in the previous section. To verify the assumptions we note that (2.22a, b, d) and (2.23a, b) have already been established. Moreover, since gDg is also elliptic, we have (3.30). Finally, the smallness of C_4 follows from the smallness of C_3 in (2.13b). This in turn can be achieved by choosing U suitably; e.g., if U is a productlike end of the manifold, we move towards infinity on U making the integral (2.13b) as small as we please.

(2) It is obvious from the proof of Theorem 3.2 that the results also hold for suitable perturbations of $\tilde{D}$.

(3) We will show below (Corollary 4.3) that

$$\dim V_0 = \dim \tilde{Q}(x).$$

In particular, $\tilde{D}$ has a unique closed extension if $\tilde{Q}(x) = 0$ (for some and hence all $x \in I$).

4. The index formula. We consider again the operator $\tilde{D}$ of the previous section, and we keep the assumptions of Theorem 3.2. In addition we require that

$$\tilde{S}(x) \text{ is constant near } x = s_0, \text{ i.e. } \tilde{S}(x) = \tilde{S}(s_0) \text{ for } x \in [\delta, s_0],\ 0 < \delta < s_0. \tag{4.1}$$

Note that this implies $\tilde{S}^0(x) = \tilde{S}^0(s_0)$, $B(x) = 0$ near s_0 in the representation (3.29). In what follows we write $\tilde{S} := \tilde{S}^0$ to simplify notation. In applications, (4.1) is not a severe restriction; see [B, Sec. 5] and Section 5 below.

The index of $\tilde{D}_{\min}$ will be computed by deforming $\tilde{D}$ to an operator with calculable index. Choose $\varphi \in C_0^\infty(-s_0, s_0)$ with $\varphi|[-\delta, \delta] = 1$. By (4.1), $B = B\varphi = \varphi B$, and it follows from (3.29b), (3.33), and (3.32) that φB is continuous $\mathscr{D}(\tilde{D}_{\min}) \to \tilde{\mathscr{H}}_2$. Thus, the operator family

$$T_\alpha^1 := \tilde{D}_{\min} - \alpha\varphi B \colon \mathscr{D}(\tilde{D}_{\min}) \to \tilde{\mathscr{H}}_2, \qquad \alpha \in [0, 1]$$

is continuous in α, with $T_0^1 = \tilde{D}_{\min}$. The proof of Theorem 3.2 shows (with the same choice of φ) that each T_α^1 is a Fredholm operator. Hence, we have

$$\operatorname{ind} \tilde{D}_{\min} = \operatorname{ind} T_1^1\,; \tag{4.2}$$

i.e., we may assume that $B(x) \equiv 0$.

Next, we define a family of operators, T_β^2, $\beta \in [0, 1]$, as follows. Let (with $\mathscr{D}_0$

defined in (2.2))

$$\tilde{S}_\beta(x) := \tilde{S}(\beta x + s_0(1-\beta)), \qquad x \in I, \tag{4.3a}$$

$$\begin{aligned} &T_\beta^2 := \text{closure in } \tilde{\mathscr{H}}_1 \text{ of the operator } \mathscr{D}_0 \to \tilde{\mathscr{H}}_2 \text{ given by} \\ &(u_i, u_b) \mapsto (f(0)Du_i, (\partial_x + x^{-1}\tilde{S}_\beta(x))u_b(x)). \end{aligned} \tag{4.3b}$$

We write $\mathscr{D}_\beta := \mathscr{D}(T_\beta^2)$, $\mathscr{D}_\beta^* := \mathscr{D}(T_\beta^{2*})$. Note that $T_1^2 = T_1^1$ and

$$T_\beta^2 u = \tilde{D} \text{ if } u_b \text{ vanishes in a neighborhood of } (0, \delta]. \tag{4.4}$$

The main difficulty in dealing with this deformation is that now $\mathscr{D}_\beta$ will vary. This can be dealt with using a result of Cordes and Labrousse [CL]; our approach follows a suggestion of R. Seeley; see [SSi].

THEOREM 4.1. *The operators T_β^2 defined above are Fredholm operators in $\tilde{\mathscr{H}}_1$, with index independent of $\beta \in [0, 1]$.*

Proof. We start with the proof of the Fredholm property which we want to deduce from Theorem 3.2. So we check first that the assumptions there are satisfied, uniformly in $\beta \in [0, 1]$. This is clear for (2.22a, b, d) and also for (2.23a, b). Consider next

$$\begin{aligned} \tilde{\alpha}_\beta(x) &:= \|\tilde{S}_\beta'(x)(|\tilde{S}_\beta(x)| + 1)^{-1}\|_H \\ &= \beta\tilde{\alpha}(\beta x + s_0(1-\beta)) \end{aligned}$$

and estimate

$$\begin{aligned} \int_0^{s_0} x\tilde{\alpha}_\beta(x)^2\,dx &= \int_0^{s_0} (\beta x)\tilde{\alpha}(\beta x + s_0(1-\beta))d\beta x \\ &= \int_{s_0(1-\beta)}^{s_0} (x - s_0(1-\beta))\tilde{\alpha}(x)^2\,dx \le C_4^2. \end{aligned}$$

If we use, in the proof of Theorem 3.2, $\varphi \in C_0^\infty(-s_0, s_0)$ with $\varphi = 1$ in a neighborhood of $[-\delta, \delta]$, then we can use the same P_φ for all $\beta \in [0, 1]$, in view of (4.4). The Fredholm property follows.

In [CL, Section 3] a metric was introduced on the space of closed operators with the property that a curve of (semi-) Fredholm operators, continuous in this metric, has constant index. It is a straightforward consequence of this definition that the curve T_β^2, $\beta \in [0, 1]$, has a constant index if the operator family formed by the orthogonal projections onto the graph of T_β^2 is continuous in $\mathscr{L}(\tilde{\mathscr{H}}_1 \oplus \tilde{\mathscr{H}}_2)$. This, in turn, is equivalent to the fact that

$$E_\beta := \begin{pmatrix} \mathbf{I} & -T_\beta^{2*} \\ T_\beta^2 & \mathbf{I} \end{pmatrix} \tag{4.5}$$

has an inverse which varies continuously with $\beta \in [0, 1]$, as a bounded operator in $\tilde{\mathscr{H}}_1 \oplus \tilde{\mathscr{H}}_2$. Note that E_β is closed in $\tilde{\mathscr{H}}_1 \oplus \tilde{\mathscr{H}}_2$ with domain $\mathscr{D}_\beta \oplus \mathscr{D}_\beta^*$ and is bijective for each $\beta \in [0, 1]$.

Now denote by $P_{\beta,\min}$ the parametrix just constructed for T_β^2, with fixed φ as above and $\psi K_\varphi^i = \psi K_\varphi^{i*} = 0$ for some $\psi \in C_0^\infty(-s_0, s_0)$ with $\psi = 1$ near $[-\delta, \delta]$. This means that

$$P_{\beta,\min} = \psi P_\beta \varphi + P_\varphi$$

where P_β is defined by (3.21) for $\tilde{S}_\beta$. We set

$$F_\beta := \begin{pmatrix} 0 & P_{\beta,\min} \\ -P_{\beta,\min}^* & 0 \end{pmatrix}. \tag{4.6}$$

Then F_β is compact in $\tilde{\mathscr{H}}_1 \oplus \tilde{\mathscr{H}}_2$ and maps into the domain of E_β. In fact, we have by construction $P_{\beta,\min}(\tilde{\mathscr{H}}_2) \subset \mathscr{D}_\beta$, and it follows from Theorem 3.1, (5), and (3.30) that $P_{\beta,\min}^*(\tilde{\mathscr{H}}_1) \subset \mathscr{D}_\beta^*$. Now we compute, as in the proof of Theorem 3.2, (2),

$$\begin{aligned} E_\beta F_\beta &= F_\beta + \begin{pmatrix} T_\beta^{2*} P_{\beta,\min}^* & 0 \\ 0 & T_\beta^2 P_{\beta,\min} \end{pmatrix} \\ &= \mathbf{I} + F_\beta + \begin{pmatrix} \varphi' P_\beta^* \psi + K_\varphi^{1*} & 0 \\ 0 & \psi' P_\beta \varphi + K_\varphi^2 \end{pmatrix} \\ &=: \mathbf{I} + F_\beta + G_\beta . \end{aligned} \tag{4.7}$$

By Theorems 3.1 and 3.2, $F_\beta + G_\beta$ is compact. Moreover, it follows from Theorem 3.1, (3), and the construction of P_φ that

$$G_\beta(\tilde{\mathscr{H}}_1 \oplus \tilde{\mathscr{H}}_2) \subset \mathscr{D}_{\beta'} \oplus \mathscr{D}_{\beta'}^* = \mathscr{D}(E_{\beta'}) \qquad \text{for all } \beta' \in [0, 1]. \tag{4.8}$$

Now, if Z_β denotes the orthogonal projection onto $\ker F_\beta$, then we obtain from (4.7)

$$Z_\beta \subset \ker(\mathbf{I} + F_\beta + G_\beta) \tag{4.9}$$

and

$$Z_\beta = -G_\beta Z_\beta . \tag{4.10}$$

(4.9) implies that Z_β is finite-dimensional, and (4.10) together with (4.8) shows that Z_β maps into $\mathscr{D}(E_{\beta'})$ for all $\beta' \in [0, 1]$.

Now fix $\beta_0 \in [0, 1]$ and consider for β near β_0

$$E_\beta(F_\beta + Z_{\beta_0}) = \mathbf{I} + F_\beta + G_\beta + E_\beta Z_{\beta_0} =: H_\beta. \tag{4.11}$$

By construction, H_β is a Fredholm operator of index 0. Since $F_\beta = -F_\beta^*$, $F_{\beta_0} + Z_{\beta_0}$ is injective; hence H_{β_0} is invertible. But in view of the strong differentiability of $\tilde{S}_\beta$ and Lemma 4.1 below, H_β varies continuously with β; hence, is invertible near β_0 with H_β^{-1} continuous in β. Thus, $(F_\beta + Z_{\beta_0})H_\beta^{-1} = E_\beta^{-1}$ is continuous near $\beta = \beta_0$. The proof is complete. □

LEMMA 4.1. *Under the assumptions of Theorem* 4.1 *the map*

$$[0, 1] \ni \beta \mapsto P_{\beta,\min} \in \mathscr{L}(L^2(I, H))$$

is continuous.

Proof. It is enough to prove the continuity of the map $\beta \mapsto \psi P_\beta \varphi$ for ψ, $\varphi \in C_0^\infty(-s_0, s_0)$. Fix $\beta_0 \in [0, 1]$ and $\varepsilon > 0$. All estimates obtained for P_β in Section 3 are uniform in $\beta \in [0, 1]$ since the constants C_2, C_3, C_4 remain unchanged. Hence, we can use Theorem 3.1, (2), to find $\psi_1 \in C_0^\infty(-s_0, s_0)$ with $\psi_1 = 1$ near 0 such that

$$\|\psi_1 \psi P_\beta \varphi\|_{L^2(I,H)} \le \varepsilon/2, \qquad \beta \in [0, 1].$$

Hence, we obtain the estimate

$$\|\psi(P_\beta - P_{\beta_0})\varphi\|_{L^2(I,H)} \le \varepsilon \text{ for } |\beta - \beta_0| \text{ sufficiently small}$$

if we show that the map $\beta \mapsto (1 - \psi_1)\psi P_\beta \varphi$ is continuous at β_0. This will follow if we prove that the kernels defining P_β are continuous in β, with suitable estimates in $0 < \delta \le y < x \le s_0$. For $\tilde{W}_{0\beta}$ we clearly have uniform continuity. For $W_{>\beta}$ we derive from (3.6) the estimate

$$\|\tilde{W}_{>\beta}(x, y) - \tilde{W}_{>\beta_0}(x, y)\| \le C_\delta (y/x)^d (\log x/y)^{-1/3} |\beta - \beta_0|$$

which implies an analogous estimate for $W_{>\beta}$, by the proof of Lemma 3.2. This implies continuity for $P_{\beta>}$, and the analogous estimate for $\tilde{W}_\beta^>$ implies continuity also for $P_{\beta<}$. □

Using the arguments leading to Theorem 4.2 in [B], we now obtain the following index formula for $\tilde{D}_{\min}$.

THEOREM 4.2. *Assume that $\tilde{D}$ satisfies the assumptions of Theorem* 3.2 *and, in addition,* (4.1). *Then we have with $S_0 := \tilde{S}(s_0)$ and ω_D the index form of the elliptic* *operator representing D on M_1*

$$\operatorname{ind} \tilde{D}_{\min} = \int_{M_1} \omega_D - \frac{1}{2}(\eta(S_0) + \dim \ker S_0) \\ - \sum_{-1/2 < s < 0} \dim \ker(S_0 - s) - \sum_{k \geq 1} \alpha_k \operatorname{Res} \eta_{S_0}(2k). \tag{4.12}$$

Similarly, we derive from [B, Corollary 4.3] the following theorem.

THEOREM 4.3. *Under the assumptions of Theorem* 4.2 *we have*

$$\operatorname{ind} \tilde{D}_{\max} = \operatorname{ind} \tilde{D}_{\min} + \sum_{|s| < 1/2} \dim \ker(S_0 - s) = \operatorname{ind} \tilde{D}_{\min} + \dim \tilde{Q}(0). \tag{4.13}$$

Finally, we want to give the index formula for the original operator D we started with in Section 2.

THEOREM 4.4. *Assume that D satisfies the assumptions* (2.2), (2.3′), (2.7), (2.9) *through* (2.11), (2.13), *and with* $S_0 := \bar{S}(0)$

$$\bar{S}(y) = (1 - y/s(0)f(0))^{-1} S_0 \text{ near } y = 0. \tag{4.14}$$

Then we have

$$L^2\text{-ind } D = \int_{M_1} \omega_D - \frac{1}{2}(\eta(S_0) + \dim \ker S_0) \\ - \sum_{-1/2 < s < 0} \dim \ker(S_0 - s) + h_0 + h_1. \tag{4.15}$$

Here, h_0 and h_1 are given in (2.34). (*See also Theorem* 2.6.)

Proof. From Theorem 2.6 we obtain

$$L^2\text{-ind } D = \operatorname{ind} D_{g,\min} + h_0 + h_1 \tag{4.16}$$

with h_i defined in (2.34). Next, we observe that $\tilde{D}_{\min}$ is unitarily equivalent to $D_{g,\min}$ by Lemma 2.2 and that our assumptions imply the assumptions (2.22) and (2.23). We can also choose the unitary equivalence in such a way that the constant C_4 in (2.25b) is as small as we please. Moreover, (3.30) follows from the ellipticity of D, and (4.14) implies (4.1). Thus, we can apply Theorem 4.2 to arrive at the formula

$$L^2\text{-ind } D = \int_{M_1} \omega_D - \frac{1}{2}(\eta(S_0) + \dim \ker S_0) - \sum_{-1/2 < s < 0} \dim \ker(S_0 - s) \\ - \sum_{k \geq 1} \alpha_k \operatorname{Res} \eta_{S_0}(2k) + h_0 + h_1. \tag{4.17}$$

Now choose ε close to 1 and replace D by εD. Then it is easy to see that (4.16) does not change, nor does ω_D since

$$\omega_D(p) = \text{constant term in the asymptotic expansion of}$$
$$\{\mathrm{tr}_{E_1}\, e^{-tD^*D}(p,p) - \mathrm{tr}_{E_2}\, e^{-tDD^*}(p,p)\} \text{ as } t \searrow 0.$$

Transforming (2.3) under the unitary map $U_\varepsilon u(y) := \varepsilon^{1/2} u(\varepsilon y)$, we see that S in (2.3) has to be replaced by

$$S_\varepsilon(y) := \frac{1}{\varepsilon} S\left(\frac{y}{\varepsilon}\right).$$

If we now define $f_\varepsilon(y) := f(y/\varepsilon)$, then all assumptions above except possibly (4.14) are easily seen to be satisfied for εD with $\bar{S}_\varepsilon(y) = (1/\varepsilon)\bar{S}(y/\varepsilon)$. To satisfy (4.14) we have to construct g_ε in such a way that

$$s_\varepsilon(0) = \int_0^\infty \frac{dy}{g_\varepsilon(y)^2} = \int_0^\infty \frac{dy}{g(y)^2} = s(0). \tag{4.18}$$

To do so we recall first from Theorem 2.6 that a modification of g in (2.15) in compact subsets of $(0, \infty)$ does not affect (4.16) since g^2 will still equal $f(0)$ near 0 and fe^F near ∞. Constructing $\bar{g}_\varepsilon$ as in (2.15), using f_ε and ψ_ε, $\psi_\varepsilon(y) := \psi(y/\varepsilon)$, we can of course modify $\bar{g}_\varepsilon$ on a compact subset of $(0, \infty)$ such that (4.18) holds for the modified function g_ε.

It remains to study the S_0-dependent terms in (4.17). By (2.3′) and [Gi, Sec. 1.10] and 4.3, η_{S_0} is meromorphic in $\mathbb{C}$ with at most simple poles and no pole at 0. If ε is sufficiently close to 0, then the only change occurs in the residues Res $\eta_{S_0}(2k)$: Res $\eta_{\varepsilon^{-1}S_0}(2k) = \varepsilon^{2k}$ Res $\eta_{S_0}(2k)$. Thus, all residues must vanish. □

5. The signature operator. Consider a complete Riemannian manifold M of dimension $m = 4k$. On $\Omega(M)$ we have an involution τ given by $\tau\omega = (-1)^{k+p(p-1)/2} *\omega$ for $\omega \in \Omega^p(M)$. Denoting by $\Omega_\pm(M)$ the ± 1 eigenspaces of τ and by d exterior differentiation with formal adjoint δ, the signature operator is defined by

$$D_S\colon \Omega_+(M) \to \Omega_-(M), \qquad \omega \mapsto (d + \delta)\omega. \tag{5.1}$$

Since M is complete, D_S has a unique closed extension, also to be denoted by D_S. We want to investigate the finiteness of

$$L^2\text{-sign } M := L^2\text{-ind } D_S = \dim\ker D_S - \dim\ker D_S^* \tag{5.2}$$

and find a formula for it, using the theory developed above. To do so we have to restrict the geometry of M near infinity. Our first assumption is that in the decomposition (2.1), $M = M_1 \cup U$, U is productlike, i.e., that we have

$$U = (0, \infty) \times N \text{ with metric } g = dy^2 \oplus g_N(y)^2 \tag{5.3}$$

where $g_N(y)$ is a family of metrics on the compact manifold N, smoothly varying on $\mathbb{R}_+$. This will allow separation of variables and thus verification of the basic assumptions (2.2) and (2.3′). For convenience we collect some terminology. $\pi_1\colon U \to (0, \infty)$ and $\pi_2\colon U \to N$ are the natural projections. For $y \in [0, \infty)$ we write $N_y := \{y\} \times N$ and $i_y\colon N \ni p \mapsto (y, p) \in N_y$; we usually identify N with N_0 to simplify notation. With y the global coordinate defined by π_1, we put $F_0 := \partial/\partial y$, a unit vector field normal to all N_y. The second fundamental form of N_y will be denoted by

$$II_y(F) = -\nabla_F F_0 \tag{5.4}$$

and the mean curvature by

$$H_y = \operatorname{tr} II_y . \tag{5.5}$$

Since π_1 is a Riemannian submersion, we have

$$\exp_{(y,p)} tF_0 = (y + t, p). \tag{5.6}$$

It follows that the vector fields

$$\hat{E}(i_y(p)) := Ti_y(p)(E), \qquad E \in T_pN, \tag{5.7}$$

are Jacobi fields along the normal geodesic defined by p, with initial value E and initial velocity $-II_0(E)$.

In dealing with the signature operator $D_S\colon \Omega_{+,0}(U) \to \Omega_{-,0}(U)$, it will be convenient to use the Clifford bundle $\mathcal{C}\ell U$ instead of the Grassmann bundle $\wedge^* U$. It is well known (see [LM]) that there is a canonical bundle isomorphism

$$\psi\colon \mathcal{C}\ell U \to \wedge^* U \tag{5.8a}$$

with the property that for $\sigma \in \mathcal{C}\ell U$, $F \in TU$,

$$\psi(F \cdot \sigma) = F^\flat \wedge \psi(\sigma) - F \llcorner \psi(\sigma) \tag{5.8b}$$

where $\cdot$ is Clifford multiplication, $\llcorner$ interior multiplication, and $^\flat\colon TU \to T^*U$ the "musical" isomorphism (with inverse $\sharp$). If $I = \{i_1, \ldots, i_p\}$, $1 \le i_1 < \cdots < i_p \le 4k$, is a strictly ordered multi-index of length p, then we write $F_I := F_{i_1} \cdot \ldots \cdot F_{i_p}$; the set of all such multi-indices will be denoted by I^{4k}. We define an involution on $\mathcal{C}\ell U$ by multiplication with

$$\omega := F_0 \cdot F_1 \cdot \ldots \cdot F_{4k-1} =: F_0 \cdot \omega' \tag{5.9}$$

where $(F_i)_{i \geq 1}$ is any local orthogonal and oriented frame for $F_0^\perp$. If $\mathscr{C}\ell_\pm U$ denotes the ± 1 eigenspace of this involution, then D_S transforms under ψ unitarily to

$$\tilde{D}_S = \sum_{i \geq 0} F_i \cdot \nabla_{F_i}: C_0^\infty(\mathscr{C}\ell_+ U) \to C_0^\infty(\mathscr{C}\ell_- U). \tag{5.10}$$

Here, ∇ denotes the canonical Dirac connection for the Dirac bundle $\mathscr{C}\ell U$ induced by the Levi-Civita connection ∇^M; i.e., with $(F_i)_{i \geq 1}$ as above and $F \in C^\infty(TU)$, we have

$$\begin{aligned} \nabla_F F_I &= \frac{1}{2} \sum_{j<k} \langle \nabla_F^M F_j, F_k \rangle (F_j \cdot F_k \cdot F_I - F_I \cdot F_j \cdot F_k) \\ &= \frac{1}{2} \sum_{j<k} \langle \nabla_F^M F_j, F_k \rangle \operatorname{ad}(F_j \cdot F_k) F_I . \end{aligned} \tag{5.11}$$

Finally, we introduce the subbundle $\mathscr{C}\ell' U$ with fiber $(\mathscr{C}\ell N_y)_p$ at (y, p). Then we find the following lemma.

LEMMA 5.1. *$D_S: \Omega_{+,0}(U) \to \Omega_{-,0}(U)$ is unitarily equivalent to the operator*

$$\nabla_{F_0} + D_y + A_y: C_0^\infty(\mathscr{C}\ell' U) \to C_0^\infty(\mathscr{C}\ell' U). \tag{5.12}$$

The operators appearing in (5.12) *are defined as follows. If $(F_i)_{i \geq 1}$ is a local orthonormal and oriented frame for $F_0^\perp$, then, with ∇^y the canonical connection* (*corresponding to* (5.11)) *on $\mathscr{C}\ell N_y$,*

$$D_y = -\omega' \cdot \sum_{i \geq 1} F_i \cdot \nabla_{F_i}^y . \tag{5.13}$$

A_y is a smooth family of zero-order differential operators on $\mathscr{C}\ell' U$ characterized by the fact that, if $(F_i)_{i \geq 1}$ diagonalizes II_y at (y, p) with eigenvalues $(\lambda_i(y, p))_{i \geq 1}$, then $(F_I)_{I \in I^{4k}}$ diagonalizes A_y at (y, p) with eigenvalues

$$-\lambda_I(y, p) := -\sum_{i \in I} \lambda_i(y, p). \tag{5.14}$$

Proof. Note first that the maps

$$V_\pm: \mathscr{C}\ell' U \ni \sigma \mapsto \frac{1}{\sqrt{2}}(\sigma \pm \omega \cdot \sigma) \in \mathscr{C}\ell_\pm U \tag{5.15}$$

induce unitary maps with respect to the natural L^2-structures.

We now choose a local orthonormal and oriented frame $(F_i)_{i\geq 1}$ as above, which we also assume to be parallel along normal geodesics. Then we compute, noting that $F\cdot\omega' = \omega'\cdot F$ if $F \perp F_0$,

$$
\begin{aligned}
\tilde{D}_S V_+ F_I &= \sum_{i\geq 0} F_i \cdot \nabla_{F_i} V_+ F_I \\
&= \sum_{i\geq 1} \frac{1}{2} F_i \cdot \sum_{j<k} \langle \nabla^M_{F_i} F_j, F_k \rangle \Bigg(F_j \cdot F_k \cdot \frac{1}{\sqrt{2}}(F_I + \omega\cdot F_I) \\
&\quad - \frac{1}{\sqrt{2}}(F_I + \omega\cdot F_I)\cdot F_j \cdot F_k \Bigg) \\
&= \frac{1}{\sqrt{2}}(1-\omega)\cdot \sum_{i\geq 1} \frac{1}{2} F_i \cdot \sum_{1\leq j<k} \langle \nabla^{N_y}_{F_i} F_j, F_k\rangle \,\mathrm{ad}(F_j\cdot F_k) F_I \qquad (5.16) \\
&\quad + \sum_{i\geq 1} \frac{1}{2} F_i \cdot \sum_{k>0} \langle \nabla^M_{F_i} F_0, F_k \rangle \,\mathrm{ad}(F_0\cdot F_k) V_+ F_I \\
&=: V_- \Bigg(\sum_{i\geq 1} F_i \cdot \nabla^y_{F_i} F_I + \tilde{A}_y(F_I) \Bigg) \\
&=: V_-(\tilde{D}_y + \tilde{A}_y) F_I
\end{aligned}
$$

where $\tilde{D}_y$ is the Dirac operator on $\mathscr{Cl}(N_y)$ induced by ∇^{N_y}.

Next, we choose $f \in C^\infty(U)$ and compute

$$
\begin{aligned}
\tilde{D}_S V_+ f F_I &= f \tilde{D}_S V_+ F_I + \nabla f \cdot V_+ F_I \\
&= V_-(f\tilde{D}_y F_I + \tilde{A}_y(fF_I)) + \langle \nabla f, F_0 \rangle F_0 \cdot V_+ F_I \\
&\quad + V_- \nabla^y f \cdot F_I \\
&= V_-(\tilde{D}_y + \tilde{A}_y) f F_I + V_-(-(F_0 f)\omega' \cdot F_I) \\
&= V_-(-\omega' \nabla_{F_0} + \tilde{D}_y + \tilde{A}_y) f F_I .
\end{aligned}
$$

It follows that

$$
-\omega' V_-^* \tilde{D}_S V_+ = \nabla_{F_0} + D_y - \omega' \tilde{A}_y ,
$$

and it only remains to verify the assertion on $A_y := -\omega' \cdot \tilde{A}_y$. Thus, we now assume in addition that $(F_i)_{i\geq 1}$ diagonalizes II_y at (y, p). Then we compute, with (5.16) and

$\omega'^2 = 1$,

$$\sum_{i\geq 1} \frac{1}{2} F_i \cdot \sum_{k>0} \langle \nabla^M_{F_i} F_0, F_k \rangle \bigg(F_0 \cdot F_k \cdot \frac{1}{\sqrt{2}} (F_I + \omega \cdot F_I)$$

$$- \frac{1}{\sqrt{2}} (F_I + \omega \cdot F_I) \cdot F_0 \cdot F_k \bigg)(y, p)$$

$$= \sum_{i\geq 1} \left(-\frac{1}{2} \lambda_i(y, p) \right) \bigg(F_i \cdot F_0 \cdot F_i \cdot \frac{1}{\sqrt{2}} (F_I + \omega \cdot F_I)$$

$$- \frac{1}{\sqrt{2}} F_i \cdot (F_I + \omega \cdot F_I) \cdot F_0 \cdot F_i \bigg)(y, p)$$

$$= \frac{1}{\sqrt{2}} \sum_{i\geq 1} \left(-\frac{1}{2} \lambda_i(y, p) \right) (\omega \cdot \omega' \cdot F_I - \omega' \cdot F_I$$

$$+ (-1)^{|I|} \omega \cdot \omega' \cdot F_i \cdot F_I \cdot F_i - (-1)^{|I|} \omega' \cdot F_i \cdot F_I \cdot F_i)(y, p)$$

$$= V_- \sum_{i\geq 1} \left(-\frac{1}{2} \lambda_i(y, p) \right) (-\omega' \cdot F_I - (-1)^{|I|} \omega' \cdot F_i \cdot F_I \cdot F_i)(y, p).$$

Thus, we have

$$A_y(F_I)(y, p) = \sum_{i\geq 1} \left(-\frac{1}{2} \lambda_i(y, p) \right) (F_I + (-1)^{|I|} F_i \cdot F_I \cdot F_i)(y, p)$$

$$= \sum_{i\in I} (-\lambda_i(y, p)) F_I(y, p). \qquad \square$$

Next, we want to achieve the form necessary for the application of Theorem 4.4. To do so we introduce a unitary map

$$\Psi\colon L^2(\mathbb{R}_+, L^2(\mathscr{C}\ell N)) \to L^2(\mathscr{C}\ell' U) \tag{5.17a}$$

by

$$\Psi\sigma \circ i_y = \alpha(y) P_y \sigma(y) = P_y \alpha\sigma(y). \tag{5.17b}$$

Here, $P_y\colon C^\infty(\mathscr{C}\ell N) \to C^\infty(\mathscr{C}\ell N_y)$ denotes parallel transport in $\mathscr{C}\ell U$ along the geodesics normal to N; hence, $P_y\sigma(y) \in C^\infty(N, \mathscr{C}\ell N_y)$. The function $\alpha(y) \in C^\infty(N)$ is given as follows: if ω_y denotes the volume form on N_y, then

$$i_y^* \omega_y = \alpha(y)^{-2} \omega_0. \tag{5.18}$$

Then it is easy to see that Ψ is unitary:

$$\begin{aligned}
\|\Psi\sigma\|^2_{L^2(\mathscr{C}\ell'U)} &= \int_0^\infty \int_{N_y} |\Psi\sigma|^2_{\mathscr{C}\ell'U}\, dy \wedge \omega_y \\
&= \int_0^\infty dy \int_{N_y} |\alpha(y)P_y\sigma(y)|^2_{\mathscr{C}\ell'U} \circ i_y^{-1}\omega_y \qquad (5.19) \\
&= \int_0^\infty dy \int_N |\sigma(y)|^2_{\mathscr{C}\ell N}\omega_0 = \|\sigma\|^2_{L^2(\mathbb{R}_+, L^2(\mathscr{C}\ell N))}.
\end{aligned}$$

Next, we compute the transformation of the various terms in (5.12). For the first term we find

$$\begin{aligned}
\nabla_{F_0}\Psi\sigma(i_y(p)) &= \lim_{\varepsilon\to 0} \varepsilon^{-1}[P_{N_{y+\varepsilon,-\varepsilon}}\Psi\sigma(i_{y+\varepsilon}(p)) - \Psi\sigma(i_y(p))] \\
&= \lim_{\varepsilon\to 0} \varepsilon^{-1}[P_y(\alpha\sigma(y+\varepsilon)(p) - \alpha\sigma(y)(p))] \\
&= \alpha(y)(p)P_y\left(\frac{\alpha'}{\alpha}\sigma(y)(p) + \sigma'(y)(p)\right)
\end{aligned}$$

where $'$ stands for the application of $\partial/\partial y$. Thus,

$$\Psi^{-1}\nabla_{F_0}\Psi =: \frac{\partial}{\partial y} + \nabla_{F_0}\log\alpha(y). \qquad (5.20a)$$

It follows from standard arguments in Riemannian geometry that

$$\nabla_{F_0}\log\alpha(y) = \tfrac{1}{2}H_y. \qquad (5.20b)$$

We have next, denoting by $\tilde{F}_i$ the parallel extension of a local orthonormal frame (F_i) for TN_0, diagonalizing II_y at $i_y(p)$,

$$\begin{aligned}
A_y\Psi\sigma(i_y(p)) &= -\alpha(y)(p)\sum_I \sigma_I(y)(p)\lambda_I(i_y(p))\tilde{F}_I(i_y(p)) \\
&=: \Psi\tilde{A}_y\sigma(i_y(p))
\end{aligned}$$

where

$$\tilde{A}_yF_I(p) = -\lambda_I(i_y(p))F_I(p) =: (-\lambda_I(y)F_I)(p) \qquad (5.20c)$$

if $(\tilde{F}_i)$ diagonalizes II_y at $i_y(p)$. This representation allows us to control $\tilde{A}_y$ in terms of II_y.

Finally, we consider

$$D_y\Psi\sigma(i_y(p)) = -\tilde{\omega}'\cdot \sum_{i\geq 1} \tilde{F}_i \cdot \nabla^y_{\tilde{F}_i}(P_y\alpha\sigma(y)\circ i_y^{-1})(i_y(p)). \tag{5.21}$$

To deal with (5.21) we need the following general fact.

LEMMA 5.2. *For $i = 1, 2$, let N_i be a Riemannian manifold and S_i a hermitian vector bundle over N_i with projection π_i. Assume that $f\colon N_1 \to N_2$ is a diffeomorphism and $F\colon S_1 \to S_2$ a bundle map such that $\pi_2 \circ F = f \circ \pi_1$. In addition, assume that F is an isometry in each fiber. Finally, let ∇ be a metric connection for S_2. With $\Phi\colon C^\infty(S_1) \to C^\infty(S_2)$, $\sigma \mapsto F\circ\sigma\circ f^{-1}$, we define for $X \in C^\infty(TN_1)$, $\sigma \in C^\infty(S_1)$*

$$(\Phi^*\nabla)_X\sigma := \Phi^{-1}\nabla_{Tf(X)}\Phi\sigma. \tag{5.22}$$

Then $\Phi^*\nabla$ is a metric connection for S_1.

Proof. We compute with $g \in C^\infty(N_1)$

$$\begin{aligned}(\Phi^*\nabla)_X g\sigma &= \Phi^{-1}\nabla_{Tf(X)} g\circ f^{-1}\Phi\sigma\\ &= g(\Phi^*\nabla)_X\sigma + Tf(X)(g\circ f^{-1})\sigma\\ &= g(\Phi^*\nabla)_X\sigma + (Xg)\sigma.\end{aligned}$$

Thus, $\Phi^*\nabla$ is a connection. Next, we compute

$$\begin{aligned}X\langle\sigma_1,\sigma_2\rangle &= X\langle\Phi\sigma_1,\Phi\sigma_2\rangle\circ f\\ &= (Tf(X)\langle\Phi\sigma_1,\Phi\sigma_2\rangle)\circ f\\ &= \langle\nabla_{Tf(X)}\Phi\sigma_1,\Phi\sigma_2\rangle\circ f + \langle\Phi\sigma_1,\nabla_{Tf(X)}\Phi\sigma_2\rangle\circ f\\ &= \langle(\Phi^*\nabla)_X\sigma_1,\sigma_2\rangle + \langle\sigma_1,(\Phi^*\nabla)_X\sigma_2\rangle.\end{aligned}$$

This proves that $\Phi^*\nabla$ is metric. □

To continue our calculations, we let

$$\Phi_y\sigma := P_y\sigma\circ i_y^{-1}, \qquad \sigma\in C^\infty(\mathscr{C}\ell N), \tag{5.23a}$$

$$\tilde{\nabla}^y := \Phi_y^*\nabla^y. \tag{5.23b}$$

Then we find

$$\begin{aligned}
&\nabla^y_{\tilde{F}_i}(P_y \alpha\sigma(y) \circ i_y^{-1})(i_y(p)) \\
&\quad = P_y(P_y^{-1} \nabla^y_{\tilde{F}_i} \Phi_y \alpha\sigma(y) \circ i_y)(p) \\
&\quad = P_y(\Phi_y^{-1} \nabla^y_{\tilde{F}_i} \Phi_y \alpha\sigma(y))(p) \\
&\quad =: P_y(\tilde{\nabla}^y_{F_i(y)} \alpha\sigma(y))(p)
\end{aligned} \tag{5.24}$$

where we have put

$$F_i(y)(p) = (Ti_y(p))^{-1} \tilde{F}_i(i_y(p)). \tag{5.25}$$

It follows now from (5.21) that

$$\begin{aligned}
D_y \Psi\sigma(i_y(p)) &= P_y\left(-\omega' \sum_{i \geq 1} F_i \cdot \tilde{\nabla}^y_{F_i(y)} \alpha\sigma(y)\right)(p) \\
&= \Psi\left(-\alpha(y)^{-1}\omega' \sum_{i \geq 1} F_i \cdot \tilde{\nabla}^y_{F_i(y)} \alpha\sigma(y)\right)(i_y(p)).
\end{aligned} \tag{5.20d}$$

Finally, we write

$$\gamma(y) := \sum_{i \geq 1} \langle F_i(y), \nabla_N \log \alpha(y) \rangle F_i \in C^\infty(TN). \tag{5.20e}$$

Then we obtain from (5.20a, b, c, d, e) the following theorem.

THEOREM 5.3. *Under the assumption* (5.1), $D_S\colon \Omega_{+,0}(U) \to \Omega_{-,0}(U)$ *is unitarily equivalent to*

$$\begin{aligned}
&-\partial_y + \omega' \cdot \sum_{i \geq 1} F_i \cdot \tilde{\nabla}^y_{F_i(y)} + \omega' \cdot \gamma(y) - \frac{1}{2} H_y - \tilde{A}_y \\
&=: -\partial y + S(y)
\end{aligned} \tag{5.26}$$

with domain $C_0^\infty(\mathbb{R}_+, C^\infty(\mathscr{C}\ell N))$ *in* $L^2(\mathbb{R}_+, L^2(\mathscr{C}\ell N))$. *This equivalence has the properties* (2.2), (2.3′), *and* (2.7).

Proof. If we put $\tilde{\Psi} := \sqrt{-1}\Psi$ with Ψ defined in (5.17), then (5.26) follows from the identities (5.20) and Lemma 5.1. From this construction, (2.2) and (2.7) are obvious. For later purposes we note the explicit form of Φ_1: $L^2(\mathscr{C}\ell_+ U) \to L^2((0, \infty), L^2(\mathscr{C}\ell N))$: if $(F_i)_{i \geq 0}$ is a local orthonormal frame for TU as above with parallel translates

$(\tilde{F}_i)_{i\geq 0}$ in the y-direction, then we obtain from (5.17) and (5.15) that, for $\sigma \in L^2((0,\infty), L^2(\mathscr{C}\ell N))$, $\sigma(y) = \sum_I \sigma_I(y)F_I$,

$$\Phi_1^{-1}\sigma(i_y(p)) = \frac{1}{\sqrt{2}}\alpha(y)(p)\sum_I \sigma_I(y)(p)(1+\tilde{\omega})\cdot \tilde{F}_I(i_y(p)). \tag{5.27}$$

To verify (2.2) we only have to put $H := L^2(\mathscr{C}\ell N)$ and $H_1 := H^1(\mathscr{C}\ell N)$, the Sobolev space of sections with square integrable first derivatives; then all assertions are readily checked. Also, (2.7) is obvious. $S(y)$ in (5.26) is clearly a first-order differential operator, and for its principal symbol we find

$$\sigma_{S(y)}(\xi)(\eta) = \omega'\cdot(i_y^* \circ P_y)^{-1}(\xi)^\#\cdot\eta, \qquad \xi \in T_p^*N, \qquad \eta \in \mathscr{C}\ell_p N.$$

Hence, $S(y)$ is also elliptic. To see the symmetry we observe that replacing F_0 by $-F_0$ in the above calculation gives the unitary representation of $D_S^*\colon \Omega_{-,0}(U) \to \Omega_{+,0}(U)$. Thus, we derive

$$\partial_y + S(y)^* = \partial_y + S(y). \qquad \square$$

We want to apply Theorem 4.4 to D_S in the situation just described. Then we have to verify in addition the conditions (2.9) through (2.11), (2.13), and (4.14). This means that we have to find a positive function $f \in C^\infty(\mathbb{R}_+)$ such that $\lim_{y\to\infty} f'(y) = a$ with $0 \leq a < 1$ and such that $\bar{S}(y) = fS(y)$ satisfies the conditions on spectral concentration ((2.9), (2.10)) and on bounded variation ((2.13)). This will be illustrated below for multiply warped products; we hope to return to this in greater generality at another occasion.

To give (4.14) a simple form we introduce the assumption

$$g_N(y) = g_N(0) \qquad \text{for } y \text{ near } 0 \tag{5.28}$$

in (5.1); i.e., we assume that M has the product metric near ∂M_1. Note that this means no restriction if D_S is Fredholm. Then an easy calculation shows that

$$S(y) = S(0) = \omega'\cdot \sum_{i\geq 1} F_i\cdot\nabla^0_{F_i}, \qquad y \text{ near } 0, \tag{5.29}$$

where ∇^0 is the canonical connection on $\mathscr{C}\ell N$. Thus, (4.14) reduces to

$$f(y) = f(0)(1 - y/s(0)f(0))^{-1}, \qquad y \text{ near } 0. \tag{5.30}$$

It is also clear from (5.29) that $S(0)$ corresponds to $-T$, T the operator introduced in [APS, Thm. (4.14)], under the isomorphism (5.28). Hence, $\eta(S_0) = -\eta(N)$. We summarize these remarks.

THEOREM 5.4. *Assume* (5.3) *and* (5.28) *and assume that we can find a smooth positive function f with* (2.11) *and* (5.30) *such that* $\bar{S}(y) := fS(y)$ *fulfills spectral concentration* ((2.9), (2.10)) *and bounded variation* ((2.13)).

Then

$$L^2\text{-sign } M = \operatorname{sign}(M_1, \partial M_1) + \frac{1}{2} \operatorname{sign} QS(0)Q + \left(h_0 + h_1 - \frac{1}{2} \dim Q\right). \quad (5.31)$$

Proof. Since the assumptions imply the assumptions of Theorem 4.4, we need only remark that

$$\begin{aligned}
&\frac{1}{2} \dim Q - \frac{1}{2} \dim \ker \bar{S}(0) - \sum_{-1/2 < s < 0} \dim \ker(\bar{S}(0) - s) \\
&\quad = \frac{1}{2} \sum_{0 < |s| < 1/2} \dim \ker(\bar{S}(0) - s) \\
&\quad = \frac{1}{2} \operatorname{sign} Q\bar{S}(0)Q = \frac{1}{2} \operatorname{sign} QS(0)Q
\end{aligned}$$

and that, by [APS, Thm. (4.14)],

$$\int_{M_1} L_k + \eta(N) = \operatorname{sign}(M_1, \partial M_1). \qquad \square$$

The signature formula (5.31) is explicit as it stands only if we can calculate h_1. For example, if D_S is Fredholm, then $h_1 = 0$ by Lemma 2.7. One might expect that in the Fredholm case the second and third term add up to zero, but we cannot prove this for the time being. It is true, however, in the case of multiply warped products, as shown below.

Another nice case occurs if $Q = 0$. This will, however, not even be the case for simple finite-volume manifolds like quotients of hyperbolic space; see [B, Sec. 5].

It has been shown in [APS, Cor. (4.11)] that a nice formula can also be obtained if U is a cylinder. We will now generalize this result. To do so we keep the assumptions of Theorem 5.4 and introduce "extended solutions" for D_S and D_S' as follows. Put

$$\begin{aligned}
\mathscr{H}^{(\prime)} &:= \{\omega \in \Omega_{\underset{(-)}{+}}(M) \mid (d + \delta)\omega = 0, g^{-1}\omega \in L^2(\wedge^* M)\} \\
&= g \ker D^{(\prime)}_{S, g, \max};
\end{aligned} \quad (5.32)$$

then $\mathscr{H}$ and $\mathscr{H}'$ are finite-dimensional, and forms in $\mathscr{H}^{(\prime)}$ will be called extended solutions of $D_S^{(\prime)}$. We now prove that $\mathscr{H}^{(\prime)}$ consists of closed forms.

LEMMA 5.5. *For* $\omega \in \mathscr{H} \cup \mathscr{H}'$ *we have* $d\omega = 0$.

Proof. Consider $\mathscr{K}$ only, the arguments for $\mathscr{K}'$ being similar.

We show first that $\omega \in \mathscr{K}$ implies

$$(fg)^{-1}\omega \in L^2(\wedge^* M). \tag{5.33}$$

If $a > 0$ in (2.11), we have nothing to prove; so we may assume $\lim_{y\to\infty} f'(y) = 0$. Now pick $\psi \in C^\infty(M)$ with $\operatorname{supp}\psi \subset U$ and $\psi = 1$ near ∞. Then we have $\psi g^{-1}\omega \in \mathscr{D}(D_{S,g,\max})$; hence, we can find $v \in \mathscr{D}(\tilde{D}_{S,\max})$ such that

$$\psi g^{-1}\omega = \Phi_1^{-1} \circ \Psi(v).$$

From (3.25), (3.26), and Gronwall's inequality applied to $\tilde{Q}v$, we obtain

$$\|v(x)\|_{L^2(\wedge^* N)} \leq Cx^{-\alpha}, \qquad x \in (0, s_0), \tag{5.34}$$

for some α, $0 \leq \alpha < 1/2$. With $\varphi \in C_0^\infty(0, \infty)$ we then find

$$\varphi\psi(gf)^{-1}\omega = \Phi_1^{-1} \circ \Psi((\varphi f^{-1}) \circ s^{-1}v),$$

and it only remains to show that $f^{-1} \circ s^{-1}v \in L^2((0, s_0), L^2(\wedge^* N))$. We estimate with (5.34) and Lemma 2.1

$$\begin{aligned}\int_0^{s_0} (f \circ s^{-1}(x))^{-2} \|v(x)\|^2_{L^2(\wedge^* N)}\, dx &\leq C \int_0^{s_0} (f \circ s^{-1}(x))^{-2} x^{-2\alpha}\, dx \\ &\leq C_T\left(1 + \int_T^\infty e^{(2\alpha-1)F(y)} f(y)^{-2}\, dy\right).\end{aligned}$$

But for $\varepsilon > 0$ and $y \geq T = T(\varepsilon)$, we have

$$f(y)^{-2} = f(T)^{-2} \exp\left(-2\int_T^y \frac{f'(u)}{f(u)}\, du\right) \leq C_\varepsilon e^{\varepsilon F(y)}.$$

Choosing $\varepsilon = (1 - 2\alpha)/2$, we conclude that

$$\begin{aligned}\|(f \circ s^{-1})^{-1} v\|_{L^2((0,s_0), L^2(\wedge^* N))} &\leq C_T\left(1 + \int_T^\infty e^{((2\alpha-1)/2)F(y)} \frac{dy}{f(y)}\right) \\ &\leq C_T\left(1 + \int_0^\infty e^{(-(1-2\alpha)/2)u}\, du\right).\end{aligned}$$

Next, we claim that

$$\eta := d\omega \in \mathscr{K} \qquad \text{whenever } \omega \in \mathscr{K}. \tag{5.35}$$

We compute

$$
\begin{aligned}
\tau\eta_p = \tau d\omega_{p-1} &= (-1)^{k+p(p-1)/2} * d\omega_{p-1} \\
&= (-1)^{k+p(p-1)/2} * d(\tau\omega)_{p-1} \\
&= (-1)^{k+p(p-1)/2+k+(4k+1-p)(4k-p)/2} * d * \omega_{4k+1-p} \\
&= -\delta\omega_{4k+1-p} = d\omega_{4k+1-p} = \eta_{4k-p}.
\end{aligned}
$$

Next, we choose $\varphi \in C_0^\infty(M)$ and compute

$$
\begin{aligned}
\|\varphi g^{-1}\eta\|^2 &= -(\varphi^2 g^{-2}\, d\omega, \delta\omega) \\
&= -(d\varphi^2 g^{-2} \wedge d\omega, \omega) \\
&= -2(d\varphi \wedge \varphi g^{-1}\eta, g^{-1}\omega) + 2(\varphi^2 g^{-3}\, dg \wedge d\omega, \omega) \\
&= -2(d\varphi \wedge \varphi g^{-1}\eta, g^{-1}\omega) + (\varphi^2 d \log g^2 \wedge \varphi g^{-1}\eta, g^{-1}\omega).
\end{aligned}
$$

But from (2.15) we have, for large y,

$$
d \log g^2(y) = \frac{f' + 1}{f}(y)\, dy;
$$

hence, we find with (5.33) that

$$
\|\varphi g^{-1}\eta\|_{L^2(\bigwedge^* M)} \leq C_\omega(\|\varphi\|_{L^\infty} + \|d\varphi\|_{L^\infty}).
$$

Since we can make $\varphi \to 1$ with $\|\varphi\|_{L^\infty}$ and $\|d\varphi\|_{L^\infty}$ uniformly bounded, (5.35) is proved.

Now we put $d_{\mathscr{K}} := d|\mathscr{K}$, $\delta_{\mathscr{K}} := \delta|\mathscr{K}$, and we denote by $d_{\mathscr{K}}^*$ the adjoint with respect to the scalar product

$$
(\omega_1, \omega_2)_{\mathscr{K}} := (g^{-1}\omega_1, g^{-1}\omega_2)_{L^2(\bigwedge^* M)}.
$$

Using (5.33), one readily computes

$$
d_{\mathscr{K}}^* = -d_{\mathscr{K}} + (\nabla \log g^2) \llcorner =: -d_{\mathscr{K}} + \alpha, \tag{5.36}
$$

$\llcorner$ denoting interior multiplication. Since

$$
(d_{\mathscr{K}}^*)^2 = d_{\mathscr{K}}^2 = \alpha^2 = 0,
$$

we find

$$d_{\mathscr{K}}\alpha + \alpha d_{\mathscr{K}} = 0;$$

hence,

$$(d_{\mathscr{K}}^* d_{\mathscr{K}})^2 = d_{\mathscr{K}}^* d_{\mathscr{K}} \alpha d_{\mathscr{K}} = -d_{\mathscr{K}}^* d_{\mathscr{K}}^2 \alpha = 0,$$

and thus

$$d_{\mathscr{K}} = 0. \qquad \square$$

We turn to the generalization of the above mentioned result in [APS, Cor. (4.11)]. Note first that Theorem 4.3 applied to $D_{S,g}$ gives

$$\begin{aligned} \dim Q &= \operatorname{ind} D_{S,g,\max} - \operatorname{ind} D_{S,g,\min} \\ &= \dim\ker D_{S,g,\max} - \dim\ker D_{S,g,\min} \\ &\quad + \dim\ker D_{S,g,\min}^* - \dim\ker D_{S,g,\max}^* \\ &= (\dim\ker D_{S,g,\max} - \dim\ker D_{S,g,\min}) \\ &\quad + (\dim\ker D'_{S,g,\max} - \dim\ker D'_{S,g,\min}) \\ &=: k + k'. \end{aligned} \tag{5.37}$$

In general, we cannot say more about k and k'. For the signature operator, however, the precise calculation is possible.

THEOREM 5.6. *Assume the hypotheses of Theorem* 5.4. *Assume in addition* (2.20) *and assume that* $\psi Q L^2(\mathscr{C}\ell N)$ *is invariant under* $*_N$ *(with* ψ *in* (5.8)*). Then*

$$k = k' = \tfrac{1}{2}\dim Q.$$

Proof. We show that $k \le \frac{1}{2}\dim Q$; by an analogous argument one finds $k' \le \frac{1}{2}\dim Q$; so the assertion follows with (5.37).

Now we define a linear map

$$j\colon \mathscr{K} \ni \omega \mapsto Q\psi^{-1}(i_0^*\omega) \in C^\infty(\mathscr{C}\ell N)$$

where $i_0\colon N \hookrightarrow M$ is the natural inclusion and ψ the isomorphism in (5.8). We know that $\mathscr{K} = g\ker D_{S,g,\max}$, and we want to show that

$$\ker j = g\ker D_{S,g,\min}. \tag{5.38}$$

If we show also

$$\dim \operatorname{im} j \le \tfrac{1}{2} \dim Q, \tag{5.39}$$

the proof will follow from (5.38) and (5.39).

For the proof of (5.38) we pick $\omega \in \mathscr{H}$ and write $g^{-1}\omega | U = \psi \circ \Phi_1^{-1}(u)$ for some $u \in H^1(\mathbb{R}_+, L^2(\mathscr{C}\ell N))$; recall that Φ_1^{-1} was given explicitly in (5.27). It follows that

$$j\omega = \frac{g(0)}{\sqrt{2}} Qu(0).$$

Then we decompose

$$g^{-1}\omega = \psi \circ \Phi_1^{-1}(Qu + (\mathbf{I} - Q)u) =: g^{-1}(\omega_1 + \omega_2).$$

It follows from Lemma 3.4 that $g^{-1}\omega_2 \in \mathscr{D}(D_{S,g,\min})$, whereas the proof of Theorem 2.6 shows that

$$g^{-1}\omega_1 \in \mathscr{D}(D_{S,g,\min}) \qquad \text{if and only if } Qu(0) = 0.$$

Hence, we deduce (5.38).

To prove (5.39) we claim slightly more, namely that

$$\operatorname{im} \psi j \cap *_N \operatorname{im} \psi j = \{0\}. \tag{5.40}$$

Since $\psi Q L^2(\mathscr{C}\ell N)$ is $*_N$ invariant by assumption, (5.40) implies (5.39).

Now, for $\omega \in \mathscr{H}$ we have $\psi j\omega = i_0^*\omega_1 =: \eta$ for some $\omega_1 \in \mathscr{H}$ since Q is independent of y. Assume that $*_N\eta = \psi j\omega' =: i_0^*\omega_2$, too, with $\omega_2 \in \mathscr{H}$. Then

$$\begin{aligned} \|\eta\|^2 &= \int_{\partial M_1} \eta \wedge *_N\eta = \int_{\partial M_1} i_0^*\omega_1 \wedge i_0^*\omega_2 \\ &= \int_{M_1} d(\omega_1 \wedge \omega_2) = 0 \end{aligned}$$

in view of Lemma 5.5. The proof is complete. □

We can present the announced generalization of Cor. (4.11) in [APS].

THEOREM 5.7. *Under the assumptions of Theorem 5.6, assume that $h_0 = h_0' = 0$ (where $'$ denotes the quantities defined by D_S' in place of D_S). Then*

$$L^2\text{-sign } M = \operatorname{sign}(M_1, \partial M_1) + \tfrac{1}{2} \operatorname{sign} QS(0)Q. \tag{5.41}$$

Proof. In view of (5.31) and the assumption we only have to show that $h_1 = \frac{1}{2} \dim Q$ in this case.

But $h_0 = h_0' = 0$ means by definition that

$$D_{S,g,V} = D_{S,g,\min}, \qquad D'_{S,g,V'} = D'_{S,g,\min};$$

hence,

$$D^*_{S,g,V} = D'_{S,g,\max}, \qquad (D'_{S,g,V'})^* = D_{S,g,\max}.$$

Since

$$\dim \ker D^{(\prime)}_{S,\max} = \dim \ker D^{(\prime)}_{S,g,V^{(\prime)}},$$

we obtain from (2.34) and Theorem 5.6

$$h_1 = h_1' = \tfrac{1}{2} \dim Q. \tag{5.42}$$ □

We add some further comments. In the cylindrical case treated in [APS], we have $S(y) = S(0)$, and we may choose $f \equiv f(0)$, $Q = \ker S(0)$. (See [B, Sec. 5].) Then clearly $h_0 = h_0' = 0$, and (5.41) gives

$$L^2\text{-sign } M = \operatorname{sign}(M_1, \partial M_1), \tag{5.43}$$

which, therefore, is indeed a special case of Theorem 5.7.

If we know for some reason that D_S is also Fredholm, then (5.42) and Lemma 2.7 show that $Q = 0$. Thus, (5.43) also holds in this more general setting. We expect, however, that sign $QS(0)Q$ does not always vanish, but at the moment we do not know of any example.

If we apply the proof of Theorem 5.4 to the operator $-D_S'$, then we obtain

$$-L^2\text{-sign } M = -\operatorname{sign}(M_1, \partial M_1) - \tfrac{1}{2} \operatorname{sign} QS(0)Q + (h_0' + h_1' - \tfrac{1}{2} \dim Q); \tag{5.44}$$

so together with (5.31)

$$h_0 + h_0' + h_1 + h_1' = \dim Q. \tag{5.45}$$

One may conjecture that for $h_1 = h_1' = 0$ we should have

$$h_0 = h_0' = \tfrac{1}{2} \dim Q, \tag{5.46}$$

but this turns out to be wrong in general; a counterexample is provided by a noncompact finite-volume quotient of complex hyperbolic space. We will return to this example at a later occasion.

To conclude this section we treat a specific geometric example, the case of multiply warped products.

By this we mean that N in (5.3) is a product

$$N = N_1 \times \cdots \times N_\ell \tag{5.47a}$$

of compact manifolds N_i with $\dim N_i =: n_i$, $1 \le i \le \ell$, and that the metric (5.3) takes the form

$$g_N(y) = f_1(y)^2 g_1 \oplus \cdots \oplus f_\ell(y)^2 g_\ell. \tag{5.47b}$$

Here, each $f_i \in C^\infty(\mathbb{R}_+)$ is positive, and g_i is a Riemannian metric on N_i, $1 \le i \le \ell$.

To calculate the L^2-signature of M in this case, we have to verify the conditions of Theorem 5.4. We start with making (5.26) more explicit. We calculate first that

$$\alpha(y) = \prod_{i=1}^{\ell} f_i(y)^{-n_i/2}, \tag{5.48a}$$

$$\gamma(y) = 0, \tag{5.48b}$$

$$-\frac{1}{2} H_y = \sum_{i=1}^{\ell} \frac{n_i}{2} \frac{f_i'}{f_i}(y). \tag{5.48c}$$

Next, we choose local orthonormal frames $(F_i^j)_{1 \le i \le n_j}$ for TN_j, and we observe that the frames defined by

$$\tilde{F}_i^j(i_y(p)) := f_i(y)^{-1}\, Ti_y(F_i^j(p)) \tag{5.49}$$

are parallel along normal geodesics by the Koszul formula. Thus, we obtain

$$F_i^j(y) = f_i(y)^{-1} F_i^j.$$

Also,

$$\begin{aligned} II_y(\tilde{F}_i^j) &= -\nabla_{\tilde{F}_i^j} F_0 = -\nabla_{F_0} \tilde{F}_i^j + [F_0, \tilde{F}_i^j] \\ &= -\frac{f_i'}{f_i}(y); \end{aligned}$$

hence from (5.20c) and (5.14), with obvious notation,

$$\tilde{A}_y(F_{I_1}^1 \cdot \ldots \cdot F_{I_\ell}^\ell) = \left(\sum_{i=1}^{\ell} |I_i| \frac{f_i'}{f_i}(y) \right) F_{I_1}^1 \cdot \ldots \cdot F_{I_\ell}^\ell. \tag{5.50}$$

It remains to calculate the connection $\tilde{\nabla}^y$. We find for $F \in C^\infty(TN_m)$, $1 \leq m \leq \ell$,

$$\tilde{\nabla}^y_F F^1_{I_1} \cdot \ldots \cdot F^\ell_{I_\ell}(p) = P_y^{-1}(\nabla^{N_y}_{Ti_y(F)} \tilde{F}^1_{I_1} \cdot \ldots \cdot \tilde{F}^\ell_{I_\ell}(i_y(p)))$$

$$= P_y^{-1} \quad \frac{1}{2} \sum_{\substack{j<k \\ p \leq q}} \langle \nabla^{N_y}_{Ti_y(F)} \tilde{F}^p_i, \tilde{F}^q_j \rangle \operatorname{ad}(\tilde{F}^p_i \cdot \tilde{F}^q_j) \tilde{F}^1_{I_1} \cdot \ldots \cdot \tilde{F}^\ell_{I_\ell}(i_y(p))$$

$$= P_y^{-1} \left(\frac{1}{2} \sum_{j<k} \langle \nabla^N_F F^m_i, F^m_j \rangle \operatorname{ad}(\tilde{F}^m_i \cdot \tilde{F}^m_j) \tilde{F}^1_{I_1} \cdot \ldots \cdot \tilde{F}^\ell_{I_\ell}(i_y(p)) \right) \tag{5.51}$$

$$= \frac{1}{2} \sum_{j<k} \langle \nabla^N_F F^m_i, F^m_j \rangle \operatorname{ad}(F^m_i \cdot F^m_j) F^1_{I_1} \cdot \ldots \cdot F^\ell_{I_\ell}(p)$$

$$= F^1_{I_1} \cdot \ldots \cdot (\nabla_F F^m_{I_m}) \cdot \ldots \cdot F^\ell_{I_\ell}(p)$$

where ∇ denotes the canonical connection on $\mathscr{C}\ell N$. Thus, we can write in (5.26)

$$S(y) := \sum_{i=1}^{\ell} S_i(y) \tag{5.52a}$$

where

$$S_i(y) := f_i(y)^{-1} \left[\omega' \cdot D_i + f_i'(y) \bigoplus_{p \geq 0}{}_i \left(\frac{n_i}{2} - p \right) \right] \tag{5.52b}$$

$$=: S_{i0}(y) + S_{i1}(y)$$

and

$$D_i := \sum_{j=1}^{n_i} F^i_j \cdot \nabla_{F^i_j} \tag{5.52c}$$

is the canonical selfadjoint Dirac operator on $\mathscr{C}\ell N_i$. Here, $\bigoplus_{p\geq 0}{}_i$ denotes decomposition with respect to degree in $\mathscr{C}\ell N_i$. We collect the crucial properties of the operator family $S(y)$.

LEMMA 5.8. *Each $S(y)$ is a symmetric elliptic differential operator of first order on $\mathscr{C}\ell N$. If H_1 denotes the domain of the unique selfadjoint extension of $S(y)$ in $H = L^2(\mathscr{C}\ell N)$, then H_1 is independent of y and the map $\mathbb{R}_+ \ni y \mapsto S(y) \in \mathscr{L}(H_1, H)$ is strongly smooth. Finally, on $C^\infty(\mathscr{C}\ell N)$*

$$\left(\sum_{i=1}^{\ell} S_{i0}(y) \right)^2 = \sum_{i=1}^{\ell} S_{i0}(y)^2. \tag{5.53}$$

Proof. By Theorem 5.3 we only have to prove (5.53). We compute for $i \neq j$, using that ω' is parallel and a central involution, that

$$\begin{aligned} S_{i_0}(y)S_{j_0}(y) + S_{j_0}(y)S_{i_0}(y) &= \sum_{k,k'} [F_k^i \cdot \nabla_{F_k^i}(F_{k'}^j \cdot \nabla_{F_{k'}^j}) + F_{k'}^j \cdot \nabla_{F_{k'}^j}(F_k^i \cdot \nabla_{F_k^i})] \\ &= \sum_{k,k'} [F_k^i \cdot F_{k'}^j + F_{k'}^j \cdot F_k^i] \cdot \nabla_{F_k^i} \nabla_{F_{k'}^j} \\ &= 0. \end{aligned}$$

□

Next, we have to find a suitable function f and the spectral projections $Q(y)$. Now (5.52) suggests the following choice of Q: we define

$$Q = \operatorname{span}\{\eta_1 \cdot \ldots \cdot \eta_\ell | D_i \eta_i = 0, 1 \leq i \leq \ell\}. \tag{5.54}$$

THEOREM 5.9. *Assume that we can find a positive function $f \in C^\infty(\mathbb{R}_+)$ satisfying* (2.11) *and in addition*

$$f(y) \geq C_7 \max_{1 \leq i \leq \ell} f_i(y), \qquad y \geq 0, \tag{5.55a}$$

$$\sum_{i=1}^{\ell} \left| f \frac{f_i'}{f_i} \right| (y) \leq C_8, \qquad y \geq 0, \tag{5.55b}$$

$$\sum_{i=1}^{\ell} \int_0^\infty \left[f \left| \frac{f'}{f} - \frac{f_i'}{f_i} \right|^2 (y) + f^3 \left| \frac{f_i''}{f_i} \right|^2 (y) \right] dy < \infty, \tag{5.55c}$$

$$f(y) = f(0)(1 - y/s(0)f(0))^{-1} \qquad \textit{near } y = 0. \tag{5.55d}$$

Assume, moreover, that

$$f_i(y) \equiv f_i(0) = 1 \textit{ for } y \textit{ near } 0 \textit{ and } 1 \leq i \leq \ell. \tag{5.56}$$

Then, if C_7 is sufficiently large and C_8 sufficiently small, D_S has a finite L^2-index given by

$$\begin{aligned} L^2\text{-ind } D_S &= \int_{M_1} L_k + \frac{1}{2}\eta(N) + \left(h_0 + h_1 - \frac{1}{2}\dim Q\right) \\ &= \operatorname{sign}(M_1, \partial M_1) + \left(h_0 + h_1 - \frac{1}{2}\dim Q\right). \end{aligned} \tag{5.57}$$

Proof. We have to verify the assumptions of Theorem 5.4. The assumptions (5.3), (5.28), (2.11), and (5.30) are clearly satisfied. We define as before

$$\bar{S}(y) := \sum_{i=1}^{\ell} \frac{f}{f_i}(y)\left[\omega' \cdot D_i + f_i'(y) \bigoplus_{p\geq 0}{}_i \left(\frac{n_i}{2} - p\right)\right]$$
$$= \sum_{i=1}^{\ell} f(y)[S_{i0}(y) + S_{i1}(y)]$$

such that

$$S(y) = \frac{1}{f}\bar{S}(y).$$

Then

$$\bar{S}(y)Q = \sum_{i=1}^{\ell} f\frac{f_i'}{f_i}(y) \bigoplus_{p\geq 0}{}_i \left(\frac{n_i}{2} - p\right)Q; \tag{5.58}$$

so (2.9) follows from (5.55b) if $C_8 n < 1$.

Next, we estimate for $\eta \in C^\infty(\mathscr{C}\ell(N))$, using (5.53),

$$\begin{aligned}\langle \bar{S}(y)^2\eta, \eta\rangle &\geq f(y)^2\left[\frac{1}{2}\sum_{i=1}^{\ell} \langle S_{0i}^2(y)\eta, \eta\rangle - C\left(\frac{f_i'}{f_i}(y)\right)^2\langle \eta, \eta\rangle\right] \\ &= \sum_{i=1}^{\ell}\left[\frac{1}{2}\left(\frac{f}{f_i}(y)\right)^2\langle D_i\eta, D_i\eta\rangle - C\left(f\frac{f_i'}{f_i}(y)\right)^2\langle \eta, \eta\rangle\right].\end{aligned} \tag{5.59}$$

Expanding η in products of eigenforms of the Dirac operators D_i on N_i, we easily see that

$$\langle D_i\eta, D_i\eta\rangle \geq \lambda\langle \eta, \eta\rangle \qquad \text{for some } i$$

if $Q\eta = 0$; here,

$$\lambda = \min_{1\leq i\leq \ell} (\operatorname{spec} D_i^2\backslash\{0\}) > 0.$$

Hence, we obtain (2.10) if $\lambda C_7 > 1$.

It remains to check the condition (2.13a) on the bounded variation. We compute, with $\bar{S}_{i0}(y) := f(y)S_{i0}(y)$,

$$\bar{S}'(y) = \sum_{i=1}^{\ell}\left(\frac{f}{f_i}\right)'\frac{f_i}{f}(y)\bar{S}_{i0}(y) + \sum_{i=1}^{\ell}\left(\frac{f}{f_i}\right)'\frac{f_i}{f}(y)\left(f\frac{f_i'}{f_i}(y) \bigoplus_{p\geq 0}{}_i \left(\frac{n_i}{2} - p\right)\right)$$
$$+ \sum_{i=1}^{\ell}\frac{f}{f_i}f_i''(y) \bigoplus_{p\geq 0}{}_i \left(\frac{n_i}{2} - p\right)$$
$$=: I(y) + II(y) + III(y).$$

The condition (2.13a) is clearly satisfied for the terms $II(y)$ and $III(y)$ by assumptions (5.55c) and (5.55b).

To deal with $I(y)$, we observe that (5.53) implies

$$\left(\sum_{i=1}^{\ell} \bar{S}_{i0}(y)\right)^2 = \sum_{i=1}^{\ell} \bar{S}_{i0}(y)^2;$$

hence, we find

$$\sum_{i=1}^{\ell} \bar{S}_{i0}(y)^2 \leq C(\mathbf{I} + |\bar{S}(y)|^2).$$

To verify (2.13a), in view of (5.55c) it is therefore enough to estimate

$$\begin{aligned}
&\left\| I(y)\left(\mathbf{I} + \sum_{i=1}^{\ell} \bar{S}_{i0}(y)^2\right)^{-1/2} \right\|^2 \\
&= \left\| \left(\mathbf{I} + \sum_{i=1}^{\ell} \bar{S}_{i0}(y)^2\right)^{-1/2} \sum_{i=1}^{\ell} \left[\left(\frac{f}{f_i}\right)' \frac{f_i}{f}(y)\right]^2 \bar{S}_{i0}(y)^2 \left(\mathbf{I} + \sum_{i=1}^{\ell} \bar{S}_{i0}(y)^2\right)^{-1/2} \right\| \\
&\leq \sum_{i=1}^{\ell} \left[\frac{f'}{f}(y) - \frac{f_i'}{f_i}(y)\right]^2.
\end{aligned}$$

Thus, Theorem 5.4 applies and gives the formula (5.31) for L^2-sign M.

It remains to prove that

$$\operatorname{sign} QS(0)Q = 0. \tag{5.60}$$

Observe that this result will follow if we can find a bounded invertible operator $B\colon QH \to QH$ that anticommutes with $Q\bar{S}(y)$ in (5.58). Now an easy calculation shows that $B := *_N$ has this property. The proof is complete. □

Theorem 5.8 generalizes Theorem 3 in [St1] where all N_i are assumed to be flat and the warping functions are subject to further assumptions.

If, again, D_S is known to be Fredholm, then $h_1 = h_1' = 0$, from (5.46), $h_0 + h_0' = \dim Q$. The argument leading to (5.60) then shows that $h_0 = h_0'$; so we obtain again (5.43). We expect an additional contribution in the non-Fredholm case, but we have no example for the time being; note that this would need $0 < h_0 < \frac{1}{2}\dim Q$ and $h_1 \neq h_1'$.

The η-invariant of a Riemannian product can be easily evaluated; see [D].

Typically, if $f_i(y) = e^{-\alpha_i y}$, $0 < \alpha_1 \leq \cdots \leq \alpha_\ell$, then our assumptions are satisfied with $f(y) := Ce^{-\alpha_1 y}$, at least for large y. If we change f_1 to $f_1(y) \equiv 1$, however, then (5.55a) and (5.55c) cannot be satisfied together. Thus, we can deal essentially with rank-one situations only.

It is of some interest to single out the case $f_i = f_1$ for $1 \leq i \leq \ell$, i.e., the case of warped products. Special cases have been treated in [St1] and [B]. We obtain the following corollary.

COROLLARY 5.10. *Assume that* $f_1 = \cdots = f_\ell$ *in* (5.47b). *Moreover, assume* (5.56) *and assume that*

$$\lim_{y \to \infty} f_1'(y) = 0 \tag{5.61a}$$

and

$$\int_0^\infty f_1 f_1''(y)^2 \, dy < \infty . \tag{5.61b}$$

Then

$$L^2\text{-ind } D_S = \int_{M_1} L_k + \frac{1}{2}\eta(N) + \left(h_0 + h_1 - \frac{1}{2}\dim Q\right).$$

Note that the more general approach taken in this investigation yields a weaker result than [B, Corollary 5.4] if specialized to that case. In fact, condition (5.61a) does not imply (5.61b) as can be seen from the function

$$f_1(y) := \int_0^y t^{-\alpha}(2 + \sin t^2) \, dt, \qquad 0 < \alpha < 2/3.$$

If condition (5.61a) is replaced by (2.11) with $0 < a < 1$, then the situation is more complicated. Following the outline [B, Corollary 5.2], one can derive results also in this case.

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