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Hilbert Complexes

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A Hilbert complex is just a complex

$$0 \longrightarrow \mathscr{D}_0 \xrightarrow{D_0} \mathscr{D}_1 \xrightarrow{D_1} \cdots \xrightarrow{D_{N-1}} \mathscr{D}_N \longrightarrow 0,$$

where the D_j are closed operators between Hilbert spaces with domain $\mathscr{D}_j$ and $D_{j+1} \circ D_j = 0$. Although this is a fairly simple object, it reflects surprisingly much of the structure known from elliptic complexes on noncompact manifolds, the main application we have in mind. In this paper we undertake a systematic study of Hilbert complexes and their relationship with elliptic complexes. It turns out that this perspective gives a common structure to various known theorems along with generalizations and extensions. We apply the abstract machinery to the de Rham complex in several singular situations. © 1992 Academic Press, Inc.

1. INTRODUCTION

The purpose of this work is to advertise the notion of "Hilbert complex," to prove some abstract results, and to demonstrate their usefulness for global analysis on singular spaces. By a Hilbert complex we simply mean a (differential) complex formed with closed operators on Hilbert spaces. Though this notion has been implicit in the literature for some time, we do not know of any systematic treatment. It turns out that the functional analytic structure has interesting and useful consequences, mainly stemming from the possibility to use the spectral theorem. In Section 2, we begin to develop a general theory, largely in the spirit of homological algebra; special features (among others) are the notions of "Fredholm complex," "Poincaré duality," and "discreteness." We then examine the variation under complex isomorphisms of various invariants that can be associated to a Hilbert complex, like Betti numbers and Laplace spectrum. As a technically very convenient fact, we find that one can always define a "smooth" subcomplex with the same homology (Theorem 2.12).

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In Section 3 we apply the abstract notions to Hilbert complexes arising from elliptic complexes on arbitrary Riemannian manifolds, by choosing closed extensions in the respective L^2-spaces of the given differential operators. Any such choice we will call an "ideal boundary condition" (following Cheeger who apparently introduced this notion). The main feature here is that ideal boundary conditions are stable under quasi-isometries and, moreover, in many interesting cases more easy to classify than the closed extensions of the elliptic operator obtained by "rolling up" the complex; this fact became apparent already in Cheeger's fundamental work on conical singularities [C1, C3]. Elliptic complexes are in some sense a very simple application, however. A wealth of new examples arises from restricting an elliptic complex to a subcomplex, cf. Theorem 3.12.

Section 4, finally, examines the de Rham complex in various situations: on compact Riemannian manifolds equipped with Lipschitz metrics, on compact Riemannian manifolds with boundary, on compact Riemannian manifolds minus a "small" subset, and (its counterpart) for Riemannian foliations. Here we put the emphasis on the common perspective of all these "singular" complexes, introduced by our abstract framework. Thus, among other things, we give a simple and natural characterization of the so-called absolute and relative boundary conditions for the Gauß–Bonnet operator on a manifold with boundary, and we show that in this case one can always find an ideal boundary condition satisfying the Poincaré duality (which is not Fredholm, however). Many other interesting cases exist which we want to deal with in the future.

2. HILBERT AND FREDHOLM COMPLEXES

We will work in the following abstract setting. Consider (mutually orthogonal) Hilbert spaces H_i, $0 \leqslant i \leqslant N$, $H_{N+1} := \{0\}$, and for each i a closed operator $D_i \in \mathscr{C}(H_i, H_{i+1})$, the set of all closed operators with domain in H_i and image in H_{i+1}. We put $\mathscr{D}_i := \mathscr{D}(D_i)$, the domain of D_i, $\mathscr{R}_i := D_i(\mathscr{D}_i)$, the range of D_i, and with D_i^* the adjoint operator, $\mathscr{D}_i^* := \mathscr{D}(D_i^*)$, $\mathscr{R}_i^* := D_i^*(\mathscr{D}_i^*)$. We then assume that

$$\mathscr{R}_i \subset \mathscr{D}_{i+1} \tag{2.1a}$$

and

$$D_{i+1} \circ D_i = 0. \tag{2.1b}$$

Thus we obtain a complex

$$0 \longrightarrow \mathscr{D}_0 \xrightarrow{D_0} \mathscr{D}_1 \xrightarrow{D_1} \cdots \xrightarrow{D_{N-1}} \mathscr{D}_N \longrightarrow 0 \tag{2.2}$$

in the sense of the homological algebra, but with additional functional analytic structure, which we refer to as a *Hilbert complex*. If, moreover, the homology of this complex is finite, i.e., if the spaces

$$\mathscr{H}_i := \ker D_i / \operatorname{im} D_{i-1} \tag{2.3}$$

are all finite dimensional and if $\mathscr{R}_i$ is closed for all i, then we call the complex a *Fredholm complex*. We will abbreviate the complex (2.2) as $(\mathscr{D}, D)$; we put

$$\beta_i := \dim \mathscr{H}_i, \tag{2.4a}$$

(such that $0 \leqslant \beta_i \leqslant \infty$) and, in the Fredholm case,

$$\operatorname{ind}(\mathscr{D}, D) := \sum_{i=0}^{N} (-1)^i \beta_i, \tag{2.4b}$$

and we call these quantities the *geometric Betti numbers* and the *geometric index* of the complex $(\mathscr{D}, D)$, respectively.

For each Hilbert complex $(\mathscr{D}, D)$ we can introduce a dual complex, $(\mathscr{D}^*, D^*)$, as follows. It is immediate from (2.1) that the adjoint operators satisfy

$$\mathscr{R}_i^* \subset \mathscr{D}_{i-1}^* \tag{2.5a}$$

and

$$D_{i-1}^* \circ D_i^* = 0 \tag{2.5b}$$

so $(\mathscr{D}^*, D^*)$ is the complex

$$0 \longleftarrow \mathscr{D}_{-1}^* \xleftarrow{D_0^*} \mathscr{D}_0^* \xleftarrow{D_1^*} \cdots \xleftarrow{D_{N-1}^*} \mathscr{D}_{N-1}^* \longleftarrow 0. \tag{2.6}$$

The ith homology group of the dual complex is

$$\mathscr{H}_i^* = \ker D_{N-i-1}^* / \operatorname{im} D_{N-i}^*. \tag{2.3*}$$

Maps are defined in the obvious way: if $(\mathscr{D}, D)$ and $(\mathscr{D}', D')$ are Hilbert complexes and $g_i \colon H_i \to H_i'$ is a bounded linear map for each i with

$$g_i(D_i) \subset D_i', \tag{2.7a}$$

$$D_i' \circ g_i = g_{i+1} \circ D_i, \tag{2.7b}$$

then $g := \bigoplus g_i \colon (\mathscr{D}, D) \to (\mathscr{D}', D')$ is called a *map of Hilbert complexes*. The induced map on homology will be denoted by g_*. It is readily seen that the dual map $g^* = \bigoplus g_i^*$ defines a complex map $(\mathscr{D}'^*, D'^*) \to (\mathscr{D}^*, D^*)$.

Each Hilbert complex $(\mathscr{D}, D)$ defines a natural orthogonal decomposition on each Hilbert space H_i which we will refer to as the *weak Hodge decomposition*. To describe it we introduce

$$\hat{\mathscr{H}}_i := \ker D_i \cap \ker D_{i-1}^*, \qquad 0 \leqslant i \leqslant N. \tag{2.8a}$$

For the dual complex we obviously have

$$\hat{\mathscr{H}}_i^* = \hat{\mathscr{H}}_{N-i}. \tag{2.8b}$$

LEMMA 2.1. *Let $(\mathscr{D}, D)$ be a Hilbert complex. Then for each i we have an orthogonal decomposition*

$$H_i = \hat{\mathscr{H}}_i \oplus \bar{\mathscr{R}}_{i-1} \oplus \overline{\mathscr{R}_i^*}. \tag{2.9}$$

Proof. Note first that $\ker D_i$ is closed in H_i since D_i is closed. Thus we can decompose

$$\begin{aligned} H_i &= (\ker D_i)^\perp \oplus \ker D_i \\ &= (\ker D_i)^\perp \oplus \bar{\mathscr{R}}_{i-1} \oplus \ker D_i \cap \mathscr{R}_{i-1}^\perp. \end{aligned} \tag{2.10}$$

Now for any closed operator $D \in \mathscr{C}(H, H')$, we have the relation

$$(\ker D)^\perp = \overline{\mathscr{R}(D^*)}. \tag{2.11}$$

Applying this to (2.10) the proof is completed. ∎

To deal with the Fredholm properties of a Hilbert complex $(\mathscr{D}, D)$ it is enough to study a single closed operator, D, defined as follows. Put

$$H_{\text{ev}} := \bigoplus_{i \geqslant 0} H_{2i}, \qquad H_{\text{odd}} := \bigoplus_{i \geqslant 0} H_{2i+1},$$

$$\mathscr{D} := \bigoplus_{i \geqslant 0} \mathscr{D}_{2i} \cap \mathscr{D}_{2i-1}^* \subset H_{\text{ev}},$$

and define $D \in \mathscr{C}(H_{\text{ev}}, H_{\text{odd}})$ by

$$\begin{aligned} Du &:= (D_0 u_0 + D_1^* u_2, D_2 u_2 + D_3^* u_4, \ldots) \in H_{\text{odd}}, \\ u &= (u_0, u_2, \ldots) \in \mathscr{D}. \end{aligned} \tag{2.12}$$

From the complex property we derive

$$\|Du\|_{H_{\text{odd}}}^2 = \sum_{i \geqslant 0} (\|D_{2i} u_{2i}\|_{H_{2i+1}}^2 + \|D_{2i+1}^* u_{2i+2}\|_{H_{2i+1}}^2), \tag{2.13}$$

which shows that D is in fact closed with domain $\mathscr{D}$. Hence we can associate with D three self-adjoint nonnegative operators, namely

$$\Delta_{\mathrm{ev}} := D^*D, \qquad \Delta_{\mathrm{odd}} := DD^*, \qquad \Delta := \Delta_{\mathrm{ev}} \oplus \Delta_{\mathrm{odd}}. \tag{2.14a}$$

Note that $D \oplus D^*$ is also self-adjoint and that

$$\Delta = (D \oplus D^*)^2.$$

Yet another family of self-adjoint operators is of interest: we write

$$\Delta = \bigoplus_{i\geqslant 1} D_{i-1} D^*_{i-1} \oplus \bigoplus_{i \geqslant 0} D^*_i D_i =: \bigoplus_{i\geqslant 1} \Delta^1_i \oplus \bigoplus_{i\geqslant 0} \Delta^2_i =: \Delta^1 \oplus \Delta^2. \tag{2.14b}$$

Then the weak Hodge decomposition can be interpreted in terms of the Δ^j as follows (recall that the *support*, $s(T)$, of a self-adjoint operator T is defined by $s(T) := (\ker T)^{\perp}$).

Lemma 2.2. *We have*

$$\hat{\mathscr{H}}_i = \ker \Delta \cap H_i = \ker \Delta^1 \cap \ker \Delta^2 \cap H_i,$$
$$\overline{\mathscr{R}_{i-1}} = s(\Delta^1) \cap H_i,$$
$$\overline{\mathscr{R}^*_i} = s(\Delta^2) \cap H_i.$$

Moreover, $s(\Delta^j)$ *reduces* Δ *and* $\Delta|_{s(\Delta^j)} = \Delta^j$, $j = 1, 2$.

Proof. This is obvious in view of (2.12) and $\ker \Delta^1 \cap H_i = \ker D^*_{i-1}$, $\ker \Delta^2 \cap H_i = \ker D_i$. ∎

We observe next that, in fact, no information is lost by going from the Hilbert complex to the closed operator D, obtained by "rolling up" the complex.

Lemma 2.3. *The Hilbert complex* $(\mathscr{D}, D)$ *can be reconstructed from the closed operator* D *defined in* (2.12).

Proof. We have

$$\mathscr{D}_{2i} \cap \mathscr{D}^*_{2i-1} = \mathscr{D}(D) \cap H_{2i}, \tag{2.15a}$$

$$\mathscr{D}_{2i+1} \cap \mathscr{D}^*_{2i} = \mathscr{D}(D^*) \cap H_{2i+1}. \tag{2.15b}$$

We claim next that $\mathscr{D}_{2i} \cap \mathscr{D}^*_{2i-1}$ is dense in $\mathscr{D}_{2i}$ with respect to the graph

norm of D_{2i}. This can be shown directly but a very easy proof will follow from the discussion after Lemma 2.11 below: $\mathscr{E}_i$ in (2.31) is a core for D_i and D^*_{i-1}. If P_i denotes the orthogonal projection in H onto H_i we thus obtain

$$D_{2i} = \text{closure of } P_{2i+1} \circ (D|_{\mathscr{D}(D) \cap H_{2i}}). \tag{2.16a}$$

A very similar argument gives

$$D^*_{2i-1} = \text{closure of } P_{2i} \circ (D^*|_{\mathscr{D}(D^*) \cap H_{2i}}). \quad ∎ \tag{2.16b}$$

We turn to a description of the Fredholm property.

Theorem 2.4. *Let* $(\mathscr{D}, D)$ *be a Hilbert complex. The following conditions are equivalent.*

(1) $(\mathscr{D}, D)$ *is a Fredholm complex.*

(2) $\dim \mathscr{H}_i < \infty$ *for all* i.

(3) D *is a Fredholm operator.*

(4) *Denoting by* spec_e *the essential spectrum, we have* $0 \notin \operatorname{spec}_e \Delta$.

If any of these conditions is satisfied, we have

$$\operatorname{ind} D = \operatorname{ind}(\mathscr{D}, D).$$

Proof. $(1) \Rightarrow (2)$. This follows from the definition.

$(2) \Rightarrow (3)$. If $\mathscr{H}_i$ has finite dimension for all i then $\mathscr{R}_{i-1}$ is closed in $\ker D_i$ hence closed in H_i. Thus we obtain from (2.9) and the closed range theorem

$$H_i = \hat{\mathscr{H}}_i \oplus \mathscr{R}_{i-1} \oplus \mathscr{R}^*_i. \tag{2.17}$$

Since $\ker D_i = \hat{\mathscr{H}}_i \oplus \mathscr{R}_{i-1}$ we conclude

$$\mathscr{H}_i \simeq \hat{\mathscr{H}}_i. \tag{2.18}$$

From (2.13) we infer that

$$\ker D = \bigoplus_{i \geqslant 0} \hat{\mathscr{H}}_{2i} \tag{2.19}$$

hence $\ker D$ is finite dimensional.

Since $\mathscr{R}_i$ is closed for all i, the decompositions (2.15) imply that the operators

$$D_{2i} : \mathscr{R}^*_{2i} \cap \mathscr{D}_{2i} \to \mathscr{R}_{2i},$$
$$D^*_{2i-1} : \mathscr{R}_{2i-1} \cap \mathscr{D}^*_{2i-1} \to \mathscr{R}^*_{2i-1},$$

are bijective. On the other hand, the weak Hodge decomposition implies

$$\operatorname{im} D = \bigoplus_{i \geqslant 0} (\mathscr{R}_{2i} \oplus \mathscr{R}_{2i+1}^*), \tag{2.20}$$

so im D is closed.

To complete the proof, it is enough to show that D^* has finite dimensional kernel. An easy calculation gives

$$D^*: \bigoplus_{i \geqslant 0} \mathscr{D}_{2i+1} \cap \mathscr{D}_{2i}^* \to H_{\mathrm{ev}},$$
$$u = (u_1, u_3, \ldots) \mapsto (D_0^* u_1, D_1 u_1 + D_2^* u_3, \ldots).$$

Hence the analogue of (2.19) is

$$\ker D^* = \bigoplus_{i \geqslant 0} \hat{\mathscr{H}}_{2i+1}, \tag{2.21}$$

which is finite dimensional by (2.18).

(3) $\Rightarrow$ (1). If D is Fredholm then (2.20) and its analogue for D^* show that $\mathscr{R}_i$ and $\mathscr{R}_i^*$ are closed for all i. From (2.19) and (2.21) we see that $\hat{\mathscr{H}}_i$ is finite dimensional for each i. As before we then reach (2.17) and (2.18).

(3) $\Leftrightarrow$ (4). This is a well known fact. ∎

As a corollary we single out the *strong Hodge decomposition* (2.17).

COROLLARY 2.5. *If* $(\mathscr{D}, D)$ *is Fredholm then we have the decomposition*

$$H_i = \hat{\mathscr{H}}_i \oplus \mathscr{R}_{i-1} \oplus \mathscr{R}_i^*.$$

Moreover,

$$\mathscr{H}_i \simeq \hat{\mathscr{H}}_i$$

and for the geometric index we have

$$\operatorname{ind}(\mathscr{D}, D) = \sum_{i \geqslant 0} (-1)^i \dim \hat{\mathscr{H}}_i. \tag{2.22}$$

Note that (2.17) holds whenever $\mathscr{R}_i$ is closed for all $i \geqslant 0$. As another consequence we note

COROLLARY 2.6. *The Hilbert complex* $(\mathscr{D}, D)$ *is Fredholm if and only if the dual complex* $(\mathscr{D}^*, D^*)$ *is. In this case we have*

$$\hat{\mathscr{H}}_i^* \simeq \mathscr{H}_i^* \simeq \mathscr{H}_{N-i} \simeq \hat{\mathscr{H}}_{N-i}. \tag{2.23a}$$

Moreover, if D *and* $\hat{D}$ *denote the operators defined by* (2.12) *for* $(\mathscr{D}, D)$ *and* $(\mathscr{D}^*, D^*)$, *respectively, then we have*

$$\hat{D} = \begin{cases} D, & N \text{ even}, \\ D^*, & N \text{ odd}, \end{cases} \tag{2.23b}$$

and

$$\operatorname{ind} \hat{D} = \operatorname{ind}(\mathscr{D}^*, D^*) = (-1)^N \operatorname{ind} D = (-1)^N \operatorname{ind}(\mathscr{D}, D). \tag{2.23c}$$

Proof. If $(\mathscr{D}, D)$ is Fredholm then (2.3*) and the Hodge decomposition imply $\mathscr{H}_i^* \simeq \mathscr{H}_{N-i}$. Thus $(\mathscr{D}^*, D^*)$ is Fredholm by (2.19), (2.21), and Theorem 2.4, 2. The converse follows from $(\mathscr{D}^{**}, D^{**}) = (\mathscr{D}, D)$. Now (2.23a) is a consequence of (2.8b).

The remaining statements follow from straightforward applications of the definitions. ∎

We observe next that complex maps also induce maps between the spaces $\hat{\mathscr{H}}_i$.

LEMMA 2.7. *Let* $g: (\mathscr{D}, D) \to (\mathscr{D}', D')$ *be a map of Hilbert complexes, and denote by* $\hat{P}_i^{(\prime)}$, $P_i^{j(\prime)}$ *the orthogonal projections in* $H_i^{(\prime)}$ *onto* $\hat{\mathscr{H}}_i^{(\prime)}$ *and* $s(\Delta^{j(\prime)}) \cap H_i^{(\prime)}$, $j = 1, 2$. *Then* g *induces functorial homomorphisms for all* i,

$$\hat{g}_i := \hat{P}_i' \circ g_i|_{\hat{\mathscr{H}}_i}: \hat{\mathscr{H}}_i \to \hat{\mathscr{H}}'_i,$$
$$g_i^1 := g_i|_{s(\Delta^1) \cap H_i}: \overline{\mathscr{R}_{i-1}} \to \overline{\mathscr{R}'_{i-1}},$$
$$g_i^2 := P_i^{2\prime} \circ g_i|_{s(\Delta^2) \cap H_i}: \overline{\mathscr{R}_i^*} \to \overline{\mathscr{R}_i'^*}.$$

If g_i *is unitary then so are* $\hat{g}_i$, g_i^1, *and* g_i^2.

Proof. The assertion concerning g_i^1 is obvious from the definition. For the functoriality of $\hat{g}_i$ we have to show that with $h: (\mathscr{D}', D') \to (\mathscr{D}'', D'')$ a second complex map it follows that $\hat{h}_i \circ \hat{g}_i = \widehat{(h \circ g)}_i$. But this is an easy consequence of the fact that g_i maps $\ker D_i$ to $\ker D_i'$ and $\overline{\mathscr{R}_{i-1}} = s(\Delta_i^1)$ to $\overline{\mathscr{R}'_{i-1}}$. Finally, if g_i is unitary it respects orthogonality, and $g_i(\overline{\mathscr{R}_{i-1}}) = \overline{\mathscr{R}'_{i-1}}$. The proof for g_i^2 is very similar. ∎

In particular, the dimension of $\hat{\mathscr{H}}_i$ is invariant under complex isomorphisms. In case that all $\hat{\mathscr{H}}_i$ have finite dimension we call $(\mathscr{D}, D)$ a *weak Fredholm complex*. The numbers

$$\hat{\beta}_i := \dim \hat{\mathscr{H}}_i \tag{2.24}$$

are called the *analytic Betti numbers of* $(\mathscr{D}, D)$, $0 \leqslant \hat{\beta}_i \leqslant \infty$. In the weak Fredholm case, we define the *analytic index of* $(\mathscr{D}, D)$ as

$$\widehat{\operatorname{ind}}(\mathscr{D}, D) := \sum_{i \geqslant 0} (-1)^i \hat{\beta}_i$$
$$= \dim \ker D - \dim \ker D^*, \tag{2.25}$$

where we have used (2.19), (2.21). Note that D need not be Fredholm for the analytic index to be well defined.

It is often necessary to compare the homologies of $(\mathscr{D}, D)$ and $(\mathscr{D}', D')$. This can be done by means of a homotopy operator.

Definition 2.8. Let $g, h: (\mathscr{D}, D) \to (\mathscr{D}', D')$ be maps of Hilbert complexes. A *homotopy operator for* g *and* h is a collection of linear maps, A_i,

$$A_i: \mathscr{D}_i \to \mathscr{D}'_{i-1} \tag{2.26}$$

such that on $\mathscr{D}_i$

$$g_i - h_i = D'_{i-1} A_i + A_{i+1} D_i. \tag{2.27}$$

From (2.27) we obtain for $x \in \ker D_i$

$$g_i(x) = h_i(x) + D'_{i-1}(A_i x).$$

Thus we have

Lemma 2.9. *Given a homotopy operator, then the induced maps coincide on homology,* $g_* = h_*$. *We also have* $\hat{g} = \hat{h}$.

The following observation is useful in reducing homology computations to subcomplexes. Suppose we are given a Hilbert complex $(\mathscr{D}, D)$ and a complex map $h: (\mathscr{D}, D) \to (\mathscr{D}, D)$ which is homotopic to the identity; i.e., we have linear maps $A_i: \mathscr{D}_i \to \mathscr{D}_{i-1}$ such that

$$h_i = \mathrm{id}_{\mathscr{D}_i} + D_{i-1} A_i + A_{i+1} D_i. \tag{2.28}$$

Now if $(\mathscr{D}^0, D)$ is a *linear* subcomplex of $(\mathscr{D}, D)$ such that for all i

$$h_i(\mathscr{D}_i) \subset \mathscr{D}_i^0 \tag{2.29a}$$

and, moreover,

$$A_i(\mathscr{D}_i^0) \subset \mathscr{D}_{i-1}^0, \tag{2.29b}$$

then the homologies coincide. More precisely, if $j: (\mathscr{D}^0, D) \to (\mathscr{D}, D)$

denotes the natural inclusion then h induces a complex map $k: (\mathscr{D}, D) \to (\mathscr{D}^0, D)$ with $j \circ k = h$. Thus (2.28) and (2.29) simply mean that

$$j_* \circ k_* = \mathrm{id}_{\mathscr{H}}, \qquad k_* \circ j_* = \mathrm{id}_{\mathscr{H}^0}.$$

If $(\mathscr{D}^0, D)$ is a Hilbert subcomplex then it is easily seen that $\hat{j} \circ \hat{k} = \mathrm{id}_{\hat{\mathscr{H}}}$, $\hat{k} \circ \hat{j} = \mathrm{id}_{\hat{\mathscr{H}}^0}$, and

$$\hat{j}: \hat{\mathscr{H}}^0 \subsetneq \hat{\mathscr{H}} \text{ is the natural inclusion,}$$
$$\hat{k}: \hat{\mathscr{H}} \to \hat{\mathscr{H}}^0 \text{ is the orthogonal projection.}$$

Hence $\hat{\mathscr{H}}^0 = \hat{\mathscr{H}}$. Thus we have

Lemma 2.10. *Let* $(\mathscr{D}^0, D)$ *be a linear subcomplex of the Hilbert complex* $(\mathscr{D}, D)$ *and* $h: (\mathscr{D}, D) \to (\mathscr{D}, D)$ *a map of Hilbert complexes with the properties* (2.28) *and* (2.29). *Then* h_* *induces an isomorphism* $\mathscr{H}^0 \simeq \mathscr{H}$.

If, moreover, $(\mathscr{D}^0, D)$ *is a Hilbert subcomplex then* $\hat{\mathscr{H}}^0 = \hat{\mathscr{H}}$.

As an application of this construction we are going to associate to each Hilbert complex $(\mathscr{D}, D)$ a *smooth subcomplex* $(\mathscr{D}^\infty, D)$. The terminology is justified by the fact that $(\mathscr{D}^\infty, D)$ consists of smooth sections if $(\mathscr{D}, D)$ is generated by an elliptic complex, cf. Section 3.

In preparation, we need the following result on cores. (Recall that a linear subspace $\mathscr{E} \subset \mathscr{D}(D)$ is called a *core* for the closed operator $D \in \mathscr{C}(H, H')$ if $\mathscr{E}$ is dense in $\mathscr{D}(D)$ with respect to the graph norm.)

Lemma 2.11. *Let* $(\mathscr{D}, D)$ *be a Hilbert complex with Laplacian* Δ. *If* $\mathscr{E} = \bigoplus_{i \geqslant 0} \mathscr{E}_i$ *is a core for* Δ *then*

$$\mathscr{F}_i := \hat{\mathscr{H}}_i \oplus D_{i-1} \mathscr{E}_{i-1} \oplus D_i^* \mathscr{E}_{i+1} \tag{2.30}$$

is a core for D_i, *for all* $i \geqslant 0$.

Proof. By construction, $\mathscr{E}_i \cup \mathscr{F}_i \subset \mathscr{D}_i \cap \mathscr{D}_{i-1}^*$. Let $x \in \mathscr{D}_i$ be orthogonal to $\mathscr{F}_i$ with respect to the graph scalar product. Then we find for $y \in \mathscr{E}_{i+1}$

$$0 = (x, D_i^* y) + (D_i x, D_i D_i^* y)$$
$$= (D_i x, (\Delta_{i+1} + \mathrm{id}_{H_{i+1}}) y),$$

hence $D_i x = 0$ since $\mathscr{E}_{i+1}$ is a core for Δ_{i+1}.

Similarly, we find for $y \in \mathscr{E}_{i-1}$

$$0 = (x, D_{i-1} y) = (D_{i-1}^* x, y),$$

hence also $D_{i-1}^* x = 0$, i.e., $x \in \hat{\mathscr{H}}_i$. Thus $x = 0$. ∎

A convenient choice for $\mathscr{E}$ is

$$\mathscr{E}=\bigcap_{k\geqslant 1}\mathscr{D}(\Delta^k)=:\mathscr{D}^\infty. \tag{2.31}$$

This is a core for $\Delta=:\int_0^\infty \lambda\, dE_\lambda$, since for $x\in\mathscr{D}(\Delta)$

$$\lim_{n\to\infty}\int_0^n dE_\lambda x=:\lim_{n\to\infty} x_n=x$$

in the graph norm, and $x_n\in\mathscr{D}(\Delta^k)$ for all $k, n\in\mathbb{N}$.
Hence, by Lemma 2.11,

$$\mathscr{F}_i=\hat{\mathscr{H}}_i\oplus D_{i-1}\mathscr{E}_{i-1}\oplus D_i^*\mathscr{E}_{i+1}$$

is a core for D_i. Now we observe that for $x\in\mathscr{E}_i$ and $k\in\mathbb{N}$

$$D_{i-1}^*\Delta_i^k x=\Delta_{i-1}^k D_{i-1}^* x, \tag{2.32a}$$

$$\Delta_{i+1}^k D_i x=D_i\Delta_i^k x. \tag{2.32b}$$

Hence we conclude

$$\mathscr{F}_i\subset\mathscr{E}_i, \tag{2.33}$$

$$D_i\mathscr{E}_i\subset\mathscr{E}_{i+1}. \tag{2.34}$$

Thus also $\mathscr{E}_i=\mathscr{D}_i^\infty$ is a core for D_i, and we see that $(\mathscr{D}^\infty, D)$ is a subcomplex of $(\mathscr{D}, D)$, the smooth subcomplex referred to above.

THEOREM 2.12. *The natural inclusion*

$$j:(\mathscr{D}^\infty, D)\to(\mathscr{D}, D)$$

induces an isomorphism on homology.

Proof. According to Lemma 2.10 we want to construct a complex map, h, homotopic to the identity in the sense of (2.28) and (2.29). To do so we pick $\varphi\in C_0^\infty(\mathbb{R})$ with $\varphi=1$ near 0 and define

$$h:=-(\Delta+\mathrm{id})^{-1}\int_0^\infty \varphi'(t)\, e^{-t\Delta}\, dt. \tag{2.35}$$

Then, clearly, h maps H_i into $\mathscr{E}_i$. Moreover, (2.32) implies that

$$D_i h_i x=h_{i+1}D_i x,\qquad x\in\mathscr{E}_i,$$

hence h is a complex map since $\mathscr{E}_i$ is a core for D_i. Next we note that

$$\tilde{h}:=-(\Delta+\mathrm{id})^{-1}\int_0^\infty \varphi(t)\, e^{-t\Delta}\, dt$$

maps H_i into $\mathscr{D}(\Delta_i)$. We compute for $x\in\mathscr{E}_i$

$$\begin{aligned}\Delta\tilde{h}x&=(\Delta+\mathrm{id})^{-1}\int_0^\infty \varphi(t)\frac{\partial}{\partial t}e^{-t\Delta}x\, dt\\ &=-(\Delta+\mathrm{id})^{-1}x+hx\\ &=hx-x+\Delta(\Delta+\mathrm{id})^{-1}x,\end{aligned}$$

or, using again (2.32),

$$\begin{aligned}h_i x=x&+D_{i-1}D_{i-1}^*(\tilde{h}_i-(\Delta_i+\mathrm{id})^{-1})x\\ &+D_i^*(\tilde{h}_{i+1}-(\Delta_{i+1}+\mathrm{id})^{-1})D_i x.\end{aligned} \tag{2.36}$$

Thus we are led to the homotopy operator

$$A_i:=-D_{i-1}^*(\Delta_i+\mathrm{id})^{-1}\left(\mathrm{id}+\int_0^\infty \varphi(t)\, e^{-t\Delta}\, dt\right). \tag{2.37}$$

It is clear that (2.36) holds for $x\in H_i$, and that A_i maps $\mathscr{D}_i$ to $\mathscr{D}_{i-1}$ and $\mathscr{E}_i$ to $\mathscr{E}_{i-1}$. Thus the assertion follows from Lemma 2.10. ∎

Since Hilbert complexes are chain complexes we can expect that there is a product theory and a Künneth Theorem. In the special case of L^2-cohomology on singular spaces (with simple singularities like warped products) a Künneth Theorem has been proved by Zucker [Z], whose method has been the model for our approach. Apparently, this was also known to Cheeger who proves a Künneth Theorem for Riemannian pseudomanifolds in [C1].

The main technical difficulty is that tensor products of Hilbert spaces are completions of algebraic tensor products, and tensor products of closed operators need not be closed. In the sequel, we denote by $\otimes$ the algebraic and by $\hat{\otimes}$ the Hilbert space tensor product.

Consider two Hilbert complexes $(\mathscr{D}', D')$, $(\mathscr{D}'', D'')$ with $\mathscr{D}_i'\subset H_i'$, $\mathscr{D}_i''\subset H_i''$, H_i', H_i'' Hilbert spaces. We define the new Hilbert spaces

$$H_i:=\bigoplus_{k+l=i}H_k'\hat{\otimes}H_l'',\qquad H_i^0:=\bigoplus_{k+l=i}H_k'\hat{\otimes}\mathscr{H}_l'', \tag{2.38a}$$

and subspaces

$$\tilde{\mathscr{D}}_i:=\bigoplus_{k+l=i}(\mathscr{D}_k'\otimes H_l''\cap H_k'\otimes\mathscr{D}_l'')\subset H_i. \tag{2.38b}$$

Put

$$\sigma_i\colon H_i \to H_i, \qquad H'_k \otimes H''_l \ni x \otimes y \mapsto (-1)^k x \otimes y,$$
$$\tilde{D}_i := \bigoplus_{k+l=i} (D'_k \otimes \mathrm{id}_{H''_l} + \sigma_i \, \mathrm{id}_{H'_k} \otimes D''_l) \tag{2.38c}$$

and let $D_i := \bar{\tilde{D}}_i$ be the closure of $\tilde{D}_i$, with domain $\mathscr{D}_i$. Moreover, let

$$\mathscr{D}_i^0 := \mathscr{D}_i \cap H_i^0, \qquad D_i^0 := D_i|_{\mathscr{D}_i^0}. \tag{2.38d}$$

$(\mathscr{D}, D)$ and $(\mathscr{D}^0, D^0)$ are Hilbert complexes, and $(\mathscr{D}^0, D^0)$ is a Hilbert subcomplex of $(\mathscr{D}, D)$; this is proved as Lemma 3.1 below.

Denote by P_i the orthogonal projection in H_i onto H_i^0. We have

$$P_i = \bigoplus_{k+l=i} \mathrm{id}_{H'_k} \hat{\otimes} P''_l, \tag{2.39}$$

where P''_l is orthogonal projection in H''_l onto $\mathscr{H}''_l$. If we endow $\mathscr{D}'_k$ with the Hilbert space structure induced by the graph norm $\|x\|^2_{D'_k} := \|x\|^2 + \|D'_k x\|^2$ it is easy to check that

$$\mathscr{D}_i^0 = \bigoplus_{k+l=i} \mathscr{D}'_k \hat{\otimes} \mathscr{H}''_l, \tag{2.40a}$$

$$D_i^0 = \bigoplus_{k+l=i} D'_k \hat{\otimes} \mathrm{id}, \tag{2.40b}$$

and

$$\ker D_i^0 = \bigoplus_{k+l=i} \ker D'_k \hat{\otimes} \mathscr{H}''_l. \tag{2.40c}$$

LEMMA 2.13. *$\bigoplus_i P_i$ is a map of Hilbert complexes $(\mathscr{D}, D) \to (\mathscr{D}^0, D^0)$.*

Proof. We have to show that

$$P_i(\mathscr{D}_i) \subset \mathscr{D}_i^0,$$

and

$$D_i^0 P_i = P_{i+1} D_i \qquad \text{on } \mathscr{D}_i. \tag{2.41}$$

Note first that from (2.38b), (2.39), and (2.40a) we get

$$P_i(\tilde{\mathscr{D}}_i) \subset \tilde{\mathscr{D}}_i \cap H_i^0.$$

Next an easy computation using Lemma 2.1 shows that (2.41) holds on $\tilde{D}_i$. Since $\tilde{D}_i$ is dense in $\mathscr{D}_i$ with respect to the graph norm of D_i, the assertion follows. ∎

Now we assume in addition that D''_l has *closed range for all l*. This will allow us to construct a homotopy operator for P_i in the sense of Lemma 2.10. Let $B''_l\colon H''_l \to H''_{l-1}$ be given by

$$x \mapsto \begin{cases} 0, & \text{if } x \in \operatorname{im} D''_{l-1}{}^{\perp}, \\ (D''_{l-1}|_{\ker D''_{l-1}{}^{\perp}})^{-1} x, & \text{if } x \in \operatorname{im} D''_{l-1}, \end{cases} \tag{2.42a}$$

and

$$B_i := \bigoplus_{k+l=i} \mathrm{id}_{H'_k} \otimes B''_l. \tag{2.42b}$$

Thus B''_l is a generalized inverse for D''_l and B_i is a bounded operator $H_i \to H_{i-1}$. Obviously,

$$B_i(\tilde{\mathscr{D}}_i) \subset \tilde{\mathscr{D}}_{i-1}, \qquad B_i P_i|_{\tilde{\mathscr{D}}_i} = 0. \tag{2.43}$$

For $y \in \mathscr{D}''_l$ one computes that the Hodge decomposition is given by

$$y = P''_l y + D''_{l-1} B''_l y + B''_{l+1} D''_l y. \tag{2.44}$$

From this equation one easily derives the following representation for $u \in \tilde{\mathscr{D}}_i$,

$$u = P_i u + D_{i-1}(\sigma_{i-1} B_i u) + \sigma_i B_{i+1}(D_i u). \tag{2.45}$$

Defining $A_i := \sigma_{i-1} B_i\colon H_i \to H_{i-1}$ we can prove

THEOREM 2.14. *If D''_l has closed range for all l then P_i induces isomorphisms $P_{i*}\colon \mathscr{H}_i \to \mathscr{H}_i^0$. Moreover, $\hat{\mathscr{H}}_i = \hat{\mathscr{H}}_i^0$.*

Proof. To apply Lemma 2.10 we have to show that

$$A_i(\mathscr{D}_i) \subset \mathscr{D}_{i-1}, \qquad A_i(\mathscr{D}_i^0) \subset \mathscr{D}_{i-1}^0,$$

and that (2.45) holds for $u \in \mathscr{D}_i$. By definition, we find for $u \in \mathscr{D}_i$ a sequence $(u_n) \subset \tilde{\mathscr{D}}_i$ such that $u_n \to u$ and $D_i u_n \to D_i u$. From (2.43) and (2.45) we obtain that $A_i u_n \in \tilde{D}_{i-1}$ and that $D_{i-1} A_i u_n$ converges. Thus $A_i u \in \mathscr{D}_{i-1}$, and (2.45) holds for u. Moreover, we have $A_i P_i u = \lim_{n\to\infty} A_i P_i u_n = 0$, hence $A_i(\mathscr{D}_i^0) \subset \mathscr{D}_{i-1}^0$ and we are done. ∎

COROLLARY 2.15. *Assume that D''_l has closed range for all l.*

(1) *We have*

$$\hat{\mathscr{H}}_i = \bigoplus_{k+l=i} \hat{\mathscr{H}}'_k \hat{\otimes} \mathscr{H}''_l. \tag{2.46a}$$

(2) *If* $(\mathcal{D}'', D'')$ *is Fredholm then*

$$\mathcal{H}_i \simeq \bigoplus_{k+l=i} \mathcal{H}'_k \otimes \mathcal{H}''_l. \tag{2.46b}$$

(3) D_i *has closed range for all i if and only if* D'_k *has closed range for all k. In this case we have*

$$\mathcal{H}_i \simeq \bigoplus_{k+l=i} \mathcal{H}'_k \hat{\otimes} \mathcal{H}''_l. \tag{2.46c}$$

Here $\hat{\otimes}$ *is with respect to the natural Hilbert space structures on* $\mathcal{H}'_k$ *and* $\mathcal{H}''_l$.

Proof. From (2.40a)–(2.40c) we derive

$$\hat{\mathcal{H}}^0_i = \bigoplus_{k+l=i} \hat{\mathcal{H}}'_k \otimes \hat{\mathcal{H}}''_l. \tag{2.47}$$

From Theorem 2.14 we then obtain (1). If $(\mathcal{D}'', D'')$ is Fredholm, $\mathcal{H}''_l \simeq \hat{\mathcal{H}}''_l$ is finite dimensional and (2) is an immediate consequence of Theorem 2.14 and (2.40c).

(3) Since $D^0_i P_i = P_{i+1} D_i$ and because of (2.40b), we see that if D_i has closed range then D'_k has closed range, too. If D_k has closed range for all k we obtain from (2.40a)–(2.40c), $\mathcal{H}^0_i \simeq \hat{\mathcal{H}}^0_i$, and hence from (2.47), Theorem 2.14, and (1), $\hat{\mathcal{H}}_i \simeq \mathcal{H}_i$. Thus D_i has closed range by Lemma 2.1. Equation (2.46c) is now clear. ∎

Next we want to introduce the notion of *Poincaré duality* for Hilbert complexes. We say that a Hilbert complex $(\mathcal{D}, D)$ satisfies Poincaré duality if there is an isomorphism of Hilbert complexes, g, from $(\mathcal{D}, D)$ to the dual complex $(\mathcal{D}^*, D^*)$.

LEMMA 2.16. *The Hilbert complex* $(\mathcal{D}, D)$ *satisfies Poincaré duality if and only if there are invertible bounded linear maps* $g_i\colon H_i \to H_{N-i}$ *satisfying*

$$g_i(\mathcal{D}_i) = \mathcal{D}^*_{N-i-1}, \tag{2.48a}$$

$$D^*_{N-i-1} \circ g_i = g_{i+1} \circ D_i. \tag{2.48b}$$

g induces isomorphisms

$$\hat{g}_i\colon \hat{\mathcal{H}}_i \to \hat{\mathcal{H}}_{N-i}. \tag{2.49}$$

If, moreover, $(\mathcal{D}, D)$ *is Fredholm g induces also isomorphisms*

$$g_{i*}\colon \mathcal{H}_i \to \mathcal{H}_{N-i}.$$

Proof. The first assertion follows immediately from the definitions (2.6) and (2.7a), (2.7b). The second assertion is a consequence of Lemma 2.7 and (2.8b). The last assertion follows from Corollary 2.6. ∎

It is clear that, for a given Hilbert complex $(\mathcal{D}, D)$, the spectral properties of Δ provide very interesting invariants. Since in practice complex isomorphisms are a rather flexible tool it is useful to investigate how spectral properties of Δ transform. We present here only a rough comparison result for the Laplacians in case of a discrete spectrum which, however, already has some useful consequences. Certainly, subtle properties (like eigenvalues imbedded in the continuum) are not preserved under quasi-isometries but it seems that this question deserves further study.

To prepare our result, consider a complex isomorphism $g\colon (\mathcal{D}, D) \to (\mathcal{D}', D')$ and the corresponding map

$$\bigoplus_{i\geq 0} \hat{g}_i \oplus g^1_i \oplus g^2_i\colon H \to H', \tag{2.50}$$

which preserves the weak Hodge decomposition according to Lemma 2.7. Then observe that $h := (g^{-1})^*\colon (\mathcal{D}^*, D^*) \to (\mathcal{D}'^*, D'^*)$ is also a complex isomorphism. Hence we obtain maps

$$g^2_i\colon \overline{\mathcal{R}^*_i} \cap \mathcal{D}_i \to \overline{\mathcal{R}'^*_i} \cap \mathcal{D}'_i,$$
$$h^2_{N-i}\colon \overline{\mathcal{R}_{i-1}} \cap \mathcal{D}^*_{i-1} \to \overline{\mathcal{R}'_{i-1}} \cap \mathcal{D}'^*_{i-1}.$$

Now the bilinear form generated by $\Delta^{(\prime)}_i$ on $\mathcal{D}^{(\prime)}_i \cap \mathcal{D}^{(\prime)*}_{i-1}$ is simply

$$\begin{aligned} C^{(\prime)}_i(u, u) &:= \|D^{(\prime)}_i u\|^2 + \|D^{(\prime)*}_{i-1} u\|^2 \\ &= \|D^{(\prime)}_i u_2\|^2 + \|D^{(\prime)*}_{i-1} u_1\|^2, \end{aligned}$$

if $u = u_0 + u_1 + u_2 \in \hat{\mathcal{H}}^{(\prime)}_i \oplus \overline{\mathcal{R}^{(\prime)}_{i-1}} \cap \mathcal{D}^{(\prime)*}_{i-1} \oplus \overline{\mathcal{R}^{(\prime)*}_i} \cap \mathcal{D}^{(\prime)}_i$. But the map

$$k_i := \hat{g}_i \oplus h^2_{N-i} \oplus g^2_i\colon \mathcal{D}_i \cap \mathcal{D}^*_{i-1} \to \mathcal{D}'_i \cap \mathcal{D}'^*_{i-1}$$

is bijective; it follows that

$$\begin{aligned} C'_i(k_i(v), k_i(v)) &= \|D'_i g_i v_2\|^2 + \|D'^*_{i-1}(g_i^{-1})^* v_1\|^2 \\ &= \|g_{i+1} D_i v_2\|^2 + \|(g^{-1}_{i-1})^* D^*_{i-1} v_1\|^2, \end{aligned} \tag{2.51}$$

since g is a complex isomorphism. This situation calls for an application of the max-min-principle for which we have to assume, however, that Δ has a discrete spectrum; i.e., all spectral values are isolated eigenvalues of finite multiplicity, equivalently $\operatorname{spec}_e \Delta = \varnothing$. In this case we call the Hilbert complex $(\mathcal{D}, D)$ *discrete*. Thus we arrive at

LEMMA 2.17. *Discreteness is invariant under complex isomorphisms. More precisely, if* $(\mathscr{D}, D)$ *is a discrete Hilbert complex and* $g: (\mathscr{D}, D) \to (\mathscr{D}', D')$ *is a complex isomorphism then we have for the eigenvalues* λ_n, λ_n' *of* Δ *and* Δ'

$$C^{-1}\lambda_n \leqslant \lambda_n' \leqslant C\lambda_n, \qquad n \geqslant 1, \tag{2.52}$$

with some constant C independent of n.

As a useful consequence we note the invariance of trace estimates for the heat kernel.

COROLLARY 2.18. *Assume that the Laplacian* Δ *of* $(\mathscr{D}, D)$ *satisfies*

$$\operatorname{tr} e^{-t\Delta} \leqslant Ct^{\alpha}$$

for $0 < t \leqslant 1$. *If there is a complex isomorphism* $g: (\mathscr{D}, D) \to (\mathscr{D}', D')$, *then we have also*

$$\operatorname{tr} e^{-t\Delta'} \leqslant C't^{\alpha}, \qquad 0 < t \leqslant 1.$$

We summarize the more important invariance properties of Hilbert complexes as follows.

COROLLARY 2.19. *Let* $(\mathscr{D}, D)$ *be a Hilbert complex. Then any isomorphic Hilbert complex,* $(\mathscr{D}', D')$, *has the same analytic and geometric Betti numbers as* $(\mathscr{D}, D)$.

Also, $0 \in \operatorname{spec}_e \Delta'$ *or* $\operatorname{spec}_e \Delta' = \varnothing$ *iff the same property holds for* Δ.

3. ELLIPTIC COMPLEXES

Elliptic complexes on *arbitrary* Riemannian manifolds are, of course, the first application of the abstract results derived in the previous section. We will concentrate on these examples in this section since they already reveal some interesting phenomena. Thus we consider a Riemannian manifold M, hermitean vector bundles E_i over M, $0 \leqslant i \leqslant N$, and a family of differential operators, $d_i: C_0^\infty(E_i) \to C_0^\infty(E_{i+1})$, such that

$$0 \longrightarrow C_0^\infty(E_0) \xrightarrow{d_0} C_0^\infty(E_1) \xrightarrow{d_1} \cdots \xrightarrow{d_{N-1}} C_0^\infty(E_N) \longrightarrow 0 \tag{3.1}$$

is a complex. The complex is called *elliptic* if the associated symbol sequence

$$0 \longrightarrow \pi^*E_0 \xrightarrow{\sigma(d_0)} \pi^*E_1 \xrightarrow{\sigma(d_1)} \cdots \xrightarrow{\sigma(d_{N-1})} \pi^*E_N \longrightarrow 0 \tag{3.2}$$

is exact in each fiber; here $\pi: S^*M \to M$ denotes the natural projection from the cosphere bundle onto M. An elliptic complex will be denoted by $(C_0^\infty(E), d)$.

Our first task is to find Hilbert complexes associated to each elliptic complex. To do so we start from the natural Hilbert spaces

$$H_i := L^2(E_i),$$

and we recall that each operator d_i has a formal adjoint $d_i^t: C_0^\infty(E_{i+1}) \to C_0^\infty(E_i)$ which is a differential operator, too. Hence d_i has closed extensions in $\mathscr{C}(H_i, H_{i+1})$; they all lie between the closure or minimal extension, $d_{i,\min}$, and the maximal extension, $d_{i,\max}$, given by

$$d_{i,\max} := (d_{i,\min}^t)^*.$$

In particular, we can always form a second elliptic complex,

$$0 \longrightarrow C_0^\infty(E_N) \xrightarrow{d_{N-1}^t} C_0^\infty(E_{N-1}) \xrightarrow{d_{N-2}^t} \cdots \xrightarrow{d_0^t} C_0^\infty(E_0) \longrightarrow 0 \tag{3.3}$$

which we refer to as the *adjoint complex* to (3.1).

Any choice of closed extensions $d_{i,\min} \subset D_i \subset d_{i,\max}$ that produces a Hilbert complex $(\mathscr{D}, D)$ will be called an *ideal boundary condition.*

LEMMA 3.1. *Let* $(C_0^\infty(E), d)$ *be an elliptic complex. Then ideal boundary conditions exist. For example, if we put* $H_i = L^2(E_i)$ *and*

$$\mathscr{D}_i := \mathscr{D}(d_{i,\min}), \qquad D_i := d_{i,\min}, \tag{3.4}$$

or

$$\mathscr{D}_i := \mathscr{D}(d_{i,\max}), \qquad D_i := d_{i,\max}, \tag{3.5}$$

then $(\mathscr{D}, D)$ *becomes a Hilbert complex.*

Proof. Consider (3.4) first. We have to show that

$$D_i(\mathscr{D}_i) \subset \mathscr{D}_{i+1}, \qquad D_{i+1}D_i = 0.$$

For $u \in \mathscr{D}_i$ we can find a sequence $(u_n) \subset C_0^\infty(E_i)$ such that $u_n \to u$ in $L^2(E_i)$ and $d_i u_n \to D_i u$ in $L^2(E_{i+1})$. But then $d_{i+1}(d_i u_n) = 0$, thus $D_i u \in \ker D_{i+1}$.

For (3.5) we only have to note that the adjoint complex (3.3) generates a Hilbert complex, $(\mathscr{D}_{\min}^t, D_{\min}^t)$, with (3.4). The dual complex is the Hilbert complex satisfying (3.5). ∎

The main example of an elliptic complex on M is the de Rham complex, $(\Omega_0(M), d)$,

$$0 \longrightarrow \Omega_0^0(M) \xrightarrow{d_0} \cdots \xrightarrow{d_{m-1}} \Omega_0^m(M) \longrightarrow 0, \tag{3.6}$$

where $\Omega_0^p(M)$ denotes the smooth p-forms with compact support and d_p the exterior derivative. Then the Hilbert complex associated to the maximal extension of d, (3.5), defines the *L^2-cohomology of M*. In this case, the weak Hodge decomposition of Lemma 3.1 is due to Kodaira [K, Chap. II, Sect. 4]. It is, of course, of interest to relate the L^2-cohomology groups to the L^2-harmonic forms on M. Let us define, with $\delta := d^t$,

$$\mathscr{H}_i(M) := \ker d_{i,\max}/\operatorname{im} d_{i-1,\max}, \tag{3.7a}$$

$$\hat{\mathscr{H}}_i(M) := \ker d_{i,\max} \cap \ker(d_{i-1,\max})^* = \ker d_{i,\max} \cap \ker \delta_{i-1,\min}, \tag{3.7b}$$

$$\hat{\hat{\mathscr{H}}}_i(M) := \ker d_{i,\max} \cap \ker \delta_{i-1,\max}, \tag{3.7c}$$

$$\tilde{\mathscr{H}}_i(M) := \ker \Delta_{i,\max}. \tag{3.7d}$$

Then we obtain the following inclusions (cf. [C2]).

LEMMA 3.2. *We always have natural injections*

$$\hat{\mathscr{H}}_i(M) \hookrightarrow \mathscr{H}_i(M) \tag{3.8}$$

and

$$\mathscr{H}_i(M) \hookrightarrow \hat{\hat{\mathscr{H}}}_i(M) \hookrightarrow \tilde{\mathscr{H}}_i(M). \tag{3.9}$$

Relation (3.8) *is an isomorphism iff $d_{i-1,\max}$ has closed range.*

The first map in (3.9) *is an isomorphism if $d_{i,\max} = d_{i,\min}$ for all i, i.e., if* (3.6) *has a unique ideal boundary condition.*

The second map in (3.9) *is an isomorphism if Δ with domain $\Omega_0(M)$ is essentially self-adjoint.*

Proof. The assertions concerning (3.8) follow from Lemma 2.1.

It is obvious that the first map in (3.9) is an isomorphism if $(\Omega_0(M), d)$ has a unique ideal boundary condition. If Δ_i is essentially self-adjoint then all self-adjoint extensions of $\delta_i d_i + d_{i-1}\delta_{i-1}$ on $\Omega_0^i(M)$ coincide. Thus we find

$$\Delta_{i,\max} = \delta_{i,\min} d_{i,\max} + d_{i-1,\max}\delta_{i-1,\min} = \bar{\Delta}_i,$$

so $\hat{\hat{\mathscr{H}}}_i(M) = \tilde{\mathscr{H}}_i(M)$. ∎

It is useful to note the following fact which, together with Lemma 3.2, yields a well-known result of Andreotti and Vesentini [AV] (cf. Lemma 3.8).

LEMMA 3.3. *If $\Delta_0 := \delta d + d\delta$ is essentially self-adjoint on $\Omega_0(M)$ then $(\Omega_0(M), d)$ has a unique ideal boundary condition. This is the case, in particular, if M is complete.*

Proof. Let $(\mathscr{D}, D)$ be an ideal boundary condition for $(\Omega_0(M), d)$ with Laplacian $\Delta(D) = D^*D + DD^*$. Then $\Delta(D)$ is a self-adjoint extension of Δ_0, and it is not difficult to see that the map $D \mapsto \Delta(D)$ is injective. But D determines $(\mathscr{D}, D)$ by Lemma 2.2, hence essential self-adjointness of Δ implies the uniqueness of ideal boundary conditions.

The remaining statement is due to Gaffney [Ga]. ∎

Whereas essential self-adjointness of Δ is, generally speaking, a rare phenomenon in the presence of singularities, ideal boundary conditions for $(\Omega_0(M), d)$ are very often easy to classify or even unique. We want to illustrate this phenomenon and its consequences in the next section. It seems that in case of uniqueness the L^2-cohomology is the most natural analytic cohomology associated with $(\Omega_0(M), d)$.

In analogy with Section 2 we introduce complex maps for elliptic complexes. Let $(C_0^\infty(E), d)$, $(C_0^\infty(E'), d')$ be elliptic complexes over a Riemannian manifold M. A *complex map* $g: (C_0^\infty(E), d) \to (C_0^\infty(E'), d')$ is then a collection of bundle maps

$$g_i: E_i \to E_i', \tag{3.10a}$$

which induce bounded linear maps

$$g_i: L^2(E_i) \to L^2(E_i') \tag{3.10b}$$

such that

$$d_i' \circ g_i = g_{i+1} \circ d_i. \tag{3.10c}$$

We call $g := \bigoplus g_i$ a *complex isomorphism* if all g_i in (3.10a) are bundle isomorphisms and all maps in (3.10b) have bounded inverses. If only all g_i in (3.10a) are bundle isomorphisms we will refer to g as a *weak complex isomorphism*. Clearly, for isomorphic complexes there is a one-to-one correspondence between ideal boundary conditions.

That g is a complex isomorphism is equivalent to the inequalities

$$C_i^{-1} \| g_i(f) \|_{L^2(E_i')} \leq \| f \|_{L^2(E_i)} \leq C_i \, \| g_i(f) \|_{L^2(E_i')}, \tag{3.11}$$

with certain constants C_i. Inequality (3.11) follows always from the pointwise estimate

$$C_i^{-1} \, | g_i(e) |_{E_{i,p}'} \leq | e |_{E_{i,p}} \leq C_i \, | g_i(e) |_{E_{i,p}'}, \tag{3.12}$$

valid for $e \in E_{i,p}$ and $p \in M$, with constants C_i independent of p. Thus we call $g = \bigoplus g_i$ a *quasi-isometry* if (3.12) holds for all i. An important example arises for the de Rham complex and two quasi-isometric Riemannian metrics h_1, h_2 on M; i.e., for some $C>0$ and all $X \in C^\infty(TM)$ we have

$$C^{-1}h_1(X,X) \leqslant h_2(X,X) \leqslant Ch_1(X,X). \tag{3.13}$$

It is then easy to see that the two Hilbert complex structures induced by h_1 and h_2 on the de Rham complex are quasi-isometric in the sense of (3.12).

For an elliptic complex, $(C_0^\infty(E), d)$, with an ideal boundary condition, $(\mathscr{D}, D)$, we can form the smooth subcomplex $(\mathscr{D}^\infty, D)$ according to (2.31). Then, by Theorem 2.12, the cohomologies coincide; i.e., we can compute the cohomology of $(\mathscr{D}, D)$ from *smooth sections* since

$$\mathscr{D}_i^\infty = \bigcap_{k \geqslant 1} \mathscr{D}(\Delta_i^k) \subset C^\infty(E_i) \cap \mathscr{D}_i =: \mathscr{E}_i, \tag{3.14}$$

by elliptic regularity. However, in this case there is a more natural choice of "smooth subcomplex," namely $(\mathscr{E}, d)$ with $\mathscr{E}_i$ in (3.14). To extend Theorem 2.12 to this case we need a simple lemma.

LEMMA 3.4. *Let Δ be a bounded below self-adjoint operator in the Hilbert space H, and let $B_j \in \mathscr{L}(H)$, $j=1,2$, satisfy*

$$B_j(\mathscr{D}(\Delta^k)) \subset \mathscr{D}(\Delta^k), \qquad k \in \mathbb{Z}_+, \tag{3.15}$$

$$\Delta^l B_1 \Delta^m B_2 : \mathscr{D}(\Delta^{l+m}) \to H$$

extends to a bounded operator in H, for all $l, m \in \mathbb{Z}_+$. (3.16)

Then, if $f \in C^\infty(\mathbb{R}_+)$ admits an asymptotic expansion of the form

$$f(x) \sim \sum_{j \geqslant 0} f_j x^{\mu_j}, \qquad \mu_j \in \mathbb{R}, \mu_j \searrow -\infty, \tag{3.17}$$

as $x \to \infty$, $\Delta^l B_1 f(\Delta) B_2$ also extends to a bounded operator in H, for all $l \in \mathbb{Z}_+$.

Proof. We may assume $\Delta \geqslant 1$. Then it follows from complex interpolation that (3.15) and (3.16) hold for all $k, l, m \in \mathbb{R}_+$. Moreover, if $l + \mu_0 \leqslant 0$ we have

$$\Delta^l B_1 f(\Delta) B_2 = (\Delta^l B_1 \Delta^{-l})(\Delta^l f(\Delta)) B_2 \in \mathscr{L}(H).$$

Since

$$\Delta^l B_1 \sum_{j=0}^{N} f_j \Delta^{\mu_j} B_2 \in \mathscr{L}(H)$$

for all $N \in \mathbb{N}$, we can complete the proof with (3.17). ∎

The lemma applies in particular to the resolvent and the heat operator, with $f(x) = (x+\lambda)^{-1}$, $\lambda \geqslant 1$, or $f(x) = e^{-tx}$, $t \geqslant 0$. In our case we want to apply it, in view of (2.37), to the function

$$\begin{aligned} f(x) &= (1+x)^{-1}\left(1 + \int_0^\infty \varphi(t)\, e^{-tx}\, dt\right) \\ &= (1+x)^{-1}\left(1 + x^{-1}\int_0^\infty \varphi\left(\frac{t}{x}\right) e^{-t}\, dt\right) \\ &\sim (1+x)^{-1}(1+x^{-1}) \qquad \text{as} \quad x \to \infty, \end{aligned} \tag{3.18}$$

since $\varphi = 1$ near 0. We find

THEOREM 3.5. *Let $(\mathscr{D}, D)$ be an ideal boundary condition for the elliptic complex $(C^\infty(E), d)$, and consider the smooth subcomplex $(\mathscr{E}, d)$ of $(\mathscr{D}, D)$ defined by* (3.14).

Then the natural inclusion induces an isomorphism on homology.

Proof. By Lemma 2.10, we see that it is enough to prove the following: defining h and A for $(\mathscr{D}, D)$ by (2.35) and (2.37), respectively, then

$$h(\mathscr{D}_i) \subset \mathscr{E}_i \tag{3.19}$$

and

$$A_i(\mathscr{E}_i) \subset \mathscr{E}_{i-1}. \tag{3.20}$$

Since $\mathscr{D}_i^\infty \subset \mathscr{E}_i$, (3.19) follows from the proof of Theorem 2.12.

Next we pick $\varphi, \chi, \psi \in C_0^\infty(M)$ such that $\psi = 1$ in a neighbourhood of supp χ and $\chi = 1$ in a neighbourhood of supp φ. Since A_i maps $\mathscr{D}_i$ to $\mathscr{D}_{i-1}$, it is enough to prove that $\varphi A_i f \in C^\infty(E_{i-1})$ for $f \in \mathscr{E}_i$. But $\psi f \in \mathscr{D}_i^\infty$, so $\varphi A_i \psi f \in \mathscr{D}_{i-1}^\infty \subset \mathscr{E}_{i-1}$ and we are left with the proof of

$$\varphi A_i (1-\psi) f \in C^\infty(E_{i-1}). \tag{3.21}$$

This in turn will follow if we prove that

$$\varphi A_i (1-\psi) : H_i \to \mathscr{D}_{i-1}^\infty.$$

Now we have from (2.37)

$$\begin{aligned} \varphi A_i(1-\psi) = &-[\varphi, D_{i-1}^*]\, \chi(\Delta_i + \mathrm{id})^{-1}\left(\mathrm{id} + \int_0^\infty \varphi(t)\, e^{-t\Delta}\, dt\right)(1-\psi) \\ &- D_{i-1}^* \varphi(\Delta_i + \mathrm{id})^{-1}\left(\mathrm{id} + \int_0^\infty \varphi(t)\, e^{-t\Delta}\, dt\right)(1-\psi). \end{aligned}$$

Since Δ is a differential operator, we have $\chi\Delta^k(1-\psi)=0$. Thus the assertion follows from (3.18) and Lemma 3.4, applied to $B_2=(1-\psi)$ and $B_1=\varphi$ or $B_1=[\varphi, D_{i-1}^*]\chi$. ∎

For the de Rham complex $(\Omega_0(M), d)$ and the ideal boundary condition defined by $d_{\max}$, we have

$$\mathscr{E}_i=\{\omega\in\Omega^i(M)\cap L^2(\textstyle\bigwedge^i T^*M)\mid d\omega\in L^2(\bigwedge^{i+1}T^*M)\}.$$

In this special case, Theorem 3.5 is due to Cheeger [C2, Sects. 1 and 8] who uses constructions introduced by de Rham and Gaffney. Our proof seems to be more general and more perspicuous.

We turn to the product theory for elliptic complexes. The case of L^2-cohomology, i.e., the maximal complex associated to the de Rham complex, has been treated by Zucker [Z] with very similar techniques. Thus, on the Riemannian manifolds M_l, $l=1, 2$, we consider elliptic complexes $(C_0^\infty(E^l), d^l)$, and we want to associate to these data an elliptic complex $(C_0^\infty(E), d)$ over M. In view of (2.38) we define (with $\boxtimes$ the exterior tensor product)

$$E_i := \bigoplus_{j+k=i} E_j^1\boxtimes E_k^2, \tag{3.22a}$$

$$\mathscr{E}_i := \bigoplus_{j+k=i} C_0^\infty(E_j^1)\otimes C_0^\infty(E_k^2), \tag{3.22b}$$

$$d_i := \bigoplus_{j+k=i} (d_j^1\otimes \mathrm{id}+(-1)^k\mathrm{id}\otimes d_k^2): \mathscr{E}_i\to\mathscr{E}_{i+1}. \tag{3.22c}$$

Since the d_j^l are differential operators, (3.22c) defines a differential operator on $C_0^\infty(E_i)$ (note that this would not work for general pseudo-differential operators). Moreover, this operator is uniquely determined by (3.22c) since $\mathscr{E}_i$ is sequentially dense in $C_0^\infty(E_i)$ if both spaces are equipped with the usual *LF*-topology (cf. for this [Tr, Sect. 13.6 and Theorem 39.2]). Moreover, an easy calculation and a density argument show that $d_{i+1}\circ d_i=0$ on $C_0^\infty(E_i)$. Finally, it is well known that the symbol sequence of the complex $(C_0^\infty(E), d)$ is elliptic [P, Chap. IV, Sect. 8]. Thus we obtain an elliptic complex in a canonical way, which we will call the *product complex* associated with $(C_0^\infty(E^1), d^1)$ and $(C_0^\infty(E^2), d^2)$. We observe next the relation between the minimal and maximal complexes.

LEMMA 3.6. *Let $(C_0^\infty(E^l), d^l)$ be an elliptic complex over the Riemannian manifold M_l, $l=1, 2$, and denote by $(C_0^\infty(E), d)$ the product complex over $M_1\times M_2$. If $(\mathscr{D}^l, D^l)$ denotes the Hilbert complex with $D^l=d^l_{\min}$ or $D^l=d^l_{\max}$ for $l=1, 2$, then the product complex associated to $(\mathscr{D}^l, D^l)$ according to (2.38) is the Hilbert complex, $(\mathscr{D}, D)$, obtained from $(C_0^\infty(E), d)$ with $D=d_{\max}$ or $D=d_{\min}$.*

Proof. Clearly, the adjoint complex to $(C_0^\infty(E), d)$ is the product of the adjoint complexes to $(C_0^\infty(E^l), d^l)$, so, as in the proof of Lemma 3.1, we only have to prove the assertion for $D^l=d^l_{\min}$, $l=1, 2$. This will follow if we show that $\tilde{\mathscr{D}}_i$, defined in (2.38b), is contained in $\mathscr{D}(d_{i,\min})$ for all i. Obviously, $\tilde{\mathscr{D}}_i\subset\mathscr{D}(d_{i,\max})$, and a standard regularization argument shows that $z\in\tilde{\mathscr{D}}_i$ is in $\mathscr{D}(d_{i,\min})$ if $\operatorname{supp} z$ is compact. But, by construction, elements with compact support are dense in $\tilde{\mathscr{D}}_i$ with respect to the graph norm of $d_{i,\max}$, and the assertion follows. ∎

We turn to the discussion of Poincaré duality. In view of Lemma 2.16 we say that an elliptic complex $(C_0^\infty(E), d)$ has *weak Poincaré duality* if there are bundle isomorphisms

$$g_i: E_i\to E_{N-i} \tag{3.23a}$$

which induce invertible bounded linear maps

$$g_i: L^2(E_i)\to L^2(E_{N-i}) \tag{3.23b}$$

such that

$$d^t_{N-i-1}\circ g_i=g_{i+1}\circ d_i, \tag{3.23c}$$

i.e., if $(C_0^\infty(E), d)$ is isomorphic to its adjoint complex $(C_0^\infty(E), d^t)$.

This definition raises the question whether we can associate to each elliptic complex with weak Poincaré duality an ideal boundary condition with Poincaré duality. This question has been dealt with by Cheeger [C3] in the context of conic singularities.

LEMMA 3.7. *Let $(C_0^\infty(E), d)$ be an elliptic complex with weak Poincaré duality. If $N\equiv 0 \pmod 2$ we put*

$$\mathscr{D}_i := \mathscr{D}(d_{i,\min}),\qquad D_i := d_{i,\min},\ 0\leqslant i< N/2, \tag{3.24a}$$

and

$$\mathscr{D}_i := (d_{i,\max}),\qquad D_i := d_{i,\max},\ N/2\leqslant i<N. \tag{3.24b}$$

Then $(\mathscr{D}, D)$ becomes a Hilbert complex with Poincaré duality.

If $N\equiv 1 \pmod 2$ we put with $\nu := (N-1)/2$

$$\mathscr{D}_i := \mathscr{D}(d_{i,\min}),\qquad D_i := d_{i,\min},\ 0\leqslant i<\nu, \tag{3.25a}$$

and

$$\mathcal{D}_i := \mathcal{D}(d_{i,\max}), \qquad D_i := d_{i,\max}, \ \nu+1 \leqslant i < N. \tag{3.25b}$$

Then, with every closed extension D_ν of d_ν satisfying

$$D_\nu^* \circ g_\nu = g_{\nu+1} \circ D_\nu, \tag{3.26}$$

$(\mathcal{D}, D)$ beomes a Hilbert complex with Poincaré duality. If $g_{\nu+1}$ is unitary with $g_{\nu+1} = g_\nu^*$, then (3.26) is equivalent to the existence of self-adjoint extensions of $g_{\nu+1} \circ d_\nu$.

Proof. Assume first that $N \equiv 0 \pmod 2$. That $(\mathcal{D}, D)$ forms a Hilbert complex follows as in the proof of Lemma 3.1. From (3.23c) we obtain immediately

$$d^t_{N-i-1,\min} \circ g_i = g_{i+1} \circ d_{i,\min}, \qquad 0 \leqslant i < N/2, \tag{3.27a}$$

and

$$d^t_{N-i-1,\max} \circ g_i = g_{i+1} \circ d_{i,\max}, \qquad N/2 \leqslant i < N, \tag{3.27b}$$

thus

$$D^*_{N-i-1} \circ g_i = g_{i+1} \circ D_i, \qquad 0 \leqslant i < N, \tag{3.28}$$

and we are done by Lemma 2.16.

Consider next the case $N \equiv 1 \pmod 2$. Since $D_{\nu-1} = d_{\nu-1,\min}$ and $D_{\nu+1} = d_{\nu+1,\max}$, it is clear that $(\mathcal{D}, D)$ forms a Hilbert complex for any closed extension D_ν of d_ν. As above we get

$$D^*_{N-i-1} \circ g_i = g_{i+1} \circ D_i, \qquad i \neq \nu. \tag{3.29}$$

Hence (3.26) is the only condition on D_ν. The last assertion is obvious. ∎

We will illustrate this result by some examples in Section 4. It is interesting to note that one can always achieve Poincaré duality for the de Rham complex unless $\dim M \equiv 1 \bmod 4$, cf. Section 4.

For an elliptic complex $(C_0^\infty(E), d)$ it is readily seen that by the prescription (2.12) we obtain an elliptic operator

$$d : C_0^\infty(E_{\mathrm{ev}}) \to C_0^\infty(E_{\mathrm{odd}}), \tag{3.30}$$

where $E_{\mathrm{ev}} := \bigoplus_{i \geqslant 0} E_{2i}$, $E_{\mathrm{odd}} := \bigoplus_{i \geqslant 0} E_{2i+1}$. For the corresponding elliptic differential operator (2.14) we will continue to use the notation Δ,

$$\Delta = d^t d + d d^t : C_0^\infty(E) \to C_0^\infty(E), \tag{3.31}$$

$E := E_{\mathrm{ev}} \oplus E_{\mathrm{odd}}$.

In the general case of a noncompact Riemannian manifold we have to expect many different closed extensions of d. Besides the minimal and the maximal extensions, $d_{\min}$ and $d_{\max}$, Lemma 3.1 gives two more geometrically significant closed extensions. For the de Rham complex on a manifold with boundary, these extensions correspond to the relative and absolute boundary conditions; this will be explained in Section 4. Therefore, we call them the *relative* and *absolute extensions*, to be denoted by d_r and d_a.

It is important to note that there are closed extensions of d which do not come from an ideal boundary condition. One would, however, expect that the geometrically most significant closed extensions of d are those arising from an ideal boundary condition satisfying Poincaré duality. In general, even this class will be very complicated and hard to characterize. So it will be useful to find conditions under which one has uniqueness of the ideal boundary condition (with Poincaré duality) or even of the closed extension of d.

The most simple case certainly occurs if there is a unique closed extension of d. Then, by Lemma 2.2, there is also a unique ideal boundary condition (given by (3.4) or (3.5)). In this case we have $d_r = d_a$ and Poincaré duality follows automatically if weak Poincaré duality is fulfilled.

The closed extensions of d are often more conveniently studied via the self-adjoint extensions of Δ. Thus, to a closed extension, D, of d we associate the operator

$$\Delta(D) := D^*D \oplus DD^*. \tag{3.32}$$

This operator is clearly a symmetric extension of Δ and self-adjoint, by a well known result of von Neumann. Moreover, it is easy to see that the map

$$D \mapsto \Delta(D) \tag{3.33}$$

is injective. We thus obtain from the above mentioned special closed extensions of d corresponding self-adjoint extensions of Δ, to be denoted by $\Delta_{\min}, \Delta_r, \Delta_a, \Delta_{\max}$. A very simple situation occurs, of course, if Δ is essentially self-adjoint, i.e., Δ has only one self-adjoint extension, which then equals its closure in $L^2(E)$ and also coincides with Δ^*.

LEMMA 3.8. *If the (symmetric nonnegative) elliptic differential operator Δ, associated to the elliptic complex $(C_0^\infty(E), d)$ by (3.31), is essentially self-adjoint then there is a unique Hilbert complex $(\mathcal{D}, D)$ associated to $(C_0^\infty(E), d)$; i.e., the elliptic complex has a unique ideal boundary condition.*

In partiular, the operators $d_{i,\min}$ and $d_{i,\max}$ coincide for all i. Moreover, $u \in \mathcal{D}(\Delta^*) \cap H_i$ satisfies $\Delta^* u = 0$ if and only if $u \in \mathcal{D}(d_{i,\min}) \cap \mathcal{D}(d^t_{i-1,\min})$ and $d_i u = d^t_{i-1} u = 0$.

Proof. Any Hilbert complex associated with $(C_0^\infty(E), d)$ defines a closed extension, D, of d which gives a self-adjoint extension, $\Delta(D)$, of Δ via (3.19). Since this map is injective, all closed extensions of d must coincide with the closure, $D=\bar{d}$. Since D determines the Hilbert complex, by Lemma 2.2, there is only one.

In particular, the complexes defined in (3.4) and (3.5) have to coincide which means precisely that $d_{i,\min}=d_{i,\max}$, for all i.

Finally, for the (unique) complex given by $d_{i,\min}$ $(=d_{i,\max})$ we have by (2.19), (2.21), and (2.8)

$$\ker D \oplus \ker D^* = \bigoplus_{i \geqslant 0} \mathscr{H}_i = \bigoplus_{i \geqslant 0} \ker d_{i,\max} \cap \ker d^*_{i-1,\max}.$$

On the other hand, $\Delta^* = \bar{\Delta} = DD^* \oplus D^*D$, so

$$\ker \Delta^* = \ker D \oplus \ker D^*. \quad \blacksquare$$

Of course, it is necessary to find conditions ensuring essential self-adjointness of Δ. The following result is convenient in applications and well known (for a nice proof cf. [Ch]).

LEMMA 3.9. *Let $(C_0^\infty(E), d)$ be an elliptic complex with*

$$\sigma(\Delta)(\xi) = |\xi|^2, \qquad \xi \in S^*M, \tag{3.34}$$

where $\sigma(\Delta)$ denotes the principal symbol of Δ. If M is complete then Δ is essentially self-adjoint.

Essential self-adjointness of Δ is a rather restrictive condition. Below we will give examples where d has many closed extensions but, nevertheless, there is a unique ideal boundary condition. An easy but useful criterion for this to happen is

LEMMA 3.10. *Let $(C_0^\infty(E), d)$ be an elliptic complex. If*

$$\mathscr{D}(d_{2i,\max}) \cap \mathscr{D}(d^t_{2i-1,\max}) = \mathscr{D}(d_{2i,\min}) \cap \mathscr{D}(d^t_{2i-1,\min}) \tag{3.35a}$$

or

$$\mathscr{D}(d_{2i+1,\max}) \cap \mathscr{D}(d^t_{2i,\max}) = \mathscr{D}(d_{2i+1,\min}) \cap \mathscr{D}(d^t_{2i,\min}) \tag{3.35b}$$

for all i, then there is a unique ideal boundary condition, i.e., the relative and absolute extensions coincide.

Proof. We assume condition (3.35a), the proof for (3.35b) is similar. Consider

$$d_{\mathrm{ev}} := \bigoplus_{i \geqslant 0} d_{2i} \colon C_0^\infty(E_{\mathrm{ev}}) \to C_0^\infty(E_{\mathrm{odd}}) \tag{3.36a}$$

and

$$d_{\mathrm{odd}} := \bigoplus_{i \geqslant 0} d_{2i+1} \colon C_0^\infty(E_{\mathrm{odd}}) \to C_0^\infty(E_{\mathrm{ev}}). \tag{3.36b}$$

Then one easily checks that

$$d_r = d_{\mathrm{ev},\min} \oplus d^t_{\mathrm{odd},\max} \tag{3.37a}$$

and

$$d_a = d_{\mathrm{ev},\max} \oplus d^t_{\mathrm{odd},\min}. \tag{3.37b}$$

From (3.35a) we obtain

$$\begin{aligned} \mathscr{D}(d_r) &= \mathscr{D}(d_{\mathrm{ev},\min}) \cap \mathscr{D}(d^t_{\mathrm{odd},\max}) \\ &= \mathscr{D}(d_{\mathrm{ev},\max}) \cap \mathscr{D}(d^t_{\mathrm{odd},\min}) = \mathscr{D}(d_a), \end{aligned} \tag{3.38}$$

hence $d_r = d_a$. By Lemma 2.3, the Hilbert complexes defined in (3.4) and (3.5) coincide, and the assertion follows. $\blacksquare$

If we are given an elliptic complex $(C_0^\infty(E), d)$ on a *compact* Riemannian manifold then there is always a unique ideal boundary condition; in fact, elliptic regularity implies that Δ with domain $C_0^\infty(E)$ is essentially self-adjoint in $L^2(E)$. In various applications (cf. Section 4) one is interested in subcomplexes of $(C_0^\infty(E), d)$ and the corresponding Hilbert complexes. Therefore, we add a few remarks concerning this situation in general. The main feature that emerges from this discussion is a close connection between nonuniqueness of ideal boundary conditions, Fredholm properties, and regularity questions.

Thus we consider an elliptic complex, $(C_0^\infty(E), d)$, on the compact Riemannian manifold M. Assume that we are given a subcomplex $(\mathscr{E}, d)$; i.e., we have subspaces $\mathscr{E}_i \subset C_0^\infty(E_i)$ with $d_i(\mathscr{E}_i) \subset \mathscr{E}_{i+1}$, $i \geqslant 0$. We denote by $\bar{\mathscr{E}}_i$ the closure of $\mathscr{E}_i$ in $L^2(E_i)$; then we have

$$\mathscr{E}_i \subset \bar{\mathscr{E}}_i \cap C_0^\infty(E_i) \subset C_0^\infty(E_i).$$

In many applications we have equality for one of these inclusions, so we will assume that either

$$\bar{\mathscr{E}}_i \cap C_0^\infty(E_i) = \mathscr{E}_i \tag{3.39}$$

or

$$\bar{\mathscr{E}}_i \cap C_0^\infty(E_i) = C_0^\infty(E_i). \tag{3.40}$$

As in the proof of Lemma 3.1 we then see that $(\mathscr{E}, d)$ admits ideal boundary conditions. The analytic difficulty encountered in dealing with this subcomplex consists in the fact that, in general, d^t does not restrict to the adjoint operator or, in other words, the adjoint complex does not restrict to $\mathscr{E}$. For clarity, we write

$$d_{\mathscr{E}} := d\,|_{\mathscr{E}},$$

and we denote by $d^t_{\mathscr{E}}$ the (formal) adjoint operator, i.e.,

$$(d^t_{\mathscr{E}}\omega, \eta) := (\omega, d_{\mathscr{E}}\eta) = (\omega, d\eta)$$

for $\omega, \eta \in \mathscr{E}$. That is, $d^t_{\mathscr{E}}$ is the projection of d^t to $\bar{\mathscr{E}}$. Very often, however, the following is satisfied:

$$\begin{aligned}&\text{there is a differential operator, } \tilde{d}^t_i\colon C_0^\infty(E_{i+1}) \to C_0^\infty(E_i),\\ &\text{such that } d^t_{\mathscr{E},i} = \tilde{d}^t_i\,|_{\mathscr{E}}.\end{aligned} \tag{3.41}$$

For our purposes it is more convenient to assume a slightly stronger condition, namely

$$\begin{aligned}&\text{there is a first order elliptic differential operator,}\\ &\tilde{t}\colon C_0^\infty(E) \to C_0^\infty(E), \text{ such that } \tilde{t}\,|_{\mathscr{E}} = d_{\mathscr{E}} + d^t_{\mathscr{E}} =: t.\end{aligned} \tag{3.42}$$

Note that we do not assume $\tilde{t}$ symmetric (which is in fact not satisfied in interesting cases). We identify $\tilde{t}$ with its closure since this is the unique closed extension.

Lemma 3.11. *Denote by $t_{\min}$ the closure of t in $\bar{\mathscr{E}}$. Then $\mathscr{R}(t_{\min})$ is closed and* $\ker t_{\min}$ *has finite dimension.*

Thus, if also $\ker t_{\max}$ *has finite dimension, where $t_{\max} := t^*_{\min} \supset t_{\min}$, then all closed extensions of t are Fredholm operators.*

Proof. We only have to note that $t_{\min} = \tilde{t}\,|_{\mathscr{D}(t_{\min})}$. Thus the assertion on $t_{\min}$ follows from the ellipticity of $\tilde{t}$, Rellich's compactness theorem, and e.g., [H, Lemma 19.1.3].

Since every closed extension, $\check{t}$, of t satisfies $t_{\min} \subset \check{t} \subset t_{\max}$ the second statement is obvious if we know that $t_{\min}$ and $t_{\max}$ are Fredholm operators. Since $t_{\max} = t^*_{\min}$ we only need to know in addition that $\dim \ker t_{\max} < \infty$. ∎

In the case (3.39) we may ask whether we have a *smooth Hodge decomposition* in the sense that

$$\mathscr{E}_i = \mathscr{H}^\infty_i \oplus d_{i-1}\mathscr{E}_{i-1} \oplus d^t_i \mathscr{E}_{i+1}, \tag{3.43}$$

where

$$\mathscr{H}^\infty_i := \{\omega \in \mathscr{E}_i \mid d_i\omega = d^t_{i-1}\omega = 0\}, \tag{3.44}$$

and decomposition is with respect to the scalar product in $\bar{\mathscr{E}}_i$. Then we would have a very close analogy to the case of the full complex, $\mathscr{E}_i = C_0^\infty(E_i)$. We have the following result.

Theorem 3.12. *Assume* (3.39) *and* (3.42). *Then the following conditions are equivalent.*

(1) *$(\mathscr{E}, d)$ admits the smooth Hodge decomposition* (3.43).

(2) *t with domain $\mathscr{E}$ is essentially self-adjoint in $\bar{\mathscr{E}}$.*

(3) *t^2 with domain $\mathscr{E}$ is essentially self-adjoint in $\bar{\mathscr{E}}$.*

(4) $\ker t_{\max} \subset \mathscr{E}$.

Proof. (1) ⇒ (2). Assuming (3.43) we want to prove the inequalities

$$\dim \ker t_{\max} \leqslant \dim \mathscr{H}^\infty_i \leqslant \dim \ker t_{\min}. \tag{3.45}$$

Here the second inequality is obvious, so consider $\ker t_{\max} = \mathscr{R}(t_{\min})^\perp$. It is enough to show that $\mathscr{R}(t)$ is dense in $\mathscr{H}^{\infty\perp}_i = \overline{d_{i-1}\mathscr{E}_{i-1} \oplus d^t_i \mathscr{E}_{i+1}}$, by (3.43). But if $\eta \in d_{i-1}\mathscr{E}_{i-1} \oplus d^t_i\mathscr{E}_{i+1}$ we can write $\eta = d_{i-1}\zeta_{i-1} + d^t_i\zeta_{i+1}$, furthermore

$$\zeta_{i-1} = d_{i-2}\theta_{i-2} + d^t_{i-1}\theta_i, \qquad \zeta_{i+1} = d_i\omega_i + d^t_{i+1}\omega_{i+2},$$

hence

$$\eta = d_{i-1}\, d^t_{i-1}\theta_i + d^t_i\, d_i\omega_i = t(d^t_{i-1}\theta_i + d_i\omega_i).$$

Thus we have established (3.45) which implies

$$\dim \ker t_{\max} = \dim \ker t^*_{\min} = \dim \ker t_{\min}.$$

As in the proof of Lemma 3.11 we now conclude that $t_{\min}$ and $t_{\max}$ are Fredholm operators with

$$\operatorname{ind} t_{\min} = \operatorname{ind} t_{\max} = 0.$$

But this implies $t_{\max}=t_{\min}$, i.e., t with domain $\mathscr{E}$ is essentially self-adjoint in $\bar{\mathscr{E}}$.

(2) ⇒ (3). Since t^2 admits self-adjoint extensions, it is enough to show that the equation $t^{2*}\omega=i\omega$ has no nontrivial solution. Thus assume that we are given $\omega\in\mathscr{D}(t^{2*})$ with $t^{2*}\omega=i\omega$. Any $\eta\in\mathscr{E}$ can be written as $\eta=\eta_1+t\eta_2$ with $\eta_i\in\mathscr{E}$ for $i=1,2$ and $\eta_1\in\ker t_{\min}$, $\|\eta_2\|\leqslant C\,\|t\eta_2\|$. This holds, since $t_{\min}=\tilde{t}\,|_{\mathscr{D}(t_{\min})}$ is a self-adjoint Fredholm operator and $\tilde{t}$ is elliptic. Thus we find

$$(\omega, t\eta)=(\omega, t^2\eta_2)=(t^{2*}\omega, \eta_2)=i(\omega, \eta_2)$$

hence

$$|(\omega, t\eta)|\leqslant \|\omega\|\;\|\eta_2\|\leqslant C\,\|\omega\|\;\|t\eta_2\|\leqslant C\,\|\omega\|\;\|\eta\|.$$

This implies $\omega\in\mathscr{D}(t_{\min})$. Again with $\eta\in\mathscr{E}$ we then find

$$(t_{\min}\omega, t\eta)=(\omega, t^2\eta)=i(\omega, \eta)$$

which implies $t_{\min}\omega\in\mathscr{D}(t_{\min})$. Thus $\omega\in\mathscr{D}(t^2_{\min})$ hence $\omega=0$.

(3) ⇒ (4). For $\omega\in\ker t_{\max}$, $\eta\in\mathscr{E}$ we find

$$0=(t_{\max}\,\omega, t\eta)=(\omega, t^2\eta)$$

hence $\omega\in\ker t^{2*}=\ker t^2_{\min}=\ker t_{\min}\subset\mathscr{E}$, by (3.39) and (3.42).

(4) ⇒ (1). If $\ker t_{\max}\subset\mathscr{E}$ then we obtain as before $\ker t_{\max}=\ker t_{\min}$ and the essential self-adjointness of t, moreover, $t_{\min}$ is Fredholm. Thus the complex $(\mathscr{E}, d)$ has a unique ideal boundary condition which is Fredholm. Consider the corresponding strong Hodge decomposition according to Corollary 2.5,

$$\bar{\mathscr{E}}_i=\mathscr{H}_i\oplus\mathscr{R}_{i-1}\oplus\mathscr{R}^*_i. \tag{3.46}$$

Then $\bigoplus_{i\geqslant 0}\mathscr{H}_i=\ker t_{\min}$, and any $\omega\in\mathscr{E}_i$ can be decomposed as $\omega=\omega_0+t_{\min}\omega_1$. By elliptic regularity and (3.39) we conclude that $\omega_j\in\mathscr{E}_i$, $j=0,1$. But then we have a representation of the form (3.43) which is unique in view of (3.46). ▮

We obtain only a partial result in the case (3.40).

THEOREM 3.13. *Assume* (3.40) *and* (3.42). *If*

$$\ker t_{\max}\subset C_0^\infty(E),$$

then all closed extensions of t are Fredholm operators.

Proof. Let $\omega\in\ker t_{\max}\subset C_0^\infty(E)$ then for $\eta\in\mathscr{E}$ we find

$$0=(\omega, t\eta)=(\omega, \tilde{t}\eta)=(\tilde{t}^*\omega, \eta).$$

Since $\mathscr{E}$ is dense in $\bar{\mathscr{E}}=L^2(E)$ (which follows from (3.40)) we conclude that $\tilde{t}^*\omega=0$. Hence the ellipticity of $\tilde{t}^*$ implies

$$\dim\ker t_{\max}\leqslant\dim\ker\tilde{t}^*<\infty.$$

Then it follows as in the proof of Lemma 3.11 that all closed extensions of t are Fredholm. ▮

We will give examples for interesting subcomplexes in Section 4 below. It is interesting to note that Fredholmness appears closely linked with regularity properties in the above theorems.

4. SOME APPLICATIONS

In this section we want to illustrate the somewhat abstract results of the preceding sections in concrete geometric situations. For simplicity, we will focus our attention on the de Rham complex, $(\Omega_0(M), d)$, on an oriented but otherwise arbitrary Riemannian manifold, and look also at some closely related complexes with Poincaré duality. For convenience, we summarize the properties which this complex enjoys on *any* Riemannian manifold.

$(\Omega_0(M), d)$ always admits ideal boundary conditions (by Lemma 3.1) which may be unique or not, some of them may be Fredholm, others not. The cohomology of any ideal boundary condition can be computed using smooth forms only (by Theorem 3.5). If N is a second Riemannian manifold and $f\in C^\infty(M, N)$ then the pull back f^* may give rise to complex maps; we will mainly look at various metrics on $\Omega_0(M)$, e.g., pull backs under smooth maps, such that the identity map becomes a quasi-isometry in the sense of (3.12). Considering the adjoint complex as in (3.3) we are led to the question of Poincaré duality. Of course, the (slightly modified) Hodge *-operator satisfies (3.23) so weak Poincaré duality always holds. We will indicate that the condition in Lemma 3.7 cannot always be satisfied; i.e., there is not always an ideal boundary condition with Poincaré duality.

To start the discussion let us recall that, by standard elliptic theory, for *M compact* there is a unique ideal boundary condition which is Fredholm and satisfies Poincaré duality. The cohomology is isomorphic to the de Rham cohomology, and Theorem 3.7 implies the smooth Hodge decomposition.

To obtain a Hilbert complex it is, however, not necessary to start with a smooth structure. Thus we can envisage a compact Lipschitz manifold as in [T]; this is a very general class since any topological manifold with dimension different from four admits a unique Lipschitz structure [S]. The work of Teleman contains the essential steps to establish the Hodge theorem and Poincaré duality also in this case: the Lipschitz structure is used to define the complex of L^2-differential forms, and an easy regularity result [T, Lemma 4.1] shows that the exterior differentials are closed operators in this Hilbert space structure; i.e., we are dealing with a Hilbert complex. The Fredholm property of the complex follows from the analogue of Rellich's theorem [T, Sect. 7] which establishes then the strong Hodge decomposition. Poincaré duality follows since the operator δ_r introduced in [T, (4.1)] equals d^*_{r-1}, in view of the regularity lemma. Of course, to identify the homology additional information is necessary.

Next we consider the case of a compact manifold, M, with boundary $N := \partial M \neq \emptyset$. This case (which is well understood in many respects) already differs rather drastically from the situation without a boundary. The two ideal boundary conditions introduced in (3.4) and (3.5) will not coincide in this case but, nevertheless, they are a good substitute for the unique ideal boundary condition in the compact case. For reasons which will become clear below we introduce the notion $(\mathscr{D}^r, D^r)$ and $(\mathscr{D}^a, D^a)$ for the ideal boundary conditions corresponding to (3.4) and (3.5), respectively, calling them the *relative* and *absolute* ideal boundary conditions henceforth. The following result relates our point of view to the standard approach (cf. [Gi, Sects. 4.1, 4.2]).

THEOREM 4.1. *Let M be the interior of a compact Riemannian manifold $\bar{M}$ with boundary N.*

(1) *The relative and absolute ideal boundary conditions are Fredholm complexes. The corresponding closed extensions of the Gauß–Bonnet operator are given by the elliptic boundary value problems defined by relative and absolute boundary conditions (in the sense of [Gi]).*

(2) *The homology of the relative and absolute boundary conditions can be computed from the smooth subcomplexes $(\mathscr{D}^{r/a} \cap \Omega(\bar{M}), d)$ where*

$$\mathscr{D}^r \cap \Omega(\bar{M}) = \{\omega \in \Omega(\bar{M}) \mid \quad \omega \mid N = 0\}, \tag{4.1}$$

$$\mathscr{D}^a \cap \Omega(\bar{M}) = \Omega(\bar{M}). \tag{4.2}$$

In particular,

$$\mathscr{H}(\mathscr{D}^a, D^a) \simeq H^*(M; \mathbb{C}), \tag{4.3}$$

$$\mathscr{H}(\mathscr{D}^r, D^r) \simeq H^*(M, N; \mathbb{C}). \tag{4.4}$$

(3) *We have*

$$\operatorname{ind}(\mathscr{D}^a, D^a) = \chi(M) = \int_M \omega_{GB} - \int_N \alpha_{GB}, \tag{4.5}$$

(where the last equality holds only if m is even) and

$$\operatorname{ind}(\mathscr{D}^r, D^r) = \operatorname{ind}(\mathscr{D}^a, D^a) - \chi(N). \tag{4.6}$$

Here $\chi(M)$, $\chi(N)$ denote the Euler numbers, ω_{GB} the Chern–Gauß–Bonnet form, and α_{GB} the transgression of ω_{GB} to some metric which is a Riemannian product near N.

Proof. Let us remark first that it is enough to prove all statements for any special metric on M. Thus we introduce the compact double, $\tilde{M}$, of M with the natural involutive diffeomorphism α interchanging the two copies of M. Then we choose a metric, g, on $\tilde{M}$ such that $\alpha^* g = g$ and there is a collar, U, of N isometric to $(-1, 1) \times N$ with the product metric, $dx^2 \oplus g_N$.

It will be useful to recall the separation of variables in U (cf. [B, Sect. 2]); writing $\omega \in \Omega^i(U)$ as

$$\omega = \omega_0(x) + \omega_1(x) \wedge dx, \qquad \omega_j \in C^\infty((-1, 1), \Omega^{i-j}(N)) \text{ for } j = 0, 1, \tag{4.7}$$

we find

$$d_i \omega = d_N \omega_0(x) + ((-1)^i \omega_0'(x) + d_N \omega_1(x)) \wedge dx, \tag{4.8}$$

$$d^t_{i-1} \omega = (\delta_N \omega_0(x) + (-1)^i \omega_1'(x)) + \delta_N \omega_1(x) \wedge dx, \tag{4.9}$$

$$\Delta \omega = (-\omega_0''(x) + \Delta_N \omega_0(x)) + (-\omega_1''(x) + \Delta_N \omega_1(x)) \wedge dx. \tag{4.10}$$

Here we have used the notation d_N, $\delta_N := d^t_N$, Δ_N for the intrinsic operations on N.

(1) We bring in the unique ideal boundary condition, $(\tilde{\mathscr{D}}, \tilde{D})$, for $(\Omega(\tilde{M}), d)$. Then α induces a complex map, α^*, on $(\tilde{\mathscr{D}}, \tilde{D})$ which is an involution, too, and we can decompose

$$(\tilde{\mathscr{D}}, \tilde{D}) = (\tilde{\mathscr{D}}^r, \tilde{D}^r) \oplus (\tilde{\mathscr{D}}^a, \tilde{D}^a), \tag{4.11}$$

corresponding to the -1 and $+1$ eigenspaces of α^*, respectively. Then we claim that

$$\mathscr{D}^r = \tilde{\mathscr{D}}^r \mid M, \qquad \mathscr{D}^a = \tilde{\mathscr{D}}^a \mid M. \tag{4.12}$$

To prove the first identity it is enough to show $\tilde{\mathscr{D}}^r \mid M \subset \mathscr{D}^r$. So pick $\omega \in \tilde{\mathscr{D}}^r \mid M$ with extension $\tilde{\omega} \in \tilde{\mathscr{D}}^r$. Then we can find a sequence $(\tilde{\omega}_n) \subset \Omega(\tilde{M}) \cap \tilde{\mathscr{D}}^r$ with $\tilde{\omega}_n \to \tilde{\omega}$, $d\tilde{\omega}_n \to \tilde{D}\tilde{\omega}$ in L^2. Thus we may assume $\tilde{\omega} \in \Omega(M) \cap \tilde{\mathscr{D}}^r$; then, on U, we have

$$\tilde{\omega} = \omega_0(x) + \omega_1(x) \wedge dx$$

with $\omega_j(-x) = (-1)^{j+1}\omega_j(x)$, $x \in (-1, 1)$. Now pick $\varphi \in C_0^\infty(-1, 1)$ with $\varphi = 1$ near 0 and put $\varphi_n(x) := \varphi(nx)$,

$$\tilde{\omega}_n := (1 - \varphi_n)\tilde{\omega}.$$

Then $\tilde{\omega}_n \to \tilde{\omega}$ in L^2, and $d\tilde{\omega}_n = (1 - \varphi_n)\, d\tilde{\omega} - \varphi_n'\, dx \wedge \omega_0(x) =: (1 - \varphi_n)\, d\tilde{\omega} + \tilde{\eta}_n$. So we have to show that $\tilde{\eta}_n \to 0$ in L^2 which follows easily from $\omega_0(0) = 0$.

The second identity in (4.12) will follow from $\mathscr{D}^a \subset \tilde{\mathscr{D}}^a \mid M$. Pick $\omega \in \mathscr{D}^a$ and denote by $\tilde{\omega}$ the extension to $\tilde{M}$ with $\alpha^*\tilde{\omega} = \tilde{\omega}$. To show that $\tilde{\omega} \in \tilde{\mathscr{D}}^a$ we verify that

$$|(\tilde{\omega}, d^t\tilde{\eta})| \leqslant C\, \|\tilde{\eta}\|, \qquad \tilde{\eta} \in \Omega(M) \cap \tilde{\mathscr{D}}^a. \tag{4.13}$$

By assumption, this holds for all $\tilde{\eta}$ which vanish in a neighborhood of N, with C independent of $\tilde{\eta}$. Thus it is enough to prove that $d^t(1 - \varphi_n)\tilde{\eta} \to d^t\tilde{\eta}$ in L^2, for all $\tilde{\eta} \in \Omega(M) \cap \tilde{\mathscr{D}}^a$. If $\tilde{\eta} = \eta_0(x) + \eta_1(x) \wedge dx$ then we have $\eta_1(0) = 0$; moreover, with $\llcorner$ denoting interior multiplication,

$$\begin{aligned} d^t(1 - \varphi_n)\tilde{\eta} &= (1 - \varphi_n)\, d^t\tilde{\eta} + \nabla\varphi_n \llcorner \tilde{\eta} \\ &=: (1 - \varphi_n)\, d^t\tilde{\eta} + \tilde{\eta}_n. \end{aligned}$$

As before we see that $\tilde{\eta}_n \to 0$ in L^2.

Since we have complex maps $(\tilde{\mathscr{D}}, \tilde{D}) \to (\tilde{\mathscr{D}}^{r/a}, \tilde{D}^{r/a})$ which are surjective on homology and since the restriction maps $(\tilde{\mathscr{D}}^{r/a}, \tilde{D}^{r/a}) \to (\mathscr{D}^{r/a}, D^{r/a})$ are complex isomorphisms, the Fredholm property follows.

Next we introduce

$$\Omega^{r/a}(\tilde{M}) := \Omega(\tilde{M}) \cap \tilde{\mathscr{D}}^{r/a}, \tag{4.14a}$$

$$\Omega^{r/a}(M) := \Omega^{r/a}(\tilde{M}) \mid M. \tag{4.14b}$$

Note that for $\omega \in \Omega^{r/a}(\tilde{M})$ we will have $\omega_j(-x) = (-1)^{j + \mu(a)/\mu(r)}\omega_j(x)$ in (4.7), for $j = 0, 1$, where $\mu(a) = 0$, $\mu(r) = 1$. Now we claim that $\Omega^{r/a}_{\mathrm{ev}}(M)$ is dense in $\mathscr{D}(D^{r/a}_{GB})$ with respect to the graph norm, where $D^{r/a}_{GB}$ denotes the closed extension of the Gauß–Bonnet operator associated with $(\mathscr{D}^{r/a}, D^{r/a})$ according to (2.12). Thus consider $\omega \in \mathscr{D}(D^{r/a}_{GB})$ such that

$$0 = (D^{r/a}_{GB}\omega, D^{r/a}_{GB}\eta) + (\omega, \eta), \qquad \eta \in \Omega^{r/a}_{\mathrm{ev}}(M).$$

Since $D^{r/a}_{GB}(\Omega^{r/a}_{\mathrm{ev}}(M)) \subset \Omega^{r/a}_{\mathrm{odd}}(M)$ we conclude that

$$0 = (\tilde{\omega}, (\Delta + 1)\eta), \qquad \eta \in \Omega^{r/a}_{\mathrm{ev}}(\tilde{M}),$$

if $\tilde{\omega}$ is the extension of ω in $L^2(\Lambda^*\tilde{M})^{r/a}$. It follows that $\omega = 0$. We observe next that the graph norm of $D^{r/a}_{GB}$ is equivalent to the norm of $\bar{H}^1(\Lambda^{\mathrm{ev}}T^*M)$ (cf. [H, III, Appendix B] for the definition) on $\Omega^{r/a}(M)$. Thus, if we show that $\Omega^{r/a}(M)$ is also dense in the space

$$\{\omega \in \bar{H}^1(\Lambda^{\mathrm{ev}}T^*M) \mid B_{r/a}\omega := \omega_{\mu(a/r)}(0) = 0 \text{ in } (4.7)\}, \tag{4.15}$$

then we conclude the equality of this latter space with $\mathscr{D}(D^{r/a}_{GB})$. Here $(D_{GB}, B_{r/a})$ is the elliptic boundary value problem on $\Omega^{\mathrm{ev}}(\bar{M})$ defined by relative and absolute boundary conditions, respectively (cf. [H, Sect. 20.1] and [Gi, Sect. 4.2]).

To see this latter density we pick $\omega \in \bar{H}^1(\Lambda^{\mathrm{ev}}T^*M)$ with $B_a\omega = \omega_1(0) = 0$. We denote by $\tilde{\omega}$ the "even" extension of ω (with components $\tilde{\omega}^i$) in $L^2(\Lambda^{\mathrm{ev}}\tilde{M})$; with $\tilde{M}_\varepsilon := \{p \in \tilde{M} \mid \mathrm{dist}(p, N) \geqslant \varepsilon\}$ and $\eta \in \Omega^a(\tilde{M})$ (with components η^i) we then compute

$$\begin{aligned} &(D_{GB}\tilde{\omega}, \eta)_{\tilde{M}_\varepsilon} - (\tilde{\omega}, D_{GB}\eta)_{\tilde{M}_\varepsilon} \\ &\quad = 2 \sum_{i \geqslant 0} (-1)^i [\langle \tilde{\omega}^i_0, \eta^i_1 \rangle(\varepsilon) + \langle \tilde{\omega}^{i-1}_1, \eta^{i-1}_0 \rangle(\varepsilon)]. \end{aligned} \tag{4.16}$$

Since $\omega_i \colon [-1, 0] \to L^2(N)$ is continuous with $\omega_1(0) = 0$, by assumption, we infer that $\tilde{\omega} \in \mathscr{D}(D_{GB,\max}) = H^1(\Lambda^{\mathrm{ev}}\tilde{M})$. Hence we can find a smooth approximation in $\Omega(\tilde{M})^a$. The argument for relative boundary conditions is completely analogous.

(2) We study the homology of $(\mathscr{D}^a, D^a)$ first. In view of Theorem 2.12 we want to show that, with $T := D^a_{GB} \oplus (D^a_{GB})^*$, we have for all $k \in \mathbb{Z}_+$

$$\mathscr{D}(T^k) \subset \bar{H}^k(\Lambda^*M).$$

This, in turn, follows if we show that $\omega \in \mathscr{D}(T)$, $T\omega \in \bar{H}^k(\Lambda^*M)$ implies $\omega \in \bar{H}^{k+1}(\Lambda^*M)$. But the arguments above show that $\mathscr{D}(T) \subset \bar{H}^1(\Lambda^*M)$ hence the assertion follows from [H, Theorems 20.1.2 and 20.1.7].

Thus we can use Theorem 3.5 to obtain a subcomplex $(\tilde{\Omega}(\bar{M}), d)$ of $(\Omega(\bar{M}), d)$ such that the inclusion in $(\mathscr{D}^a, D^a)$ induces an isomorphism on homology. The same will be true for $(\Omega(\bar{M}), d)$ if we show that the inclusion $\beta: (\Omega(\bar{M}), d) \to (\mathscr{D}^a, D^a)$ is injective on homology. Thus consider $\omega \in \Omega^i(\bar{M})$ with $\omega = d\eta$, for some $\eta \in \mathscr{D}^a$. It is easy to see from $\mathscr{D}^a = \tilde{\mathscr{D}}^a \mid M$ that we may assume $\eta \in \Omega(M)$. Near N, we write $\eta = \eta_0(x) + \eta_1(x) \wedge dx$, $\omega = \omega_0(x) + \omega_1(x) \wedge dx = d_N\eta_0(x) + ((-1)^{i-1}\eta_0'(x) + d_N\eta_1(x)) \wedge dx$. We use the smooth Hodge decomposition on N to write

$$\eta_0(x) = \eta_0^h(x) + d_N\zeta_-(x) + \delta_N\zeta_+(x),$$

where $\eta_0^h, \zeta_-, \zeta_+$ are smooth in $(-1, 0)$. Then we deduce from

$$\omega_1(x) = d_N(\eta_1(x) + (-1)^{i-1}\zeta_-'(x)) + (-1)^{i-1}(\eta_0^h)'(x) + \delta_N\zeta_+'(x))$$

that η_0^h and $\delta_N\zeta_+$ are smooth in $(-1, 0]$. We may also assume that ζ_- is smooth in $(-1, 0]$ replacing, if necessary, η by $\eta - \tilde{\eta}$, where with $\varphi \in C_0^\infty(-1, 1)$, $\varphi = 1$ near 0,

$$\tilde{\eta}_0(x) := \varphi(x)\, d_N\zeta_-(x), \qquad \tilde{\eta}_1(x) := (-1)^i(\varphi'(x)\,\zeta_-(x) + \varphi(x)\,\zeta_-'(x)).$$

But then, again from the smooth Hodge decomposition, we may also assume that $\eta_1(x) = \delta_N\gamma_+(x)$ and $d_N\gamma_+(x) \equiv 0$, without affecting $d\eta = \omega$. But then $\eta \in \Omega(\bar{M})$, as requested.

To compute the homology of $(\tilde{\mathscr{D}}^a, \tilde{D}^a)$ we observe that $H_*(\bar{M}, \mathbb{R}) \simeq H_*(\tilde{M}, \mathbb{R})^{\mathbb{Z}_2}$, where the $\mathbb{Z}_2$-action is given by α. It follows that $H^*(\bar{M}, \mathbb{R}) \simeq H^*(\tilde{M}, \mathbb{R})^{\mathbb{Z}_2}$, and the de Rham theorem implies easily that the latter cohomology coincides with the cohomology of the complex $(\Omega(\tilde{M})^{\mathbb{Z}_2}, d)$ hence with that of $(\tilde{\mathscr{D}}^a, \tilde{D}^a)$ and $(\mathscr{D}^a, D^a)$.

We turn to the relative boundary condition $(\mathscr{D}^r, D^r)$. Then we see from the discussion above that

$$\Omega(\bar{M}) \cap \mathscr{D}^r = \Omega(\bar{M}) \cap \mathscr{D}(D_{GB}^r) = \{\omega \in \Omega(\bar{M}) \mid \quad \omega \mid N = 0\},$$

as claimed. That the inclusion into $(\mathscr{D}^r, D^r)$ induces an isomorphism on homology is proved along the same lines as above: if $\omega = d\eta$, $\eta \in \mathscr{D}^r$, then we can achieve $\eta \in \mathscr{D}^r \cap \Omega(\bar{M})$, hence the inclusion is injective on homology.

The homology computation in (4.4) is now an easy consequence of our calculations which imply the fact that the following sequence of complex maps is exact:

$$0 \longrightarrow \Omega(\bar{M}) \cap \mathscr{D}^r \xrightarrow{j} \Omega(\bar{M}) \xrightarrow{i_N^*} \Omega(N) \longrightarrow 0, \tag{4.17}$$

where j is the natural inclusion, $i_N: N \to \bar{M}$ the embedding.

(3) The first equality in (4.5) follows directly from (4.3), and (4.6) is a consequence of the long exact homology sequence associated with (4.17) and (2).

If m is even then we obtain from (1), Corollary 2.6, (4.5), and the Gauß–Bonnet theorem for $\tilde{M}$, using the special metric again,

$$\begin{aligned}\int_M \omega_{GB} &= \frac{1}{2}\int_{\tilde{M}} \omega_{GB} = \frac{1}{2}\operatorname{ind}(\tilde{\mathscr{D}}, \tilde{D})\\ &= \frac{1}{2}(\operatorname{ind}(\tilde{\mathscr{D}}^a, \tilde{D}^a) + \operatorname{ind}(\tilde{\mathscr{D}}^r, \tilde{D}^r))\\ &= \frac{1}{2}(\operatorname{ind}(\mathscr{D}^a, D^a) + \operatorname{ind}(\mathscr{D}^r, D^r))\\ &= \operatorname{ind}(\mathscr{D}^a, D^a) = \chi(M).\end{aligned}$$

Now (4.5) in general follows by transgressing the Euler form from any given metric to the special one. ∎

The material presented in Theorem 4.1 deserves some further comments. Let us elaborate first on the restrictions imposed by the special choice of metric: the absolute boundary conditions depend on the metric, and our statement proves equality only for the special metric, whereas the relative boundary condition is obviously independent of the metric. But the Hodge operator $*: \Omega_0(M) \to \Omega_0(M)$ induces a complex isomorphism

$$(\mathscr{D}^r, D^r) \xrightarrow{*} (\mathscr{D}^{a*}, D^{a*}). \tag{4.18}$$

Combining this with Corollary 2.6 it is easy to see that the domain of D_{GB}^a is given by the absolute boundary conditions for any metric. This argument works in general if $(\mathscr{D}^r, D^r)$ is Fredholm, so we find

LEMMA 4.2. *Let M be an arbitrary orientable Riemannian manifold and denote again by $(\mathscr{D}^{r/a}, D^{r/a})$ the relative and absolute (ideal) boundary condition for the de Rham complex $(\Omega_0(M), d)$.*

Then $(\mathscr{D}^r, D^r)$ is Fredholm if and only if $(\mathscr{D}^a, D^a)$ is, and we have in this case

$$\mathscr{H}_i(\mathscr{D}^r, D^r) \simeq \mathscr{H}_{m-i}(\mathscr{D}^a, D^a), \tag{4.19}$$

$$\operatorname{ind}(\mathscr{D}^r, D^r) = (-1)^m \operatorname{ind}(\mathscr{D}^a, D^a). \tag{4.20}$$

If M is compact with boundary then (4.19) is usually referred to as *Poincaré duality for manifolds with boundary.* Returning to this situation we may ask, however, whether we can find another ideal boundary condition satisfying Poincaré duality, in the sense of Lemma 3.7, induced by the Hodge operator.

LEMMA 4.3. *Let M be the interior of a compact manifold with boundary. Then one can always find an ideal boundary condition, $(\mathscr{D}, D)$, for the de Rham complex which satisfies Poincaré duality.*

$(\mathscr{D}, D)$ is not Fredholm; in fact, the de Rham complex admits infinitely many different non-Fredholm ideal boundary conditions.

Proof. Let us remark first that we have to choose in (3.23b)

$$g_i := (\sqrt{-1})^{\beta(i)} *_i, \qquad \beta(i) \in \mathbb{Z}, \tag{4.21a}$$

where (3.23c) is satisfied if and only if

$$\beta(i+1) - \beta(i) = 2(i+1). \tag{4.21b}$$

If m is odd then we also want the relation

$$g_{\nu+1} = g_\nu^*, \qquad \nu := (m-1)/2,$$

which can be satisfied if we require in addition

$$\beta(\nu) := \begin{cases} 2, & m \equiv 3 \bmod 4, \\ 1, & m \equiv 1 \bmod 4, \end{cases} \tag{4.22}$$

and then determine all other $\beta(i)$ from (4.21b).

Thus it follows from Lemma 3.7 that we can find an ideal boundary condition with Poincaré duality induced from (4.21) if m is even.

If $m \equiv 3 \bmod 4$ then the operator $T := g_{\nu+1} \circ d_\nu = (\sqrt{-1})^{\beta(\nu+1)} *_{\nu+1} d_\nu = -*_{\nu+1} d_\nu$ is real, hence admits self-adjoint extensions by a well-known result of von Neumann. This settles the question in this case, by Lemma 3.7 again.

If $m \equiv 1 \bmod 4$ then we have to determine a self-adjoint extension in $L^2(\Lambda^\nu T^*M)$ of the symmetric operator $T_0 := \sqrt{-1} *_{\nu+1} d_\nu$ with domain $\Omega_0^\nu(M)$. To do so we observe that for $\omega_1, \omega_2 \in \Omega^\nu(M)$

$$(T_0\omega_1, \omega_2) - (\omega_1, T_0\omega_2) = \sqrt{-1}(\omega_1 \mid N, *_N(\omega_2 \mid N))_{L^2(\Lambda^* N)}. \tag{4.23}$$

We write $\bar{\omega} := \omega \mid N$ for $\omega \in \Omega^\nu(\bar{M})$ and decompose $\bar{\omega} = \omega_+ + \omega_-$, according to the ± 1 eigenspaces of the involution $*_N$, $E_\pm^\nu(N)$, on $L^2(\Lambda^\nu T^*N)$. We can construct an isometry $I: E_+^\nu(N) \to E_-^\nu(N)$, e.g., by defining I somehow as isometry $E_+^\nu(N) \cap H^\nu(N) \to E_-^\nu(N) \cap H^\nu(N)$ (note that this is possible since sign $N = 0$); then, if $(\psi_i)_{i \in \mathbb{N}}$ denotes an orthonormal basis for $\overline{d_N \Omega^{\nu-1}(N)}$ consisting of closed eigenforms of Δ^ν, we require $I(\psi_i + *_N \psi_i) = \psi_i - *_N \psi_i$. Now we introduce

$$\Omega_I^\nu(\bar{M}) := \{\omega \in \Omega^\nu(\bar{M}) \mid \bar{\omega}_- = I\bar{\omega}_+\}. \tag{4.24}$$

It is immediate from (4.23) that $T := \sqrt{-1} * d \mid \Omega_I^\nu(\bar{M})$ is a symmetric extension of T_0; we want to show that T is essentially self-adjoint. To do so it is enough to show that any solution $\omega \in \mathscr{D}(T^*)$ of

$$T^*\omega = \pm\sqrt{-1}\omega \tag{4.25}$$

is already in $\mathscr{D}(\bar{T})$. To prove this we will assume first that the metric is nice, i.e., a product in some neighborhood of N, say in $U = (-2\varepsilon_0, 2\varepsilon_0) \times N$. As before we use separation of variables and write $\omega = \omega_0(x) + \omega_1(x) \wedge dx$ for a solution of (4.25). It is not hard to see that $(\Delta_{\max}^\nu + 1)\omega = 0$, so $\omega_i \in C^\infty((-2\varepsilon_0, 2\varepsilon_0), \Omega^{\nu-i}(N))$, and that (4.25) is equivalent to the initial value problem

$$\begin{aligned} \omega_0''(x) &= (\Delta_N^\nu + 1)\, \omega_0(x), \\ \omega_0'(\varepsilon_0) &= \pm(d_N \delta_N + 1) *_N \omega_0(\varepsilon_0), \\ \omega_1(x) &= \pm *_N d_N \omega_0(x). \end{aligned} \tag{4.26}$$

Using the basis $(\psi_i)_{i \in \mathbb{N}}$ introduced above and denoting by ω_0^h the harmonic part, we write

$$\begin{aligned} \omega_0(x) &= \omega_0^h(x) + \sum_{i \geq 1} [\alpha_i(x)\, \psi_i + \beta_i(x) *_N \psi_i], \\ \omega_0^L(x) &:= \omega_0^h(x) + \sum_{i=1}^{L} [\alpha_i(x)\psi_i + \beta_i(x) *_N \psi_i], \\ \omega_1^L(x) &:= \pm *_N d_N \omega_0^L(x), \\ \omega^L &:= \omega_0^L(x) + \omega_1^L(x) \wedge dx. \end{aligned}$$

Then, if ω solves (4.25) so does ω^L. Moreover, from (4.23) we derive

$$(\bar{\omega}, *_N \bar{\eta})_{L^2(\Lambda^* N)} = 0, \qquad \eta \in \Omega_I^\nu(\bar{M}),$$

which easily implies, with $\varphi \in C_0^\infty(-2\varepsilon_0, 2\varepsilon_0)$,

$$\varphi\omega^L \in \Omega_I^\nu(\bar{M}).$$

Since $\varphi\omega^L \to \varphi\omega$ in $L^2(\Lambda^\nu T^*M)$, an easy approximation argument now proves that $\omega \in \mathscr{D}(\bar{T})$.

To deal with a general metric we remark first that $\Omega_I^\nu(\bar{M})$ depends only on the metric induced on N. For a given metric g_1 on $\bar{M}$ we can construct a "nice" metric g_2 which coincides with g_1 on N. We introduce the complex $(d, \tilde{\Omega}(M))$ where

$$\tilde{\Omega}^i(M) := \begin{cases} \Omega^i(M), & 0 \leq i < \nu, \\ \Omega_I^\nu(\bar{M}), & i = \nu, \\ \Omega^i(\bar{M}), & i > \nu. \end{cases} \tag{4.27}$$

Then we obtain Hilbert complexes $(\mathscr{D}^j, D^j)$ by taking the closure with respect to the Hilbert structure induced by g_j, $j=1, 2$, and $(\mathscr{D}^2, D^2)$ satisfies Poincaré duality by the arguments above and Lemma 3.7. But the identity induces a complex isomorphism $h: (\mathscr{D}^1, D^1) \to (\mathscr{D}^2, D^2)$, so $(\mathscr{D}^1, D^1)$ satisfies Poincaré duality, too.

We turn to the Fredholm property. Note that we can construct an ideal boundary condition with Poincaré duality from (4.27) also in the case $m \equiv 3 \bmod 4$, and for this specific Hilbert complex, $(\mathscr{D}, D)$, we show that $\dim \mathscr{H}_v(\mathscr{D}, D) = \infty$. To do so we may again assume that the metric is "nice," i.e., a neighborhood U of N is isometric to $(-\varepsilon_0, 0] \times N$ with the product metric. Choose $\varphi \in C_0^\infty(-\varepsilon_0, \varepsilon_0)$ with $\varphi(0)=1$; then

$$\mathscr{K} := \{d_{v-1}(\varphi(x)\alpha) \mid \alpha \in \Omega^{v-1}(N)\}$$

is contained in $\Omega_r^v(\bar{M}) \cap \ker d_v$ but not in $d_{v-1}(\Omega_0^{v-1}(M))$ unless $d_N \alpha = 0$. The same construction proves non-Fredholmness in case m is even, and shows, in fact, that there are infinitely many ideal boundary conditions for $(\Omega_0(M), d)$ which are not Fredholm. ∎

Remark. Note that we can always, for arbitrary M, construct an ideal boundary condition with Poincaré duality if $m \equiv 3 \bmod 4$.

Thus a manifold with boundary already provides a very rich variety of ideal boundary conditions. This makes it a good source of counterexamples like the following.

EXAMPLE. It is possible to have two different ideal boundary conditions, $(\mathscr{D}^j, D^j)$, for $(\Omega_0(M), d)$ such that $\mathscr{D}_i^1 \subset \mathscr{D}_i^2$ and the inclusion induces an isomorphism on homology.

In fact, pick $0 \leqslant i < m-1$ and $\omega \in \mathscr{D}(d_{i,\max})$ with $d\omega \mid N \neq 0$. Then we put $(\mathscr{D}^1, D^1) := (\mathscr{D}^r, D^r)$ and

$$\mathscr{D}_j^2 := \begin{cases} \mathscr{D}_j^1, & j \neq i, i+1, \\ \mathscr{D}_i^1 + \operatorname{span} \omega, & j = i, \\ \mathscr{D}_{i+1}^1 + \operatorname{span} d\omega, & j = i+1. \end{cases}$$

A routine check shows that this construction fulfills our claim. It is not clear, however, whether there is a counterexample with $(\mathscr{D}^1, D^1) = (\mathscr{D}^r, D^r)$, $(\mathscr{D}^2, D^2) = (\mathscr{D}^a, D^a)$. This would be of great interest in the discussion of uniqueness for ideal boundary conditions.

As another remark we note that in the proof of Lemma 4.3 we have used that the signature of a compact oriented manifold N with even dimension vanishes if N bounds. Thus one may conjecture that sign N is an obstruction to self-adjoint extensions of $\sqrt{-1} * d$ on a suitable *singular* manifold M, with $m = 4k+1$, and boundary N. In fact, one can construct such an M, noncompact and with conic singularities, such that the deficiency indices, $n_\pm$, of $\sqrt{-1} * d$ satisfy the relation

$$n_+ - n_- = \operatorname{sign} N. \tag{4.28}$$

We will return to this example in a future publication.

In the above discussion we have viewed a compact manifold M with boundary N as contained in the compact double $\tilde{M}$; for most questions we could even assume that the metric on M is the restriction of a reflection-invariant metric on $\tilde{M}$. Thus we obtain in particular the fact that the elliptic complex $(\Omega_0(\tilde{M} \setminus N), d)$ has infinitely many ideal boundary conditions. This leads to the following problem: Let $\tilde{M}$ be compact Riemannian and $\Sigma \subset \tilde{M}$ closed and of measure zero. Under what conditions do we have a unique ideal boundary condition for $(\Omega_0(M), d)$ if $M := \tilde{M} \setminus \Sigma$? For this question we have the following partial answer.

THEOREM 4.4. *Let $\tilde{M}$ be compact, $m = \dim \tilde{M} > 2$, and let $\Sigma \subset \tilde{M}$ be a closed subset which is a finite union of submanifolds with codimension at least two.*

Then $(\Omega_0(M), d)$ has a unique ideal boundary condition, $M := \tilde{M} \setminus \Sigma$.

Proof. Denote by D_{GB} the Gauß–Bonnet operator on $\Omega(\tilde{M})$. Since D_{GB} has a unique closed extension (which we also denote by D_{GB}) it is enough to prove that $\Omega_0(M)$ is a core for D_{GB}. Since $\mathscr{D}(D_{GB}) = H^1(\Lambda^* M)$, we only have to show that $\Omega_0(M)$ is dense in $H^1(\Lambda^* M)$.

Now we introduce the set function "capacity" as

$$\operatorname{Cap} \Sigma := \inf\{\|u\|_{H^1}^2 \mid u \in C^\infty(\tilde{M}), u = 1 \text{ near } \Sigma\}, \tag{4.29}$$

following [M, Chap. 9]. A straightforward adaption of the arguments given in [M, pp. 396, 397] shows that $\Omega_0(M)$ is dense in $H^1(\Lambda^* M)$ if and only if $\operatorname{Cap} \Sigma = 0$. As in loc. cit. we consider also

$$\operatorname{cap} \Sigma := \inf\{\|u\|_{H^1}^2 \mid u \in C^\infty(\tilde{M}), u \geqslant 1 \text{ on } \Sigma\}. \tag{4.30}$$

If $u \in C^\infty(\tilde{M})$ and $u \geqslant 1$ near Σ then $\tilde{u} := -(1-u)_+$ satisfies $\tilde{u} \in H^1(\tilde{M})$, $\tilde{u} \mid \Sigma = 1$, and

$$\|\tilde{u}\|_{H^1} \leqslant C \, \|u\|_{H^1},$$

where C does notdepend on u. Hence $\operatorname{Cap} \Sigma = 0$ is equivalent to $\operatorname{cap} \Sigma = 0$, and if $\Sigma = \bigcup_{i=1}^L \Sigma_i$ is any finite decomposition then $\operatorname{cap} \Sigma = 0$ if and only if $\operatorname{cap} \Sigma_i = 0$, $1 \leqslant i \leqslant L$.

But then we may assume $\Sigma \subset \mathbb{R}^m$, and the proof of cap $\Sigma = 0$ follows from [M, p. 358]. ∎

We can reformulate the above result as follows: if the Hausdorff dimension of Σ is at most $m-2$ then there is a unique ideal boundary condition whereas this is certainly wrong for Hausdorff dimension $\geqslant m-1$. It seems interesting to investigate the behavior for dimensions in $(m-2, m-1)$.

In the case just discussed we are in the situation for Theorem 3.13, i.e., (3.40) holds. We want to conclude this section with an example for the situation (3.39), namely the complex of basic forms associated to a Riemannian foliation. To describe it we consider a compact Riemannian manifold M together with a foliation $\mathscr{F}$, i.e., an exact sequence of vector bundles

$$0 \to L \to TM \to Q \to 0, \tag{4.31}$$

where L is an involutive subbundle of TM. The rank of Q, q, is called the codimension of $\mathscr{F}$. Then we can introduce the *basic forms* with respect to $\mathscr{F}$ by

$$\Omega_b(M) := \{\omega \in \Omega(M) \mid X \llcorner \omega = X \llcorner d\omega| = 0 \text{ for all } X \in C^\infty(L)\}; \tag{4.32}$$

note that in a distinguished open set $U \subset M$ with distinguished coordinates $(x_1, ..., x_{m-q}, y_1, ..., y_q)$ the restriction of a basic form, ω, satisfies

$$(\omega \mid U)(x, y) = \sum_I \omega_I(y)\, dy_I.$$

Whereas this description has a global counterpart in the case of a simple foliation (generated by a submersion), $\Omega_b(M)$ becomes finite dimensional if $\mathscr{F}$ has a dense leaf. But, in any case, $(\Omega_b(M), d)$ is clearly a subcomplex of $(\Omega(M), d)$. Next we denote by $L^\perp$ the orthogonal complement to L in TM, with respect to the given metric g, and by $\pi^\perp : TM \to L^\perp$ the orthogonal projection. Then $\mathscr{F}$ is called a *Riemannian foliation* if the 2-tensor $g^\perp$, defined by $g^\perp(x, y) := g(\pi^\perp X, \pi^\perp Y)$, satisfies

$$\mathscr{L}_X g^\perp = 0 \qquad \text{for all} \quad X \in C^\infty(L); \tag{4.33}$$

in this case $g^\perp$ is called the *transverse metric* of $\mathscr{F}$.

We assume that $\mathscr{F}$ is Riemannian and that TM and $L^\perp$ are oriented. Then with any local oriented and orthonormal frame, $(F_i)_{i=1}^q$, for $L^\perp$ we can form

$$\nu := F_1^\flat \wedge \cdots \wedge F_q^\flat, \tag{4.34}$$

the *transversal volume form*. Since $\mathscr{F}$ is Riemannian, $\nu \in \Omega_b^q(M)$. We assume that the orientations are chosen in such a way that $\omega := \nu \wedge * \nu$ is the volume form on M; $\chi_{\mathscr{F}} := * \nu$ is called the *characteristic form* of $\mathscr{F}$. Moreover, we can define the *basic Hodge operator*, $*_b$, by

$$*_b\, \alpha := (-1)^{(m-q)(q-p)} * (\alpha \wedge \chi_{\mathscr{F}}), \qquad \alpha \in \Omega_b^p(M). \tag{4.35}$$

Then it can be computed (cf. [KT]) that with $d_b^p := d \mid \Omega_b^p(M)$ we have for $\alpha \in \Omega_b^{p+1}(M)$

$$\delta_b^{p+1} \alpha := (d_b^p)^t \alpha = (-1)^{q(p+2)+1} *_b (d_b - \kappa \wedge) *_b \alpha, \tag{4.36}$$

where κ is the mean curvature form of the leaves of $\mathscr{F}$. It follows from (4.35) that (3.42) holds for $(\Omega_b(M), d_b) \subsetneq (\Omega(M), d)$. Moreover, the relation (3.40) has been proved in [KT, Corollary (4.14)]. Thus Theorem 3.12 applies. It seems that one of the equivalent conditions in this theorem (or some other conditions which is easily seen to be equivalent, too) has been proved by several authors: cf. [KT] and the work quoted there, [NRT]. However, since the smooth Hodge decomposition is equivalent to the essential self-adjointness of

$$t_b := d_b + \delta_b : \Omega_b(M) \to \Omega_b(M), \tag{4.37}$$

it seems desirable to give a proof of this latter fact along the lines developed above. We will return to this problem in a forthcoming publication.

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Hardy's Inequality and Fractal Measures

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Hardy's inequality and the subsequent improvement by McGehee, Pigno, and Smith are generalized from the positive integers to sets of dimension 0, dimension 1, and in between. The asymptotic estimate obtained for the Fourier transform of fractal measures is much in the spirit of recent work by Strichartz.

1. INTRODUCTION

An interesting problem in Fourier analysis is to extend the classical inequalities of the Fourier transform, or what Hardy and Littlewood refer to as the theory of Fourier constants [5], to tempered distributions that correspond to lower-dimensional sets. Particularly important theorems are the L^1 inequality known as Hardy's inequality with the McGehee–Pigno–Smith (henceforth M.P.S.) generalization [7], the Plancherel theorem for L^2, and Payley's theorem [12] with the Pitt–Stein [8] generalizations for L^p, $1 < p \leqslant 2$. Extensions of the Plancherel theorem for measures supported on manifolds in R^n have been established by Agmon and Hormander [1], and more recently by Strichartz [9] for measures on R^n of dimension $0 \leqslant \alpha \leqslant n$, α not necessarily an integer. This paper proves a generalized Hardy inequality (henceforth g.h.i.) for fractal measures on R^1 of dimension α, $0 \leqslant \alpha \leqslant 1$. This result includes the M.P.S. version as the periodic case for $\alpha = 0$. Each of the results above for $\alpha < n$ involves a limit on the Fourier transform side and provides information in the form of an asymptotic growth estimate for the transform.

Some regularity will be required of the support of the fractal measure. Classically, Hardy's inequality and the M.P.S. version hold only for measures supported on a well-ordered set of integers, which means the transform of the measure is in H^1 of the unit circle, at least up to a multiplicative factor of $e^{in\pi}$. The well-known inequalities above, in which $p > 1$, are rearrangement-invariant, while Hardy's inequality is not. This implies